Linearization theorems, Koopman operator and its application

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April, 2014

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1. Introduction
   - Linear and nonlinear dynamics
   - Local and global linearization
2. linearization in large
   - Extensions of Hartman's theorem
   - Examples
3. The Koopman operator
   - Its introduction
   - Koopman operator and partition of the phase space
4. Applications
   - The standard map
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5. Summary

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Complex systems-its characterization

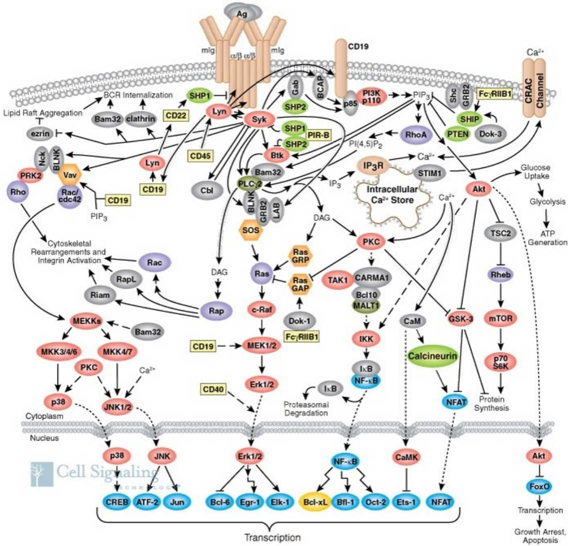
- **Hierarchical Structure**
  Many agents interconnected in a complex manner, resulting in a multi-scale, highly heterogeneous system.
- **Self-organized dynamics**
  Nonlinear, non-equilibrium dynamics leads to the emergent behavior, which is non-reducible and unpredictable from single agent behavior.
- **Adaptability**
  Robust while flexible, evolvable.
- **Uncertainties**
  The determination of parameters and relations are hard.

The internetiverse

Turbulence

Structure of macromolecules

Cell regulatory networks



Nonlinear dynamics: triumph and challenge

- **Triumph of theory of nonlinear dynamical systems**
  Local: linear stability analysis, bifurcation theory, normal form theory, ...
  Global: Asymptotic analysis, topological methods, symbolic dynamics,...
- **Troubles when treating complex systems**
  (1) Huge number of interacting agents (high-dimensional)
  (2) Heterogeneity in spatiotemporal scales (numerical challenge)
  (3) Hierarchical structure and great many dynamic modes
  (4) Lack of exact mathematical description
  (5) Uncertainty in data or parameters (noise or ignorance)

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Linear systems and their solution

- General form: $\dot{\mathbf{x}} = A\mathbf{x}$ with

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{pmatrix} .$$

- Idea: separate solutions into independent modes by assuming $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$.
- We then obtain an **eigenvalue equation**

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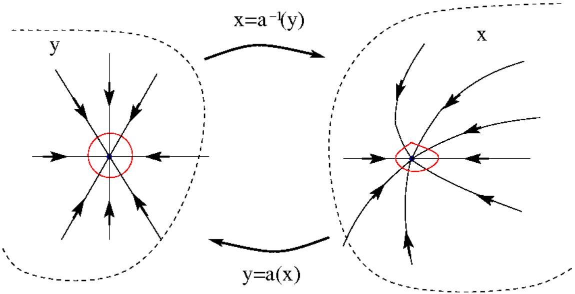

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- Global linearization: weak nonlinearity or symmetry by lie group theory.

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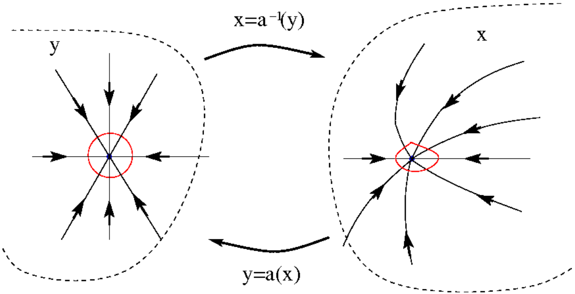

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Linearization in large

- Extension of Hartman's theorem for differential flows.
- The theorem can be extended to diffeomorphic mappings or flows with periodic driving.
- Linearization around an attractive or repulsive periodic orbit.
- How to treat saddles?
  The above theorems are applicable to flows on stable or unstable manifolds.

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Two examples

- Consider the 1-d equation $\dot{x} = x - x^3$. The transformation

$$x = b(y) = \frac{y}{\sqrt{1+y^2}}$$

results in $\dot{y} = y$, valid for $x \in [-1, 1]$.

