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Stabilized Finite Elements and Discontinuous Galerkin

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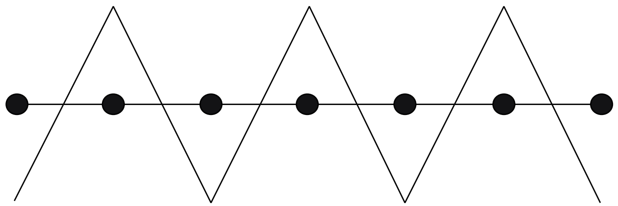
Motivation

- Using Galerkin finite-element method for convection-dominated flows is equivalent to a central-difference method
- Requires dissipation to prevent odd-even decoupling
  - Can be added through “physics” (upwind differencing)
  - Same effect can be achieved through explicit addition

$$\frac{\partial U}{\partial t} + c\frac{\partial U}{\partial x} = 0 \qquad U_i^{n+1} = U_i^n - c\Delta t\left(\frac{U_{i+1}^n - U_{i-1}^n}{\Delta x}\right)$$

$$U_i = U\left(x_i\right)$$



Zero residual using central differencing

Motivation

- Dissipation through improved physics

$$U_i^{n+1} = U_i^n - c\Delta t\left(\frac{U_i^n - U_{i-1}^n}{\Delta x}\right) \quad \text{First-order "upwind"}$$

- Dissipation through explicit addition

$$\frac{\partial U}{\partial t} + c\frac{\partial U}{\partial x} = \upsilon\frac{\partial^2 U}{\partial x^2} \quad \text{Explicitly added dissipation}$$

$$\frac{\partial U}{\partial x} \approx \left(\frac{U_{i+1}^n - U_{i-1}^n}{\Delta x}\right) \qquad \upsilon\frac{\partial^2 U}{\partial x^2} = \upsilon\frac{U_{i+1} - 2U_i + U_{i-1}}{\Delta x^2} \quad \upsilon = c\Delta x \quad \left(\frac{L^2}{t}\right)$$

$$U_i^{n+1} = U_i^n - c\Delta t\left(\frac{U_i^n - U_{i-1}^n}{\Delta x}\right)$$

- Same result obtained using both approaches
- Discontinuous Galerkin follows first approach
- Stabilized finite elements follows second approach

Outline

- Stabilized finite elements (Petrov-Galerkin (PG))
  - Streamline Upwind Petrov Galerkin (SUPG)
  - Surface integral for continuous finite elements
  - Stabilization matrix and viscous scaling
  - Accuracy/Method of Manufactured Solutions (MMS)
- Discontinuous Galerkin (DG)
- Accurate surface representation
  - Effect on accuracy
  - Treatment
- Example results
- Comparison of accuracy and efficiency between PG and DG

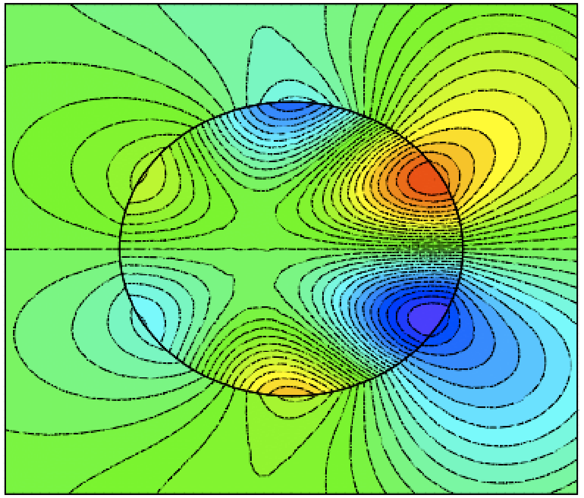
Petrov-Galerkin

$$\iiint_{\Omega_k} w_i \frac{\partial Q}{\partial t} \partial\Omega_k - \iiint_{\Omega_k} \nabla w_i \cdot \left(F\left(Q\right) - F_v\left(Q\right)\right) \partial\Omega_k +$$
$$\iiint_{\Omega_k} \left\{ \left(\frac{\partial w_i}{\partial x}\left[A\right] + \frac{\partial w_i}{\partial y}\left[B\right] + \frac{\partial w_i}{\partial z}\left[C\right] \right) \left[\tau\right] \left(\frac{\partial Q}{\partial t} + \nabla \cdot \left(F\left(Q\right) - F_v\left(Q\right)\right) \right) \right\} \partial\Omega_k +$$
$$\iint_{\Gamma_k} w_k \left(F\left(Q\right) - F_v\left(Q\right)\right) \cdot \hat{n}\, \partial\Gamma_k = 0$$

- Not widely used for compressible flow: Approximately ten times fewer papers in AIAA conferences compared with discontinuous Galerkin
- Surface integral typically not evaluated because of continuity assumptions between elements. However, assumption not required (e.g. multiple materials in electromagnetics)

Evaluation of Surface Integral

- Typically ignored due to assumed continuity across elements
- Not a required assumption, such as multiple materials or port boundary conditions in electromagnetic applications
  - Create duplicate mesh points along interface
  - Resolve jumps in field parameters using Riemann solver
  - May also be used to easily create discontinuous-Galerkin



Stabilization Matrix

- Weighted residual form for 1D steady advection with modified weighting term

$$\int_\Omega \left(N + \frac{\partial N}{\partial x} a\tau \right)\left(a\frac{\partial u}{\partial x} \right) d\Omega = 0$$

- First term in weight function results in Galerkin method
- Consider stabilization term with $\tau^{-1} = 2\frac{\partial N}{\partial x}|a|$ ($\tau = \frac{\Delta x}{2|a|}$)

$$\int_\Omega \left(\frac{\partial N}{\partial x} a\tau \right)\left(a\frac{\partial u}{\partial x} \right) d\Omega = \frac{1}{2}\int_\Omega \frac{\partial N}{\partial x} a \frac{\Delta x}{|a|}\left(a\frac{\partial u}{\partial x} \right) d\Omega = \frac{\Delta x}{2}\int_\Omega \frac{\partial N}{\partial x}|a|\frac{\partial u}{\partial x} d\Omega$$

- Note that τ scales in proportion to the mesh size

Stabilization Matrix

- Integrate last term by parts

$$\frac{\Delta x}{2}\int_{\Omega}\frac{\partial N}{\partial x}\left|a\right|\frac{\partial u}{\partial x}d\Omega = -\left|a\right|\frac{\Delta x}{2}\int_{\Omega}N\frac{\partial^2 u}{\partial x^2}d\Omega$$

- Stabilization term is dissipative with a coefficient of $\left|a\right|\frac{\Delta x}{2}$
- Similar to form given above for model problem that yielded first-order upwind scheme

Stabilization Matrix

- Scaling of stabilization different for inviscid and viscous limit

$$\iiint_{\Omega} \left[N + \left(\frac{\partial N}{\partial x} a \right) [\tau] \right] \left[a \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} \left(\nu \frac{\partial u}{\partial x} \right) \right] d\Omega = 0$$

Weighted residual

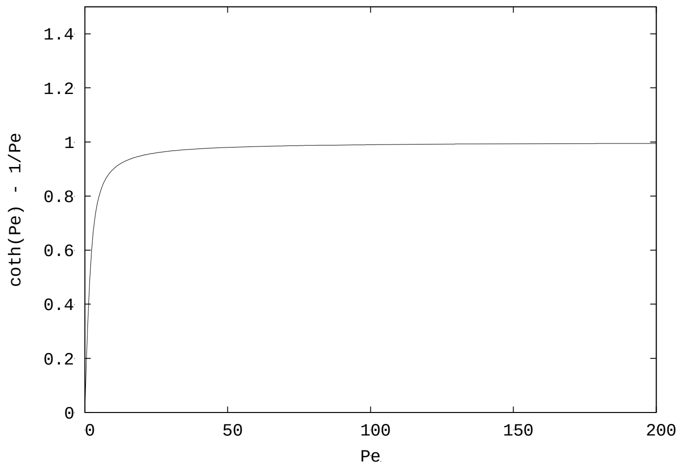
- Galerkin contribution is integrated by parts thereby lowering the order of the derivatives
- Stabilization term is not integrated by parts
- If solution converges as $O\left(h^{p+1}\right)$ then derivatives converge with order $O\left(h^{p}\right)$ and second derivatives as $O\left(h^{p-1}\right)$
  - For inviscid flows τ scales by $O\left(h\right)$
  - for viscous flow τ needs to scale as $O\left(h^{2}\right)$

Stabilization Matrix

- For scalar advection-diffusion the stabilization matrix can be scaled according to the Peclet number so that in the viscous limit τ behaves as $O\left(h^2\right)$ instead of $O\left(h\right)$

$$\tau = \frac{\Delta x}{2a}\left(\coth\left(P_e\right) - \frac{1}{P_e}\right)$$

$$P_e = \frac{a\Delta x}{2\nu}$$



- Equation above determined to obtain exact solution in 1D
- Cotangent scaling based on Peclet number for systems in multiple dimensions found to be unreliable