- Consider the 2-d system $\dot{z}_1 = 2z_1\,, \dot{z}_2 = 4z_2 + z_1^2$. The transformation

$$z_1 = y_1\,, z_2 = y_2 + t(y_1, y_2)y_1^2$$

where $t(y_1, y_2) = \frac{1}{4}\ln y_1^2$ results in

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[Y. Lan and I. Mezic, Physica D. 242, 42(2013)]

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Remarks on the linearization theorem

- According to Morse theory, the whole phase space of a hyperbolic system can be viewed as a gradient(-like) system modulo the minimal invariant sets. The phase space is a juxtaposition of linearizable patches.
- **Problems:**

  (1) Hard to identify the linearization transformation.

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Koopman operator, a way out?

- **Statistical treatment of dynamical systems**:
  Evolution of densities: the Perron-Frobenius Operator in analogy with the Schrödinger picture;

  Evolution of observables: the Koopman operator in analogy with the Heisenberg picture.
- For a map $x_{n+1} = f(x_n)$ and a function $g(x)$, the Koopman operator $U \circ g(x) = g(f(x))$
- For a flow $\phi(x,t)$ and a function $g(x)$, a semigroup of Koopman operators could be defined as $U_t \circ g(x) = g(\phi(x,t))$.

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- In fact, for C^1 observables, the most general form of the eigenfunction with eigenvalue λ^a is

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Eigenvalues and Eigenmodes (continued)

- **Examples**:

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- Suppose

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Construction of eigenmodes along trajectories

- For the map $x_{n+1} = T(x_n)$ and function $g(x)$, consider

$$g^*(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g(T^j x)\,,$$

  which is an eigenfunction of the Koopman operator with eigenvalue 1.

- Furthermore, the construction

$$g^\omega(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} e^{i2\pi j\omega} g(T^j x)$$

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Connection with Liouville operator

- In a Hamiltonian system with $H(p, q)$, the Liouville operator L may be defined as

$$L \circ f(x) = -i \sum_j \left[\frac{\partial H}{\partial p_j} \frac{\partial f}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial f}{\partial p_j} \right] ,$$

  which could be written with the Poisson bracket $L \circ f(x) = -i[f, H]$.

- It is related to Koopman operator by the exponentiation

$$U_t = \exp(itL) .$$

- On a transitive invariant set, Koopman operator is unitary, *i.e.*

$$U_t g(x) = e^{it\alpha} g(x) = e^{i(t\alpha + \arg(g))} |g(x)| .$$

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Spectral decomposition of evolution equations

- For an evolution equation in an infinite-dimensional Hilbert space $v(x)^{n+1} = N(v(x)^n, p)$ and if the attractor M is of finite dimension with the evolution $m^{n+1} = T(m^n)$.
  For an observable $g(x, m)$, we have

$$\begin{aligned} Ug(x,m) &= U_s g(x,m) + U_r g(x,m) \\ &= g^*(x) + \sum_{j=1}^{k} \lambda_j f_j(m) g_j(x) + \int_0^1 e^{i2\pi\alpha} dE(\alpha) g(x,m) . \end{aligned}$$

- U_s: the singular part of the operator corresponding to the discrete part of the spectrum, viewed as a deterministic part.
  U_r: the regular part of the operator corresponding to the continuous part of the spectrum, modeled as a stochastic process.

[I. Mezic, Nonlinear Dynamics 41, 309(2005)]

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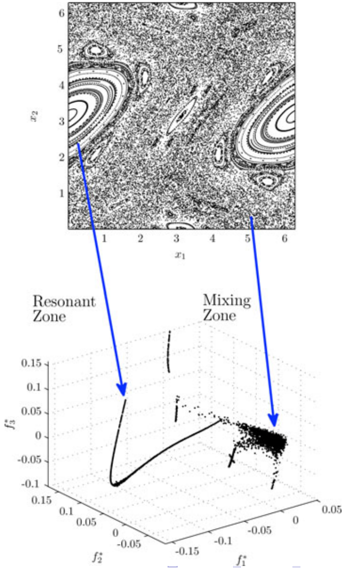
The standard map