Stabilization Matrix

- Technique due to Shakib automatically scales stabilization
- Varies as $O(h)$ for inviscid flows, $O(h^2)$ for viscous flow

$$a\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right) = 0$$

$$\tau^{-1} = \sum_i \left(\left| \frac{\partial N_i}{\partial x} a \right| + \frac{\partial N_i}{\partial x} \nu \frac{\partial N_i}{\partial x} \right) \propto \frac{1}{L^2}\left(\left| aL \right| + \nu \right)$$

$$\tau \propto \frac{L^2}{\left| aL \right| + \nu}$$

- Above heuristics suggest generalization to multiple dimensions

Stabilization Matrix

- Eigenvalue-based stabilization is “baseline”

$$\left[\tau\right]^{-1}=\sum_i\left(\left|\frac{\partial N_i}{\partial x}\left[\boldsymbol{A}\right]+\frac{\partial N_i}{\partial y}\left[\boldsymbol{B}\right]+\frac{\partial N_i}{\partial z}\left[\boldsymbol{C}\right]\right|+\frac{\partial N_i}{\partial x_j}\left[\boldsymbol{K}\right]_{jk}\frac{\partial N_i}{\partial x_k}\right)$$

- Inviscid contribution may be defined using concepts from flux-vector splitting

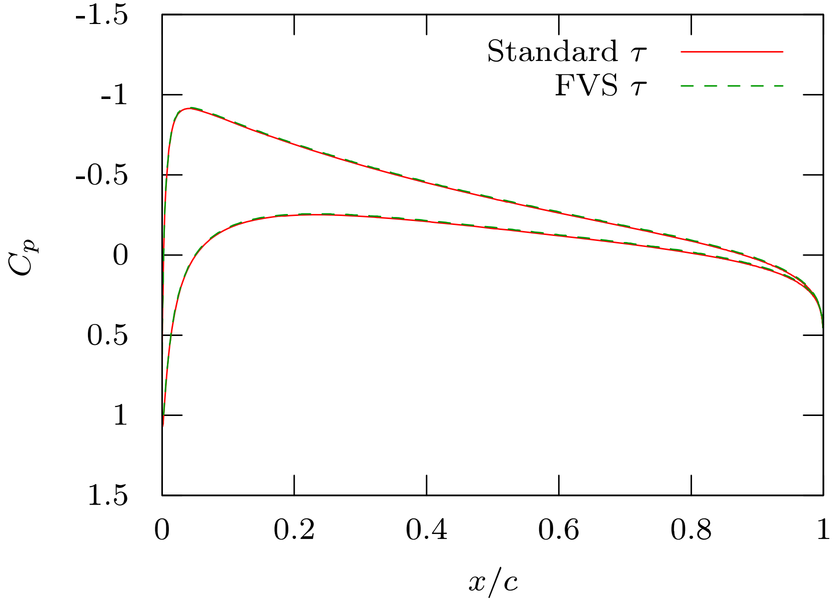
$$\left|\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{Q}}\right|=\frac{\partial \boldsymbol{F}^{+}}{\partial \boldsymbol{Q}}-\frac{\partial \boldsymbol{F}^{-}}{\partial \boldsymbol{Q}}$$

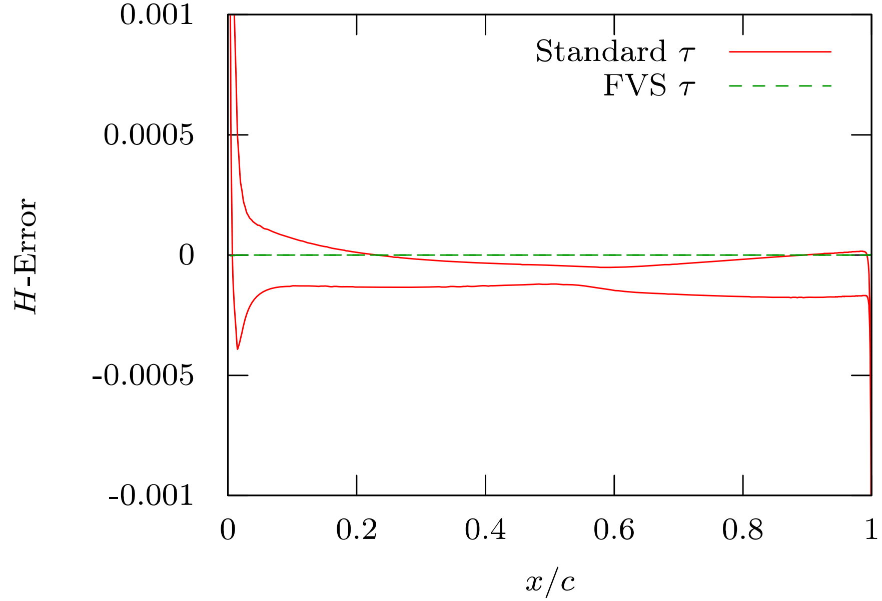
- $\frac{\partial \boldsymbol{F}^{+}}{\partial \boldsymbol{Q}}$ positive eigenvalues: $\frac{\partial \boldsymbol{F}^{-}}{\partial \boldsymbol{Q}}$ negative eigenvalues

$$\left[\tau\right]^{-1}=\sum_i\left(\frac{\partial}{\partial \boldsymbol{Q}}\left(\left(\boldsymbol{F}^{+}-\boldsymbol{F}^{-}\right)\cdot\left\{\frac{\partial N_i}{\partial \chi}\right\}\right)+\frac{\partial N_i}{\partial x_j}\left[\boldsymbol{K}\right]_{jk}\frac{\partial N_i}{\partial x_k}\right)$$

Stabilization Matrix Based on FVS

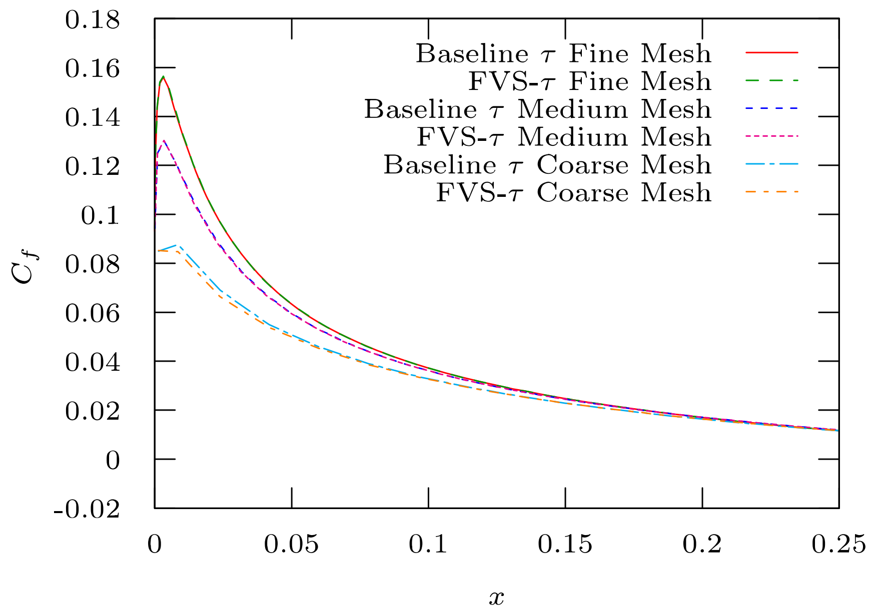
- Any flux-vector splitting formulation can be used
- Using van Leer FVS with Hanal modifications can maintain constant total enthalpy



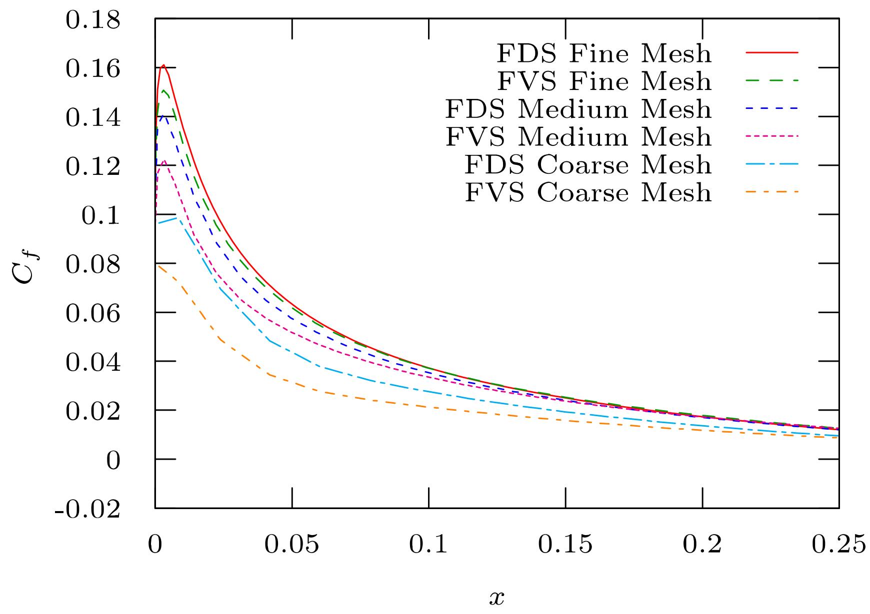


Stabilization Matrix Based on FVS

FVS-based stabilization not inferior to eigenvalue-based stabilization for viscous flows



Petrov Galerkin



Finite Volume

Manufactured Solutions

- Verification is the process of determining if the code implementation is correct
- Validation is determining whether physical model is correct
- Would like to verify implementation using nontrivial solutions
- Few solutions exists especially for turbulent flows
- Method of Manufactured Solutions (MMS) is a means to generate exact solutions