- The Chirikov standard map is

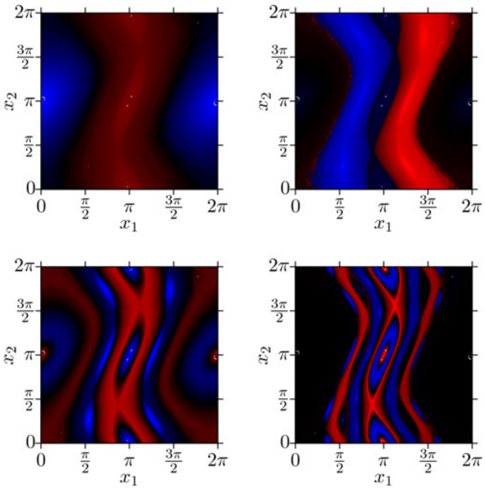
$$x_1^* = x_1 + 2\pi\epsilon \sin(x_2)(mod 2\pi)$$

$$x_2^* = x_1^* + x_2(mod 2\pi)$$

- The embedding of dynamics into space of three observables.
  [M. Budisic and I. Mezic, 48th IEEE Conference on Decision and Control]



Organizing invariant set by diffusion map



Eigenvectors for λ_1 , λ_2 , λ_7 , λ_{17} at $\epsilon = 0.133$.

Main contents

Fluid dynamics

- The Navier-Stokes equation

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{v}$$

with $\nabla \cdot \mathbf{v} = 0$ describes incompressible Newtonian fluids.

- With different Reynold's number $Re = Lv/\nu$, the system experience a series of bifurcation:
laminar $\rightarrow$ periodic $\rightarrow$ turbulent
- Turbulence is a spatiotemporal chaos with enormous space-time structures and scales.
- Jet in cross flow: turbulent but with large eddies. Could we describe it with the Koopman operator approach?

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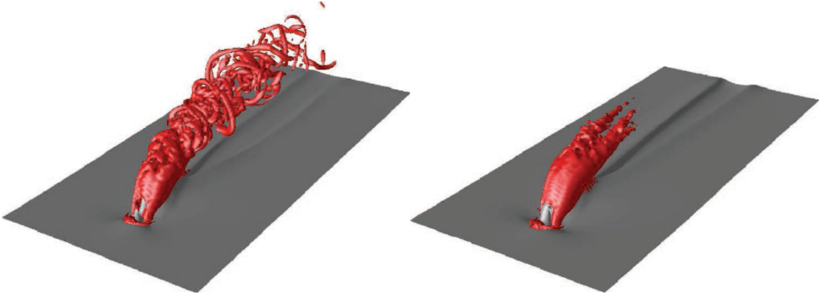

[C. W. Rowley *et al*, J. Fluid Mech. 641, 115(2009)]

The Arnoldi algorithm

- Consider a linear dynamical system $x_{k+1} = Ax_k$ and construct the matrix

$$K = [x_0\,, x_1\,, \cdots\,, x_{m-1}] = [x_0\,, Ax_0\,, \cdots\,, A^{m-1}x_0]\,.$$

If the mth iterate $x_m = Ax_{m-1} = \sum_{k=0}^{m-1} c_k x_k + r$, the we can write $AK \approx KC$, where

$$C = \begin{pmatrix} 0 & 0 & 0 & \cdots & c_0 \\ 1 & 0 & 0 & \cdots & c_1 \\ 0 & 1 & 0 & \cdots & c_2 \\ & \cdots & & \cdots & \\ 0 & \cdots & 0 & 1 & c_0 \end{pmatrix}.$$

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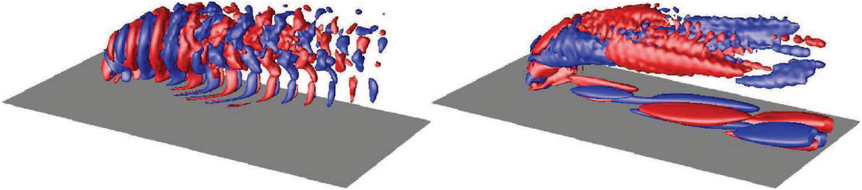
Two structure functions



The two eigenmodes

Summary

- Linearization is possible in the basin of attraction of a hyperbolic set.
- Koopman operator provides a way to identify the linearization transformation.
- On the minimal invariant set, the spectrum of the Koopman operator is on the unit circle.
- The eigenmodes could be constructed through numerical computation, revealing the most important dynamics.
- Generalizations and challenges:
  (1) Can deal with stochastic systems.
  (2) How to deal with uncertainty in complex systems.
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  (1) Can deal with stochastic systems.
  (2) How to deal with uncertainty in complex systems.
  (3) How to construct eigenmodes from pieces of information.

Summary

- Linearization is possible in the basin of attraction of a hyperbolic set.
- Koopman operator provides a way to identify the linearization transformation.
- On the minimal invariant set, the spectrum of the Koopman operator is on the unit circle.
- The eigenmodes could be constructed through numerical computation, revealing the most important dynamics.
- Generalizations and challenges:
  (1) Can deal with stochastic systems.
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