Verification of Order using MMS

- Used to generate nontrivial exact solutions to PDEs
- Substitution of specified solution into PDE results source term
- Example $\nabla^2\psi = 0$

$$\psi = x^2 - y^2 \quad \text{No source term results}$$

$$\psi = \sin\left(\pi x\right) + \cos\left(\pi y\right)$$

$$\nabla^2\psi = -\pi^2\left(\sin\left(\pi x\right) + \cos\left(\pi y\right)\right)$$

- Can generate nontrivial exact solutions to Euler and Navier-Stokes equations that can be used to verify accuracy
- Interior scheme verified with Dirchlet boundary conditions
- Boundary conditions can also be verified if MMS satisfies BC

Guidelines for Manufactured Solutions

- Composed of smooth functions such as polynomials, trigonometric and exponentials
- Should be general enough to exercise all terms in governing equations
- Sufficient number of nontrivial derivatives (e.g. linear polynomials insufficient for verifying second-order accuracy)
- Solution should not contain singularities, discontinuities, or steep gradients
- Experience also shows that while steep gradients may be theoretically acceptable, resolution requires finer grids for solution to be in linear range
- Manufactured solutions not necessarily physically meaningful

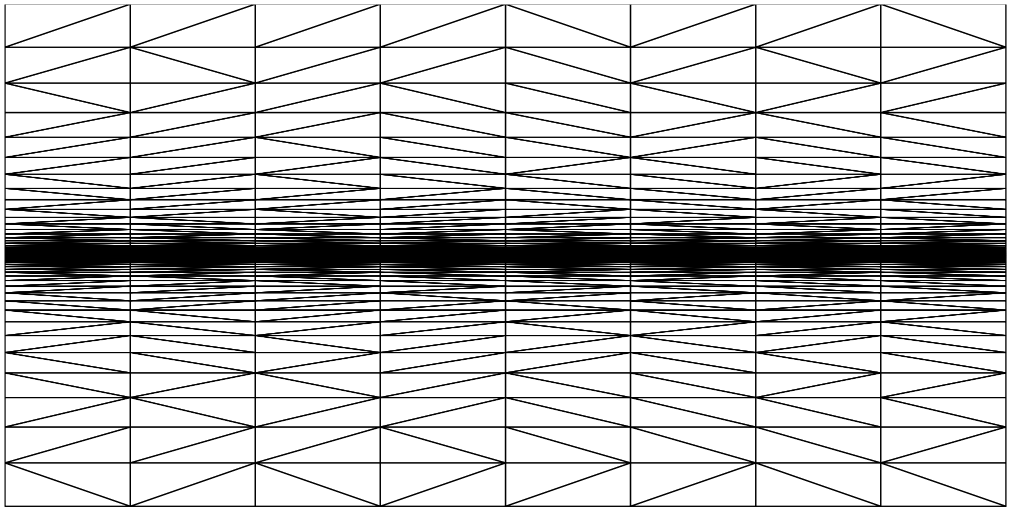
Manufactured Solutions

$$\rho(x,y) = A_\rho\left(1 + \sin\left(\pi x\right)\cos\left(\pi x\right)\sin\left(\pi y\right)\cos\left(\pi y\right)\right)$$

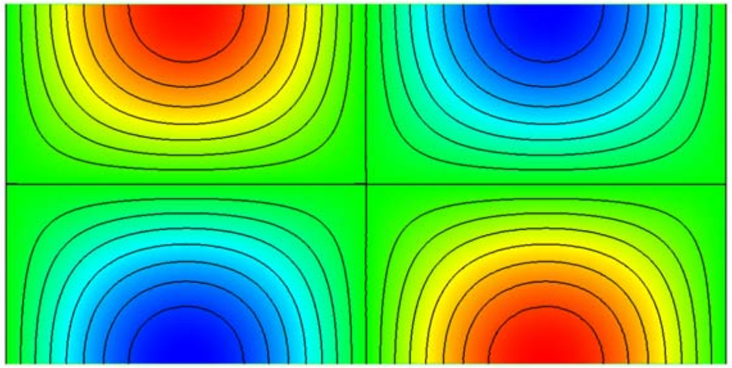
$$u(x,y) = A_u\left(1 + \sin\left(2\pi x\right)\cos\left(2\pi x\right)\sin\left(2\pi y\right)\cos\left(2\pi y\right)\right)$$

$$u(x,y) = A_v\left(1 + \cos\left(2\pi x\right)\cos\left(2\pi x\right)\cos\left(2\pi y\right)\cos\left(2\pi y\right)\right)$$

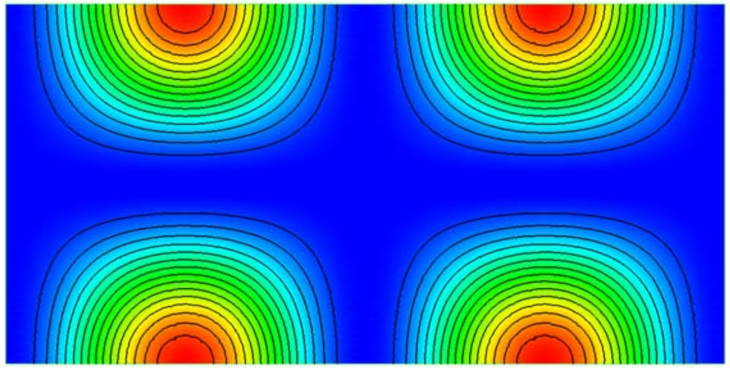
$$T(x,y) = A_T\left(1 + \sin\left(2\pi x\right)\sin\left(2\pi x\right)\sin\left(2\pi y\right)\sin\left(2\pi y\right)\right)$$



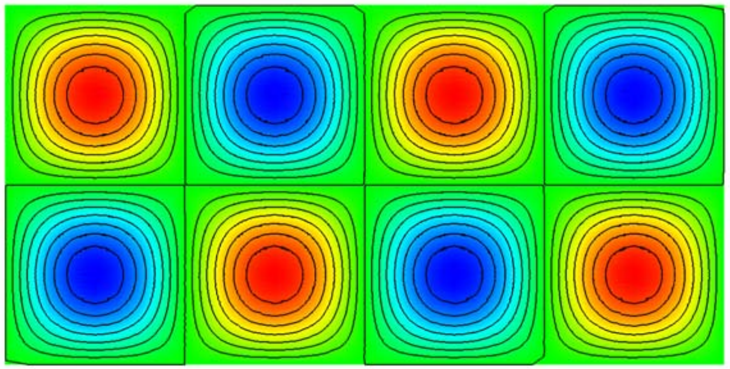
Manufactured Solutions



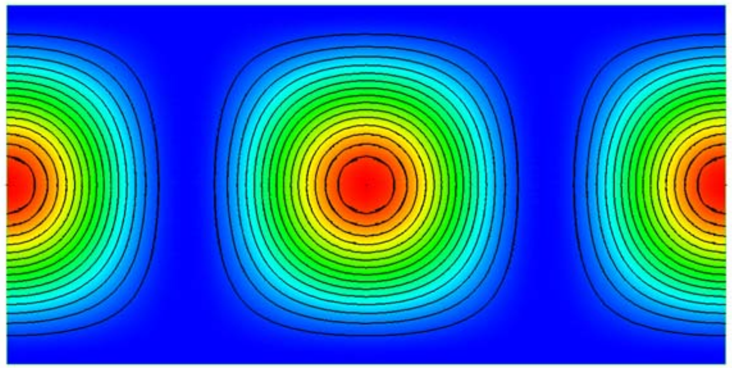
Density



Temperature



u-velocity

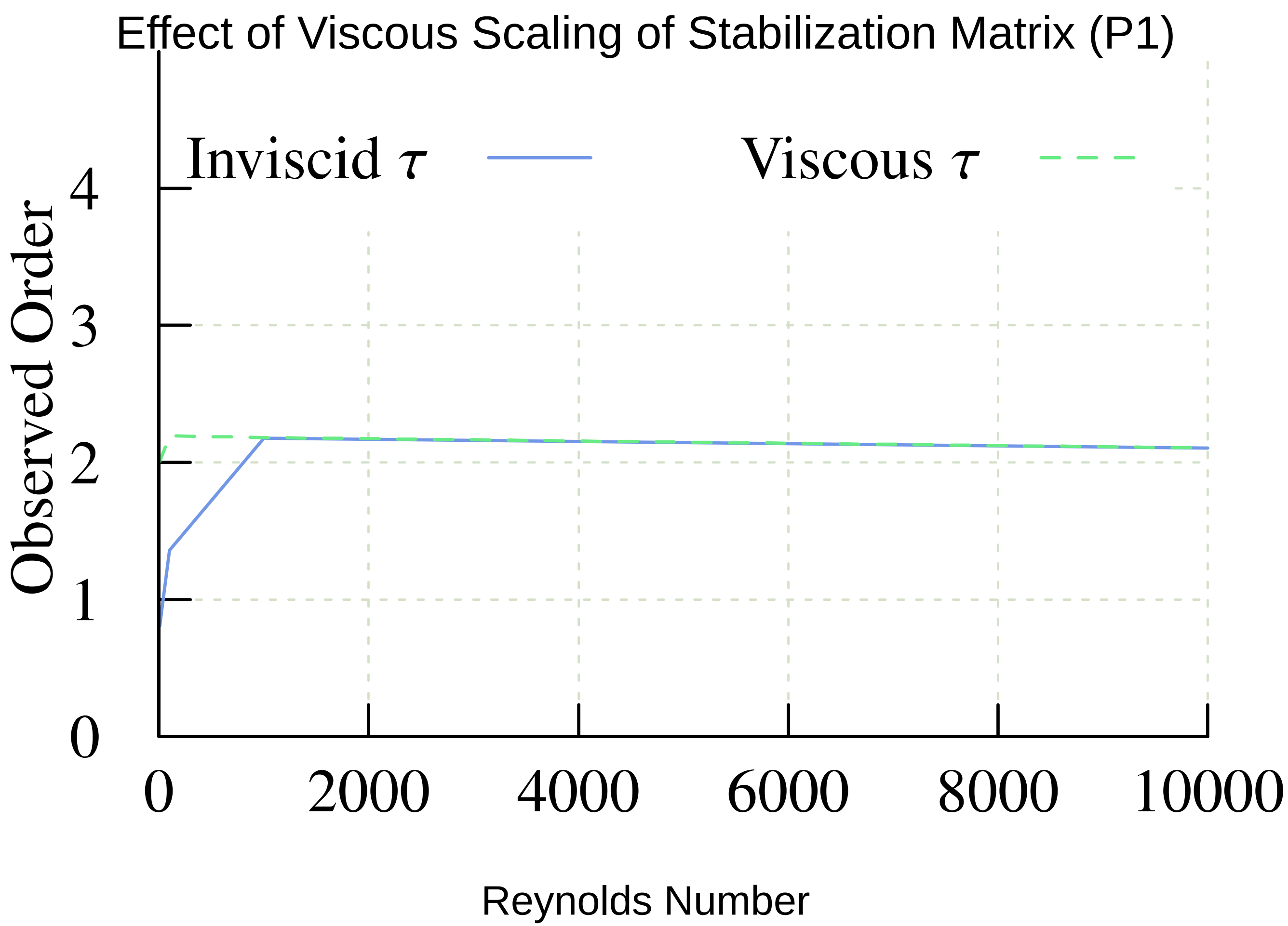


v-velocity

Order of Accuracy using Manufactured Solutions

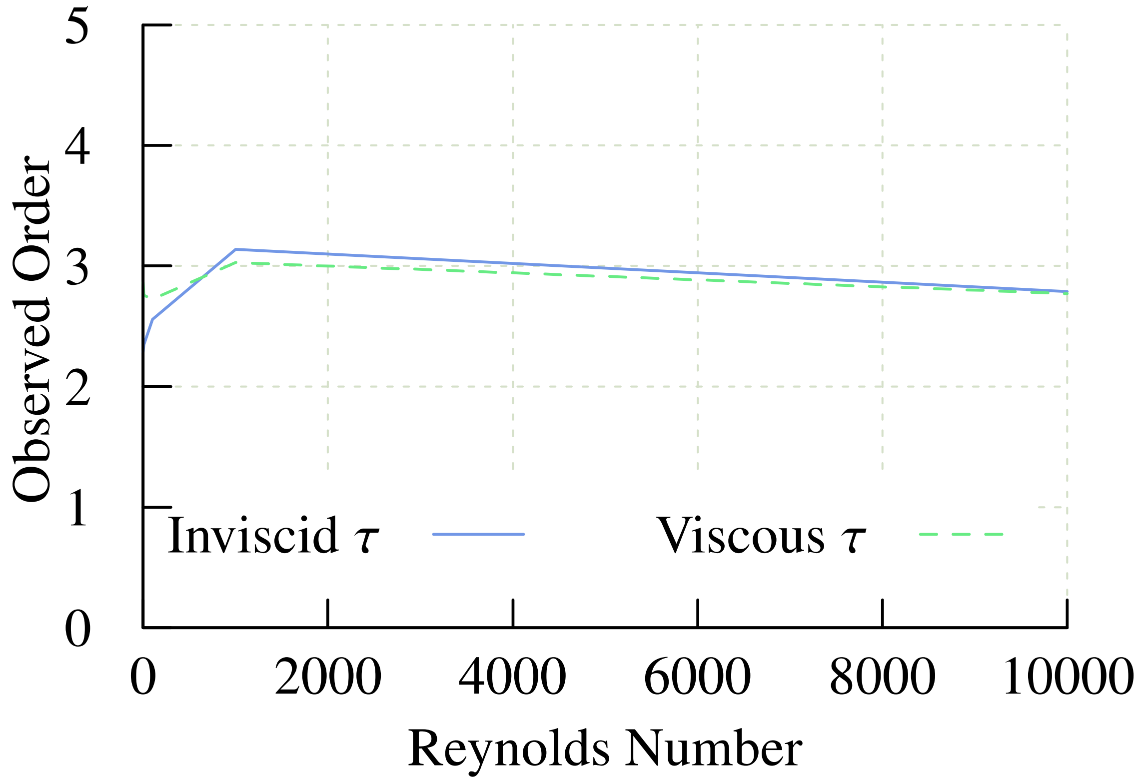
Nodes in mesh		P_1 Elements		P_2 Elements		P_3 Elements	
		L_1	L_2	L_1	L_2	L_1	L_2
2,193/8,481	ρ	2.0437	2.0839	2.8186	2.7675	4.2141	4.1781
2,193/8,481	u	2.0433	2.0634	2.6693	2.6928	4.1830	4.1818
2,193/8,481	v	2.0018	2.0148	2.4867	2.4096	4.1942	4.1866
2,193/8,481	T	2.1693	2.1739	3.0159	2.9528	4.2189	4.2144

Order of Accuracy using Manufactured Solutions



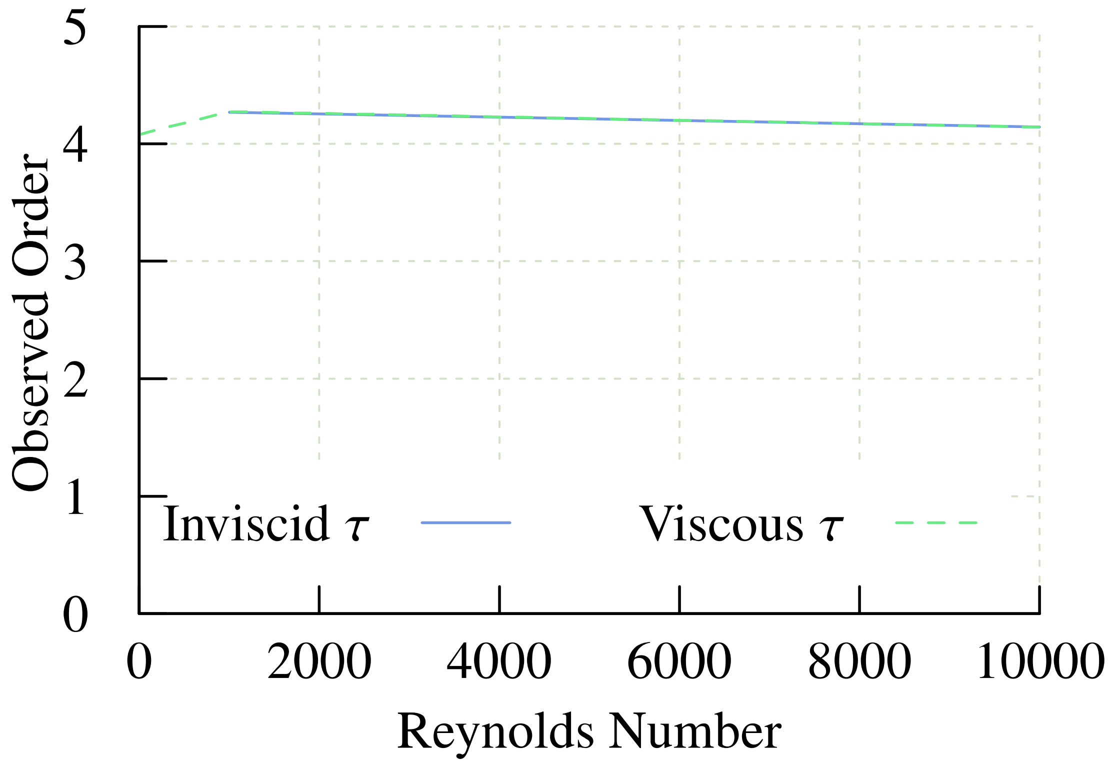
Order of Accuracy using Manufactured Solutions

Effect of Viscous Scaling of Stabilization Matrix (P2)

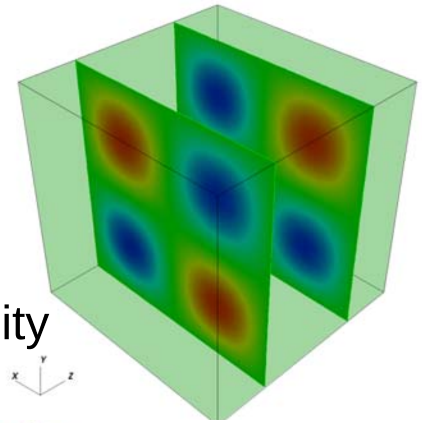


Order of Accuracy using Manufactured Solutions

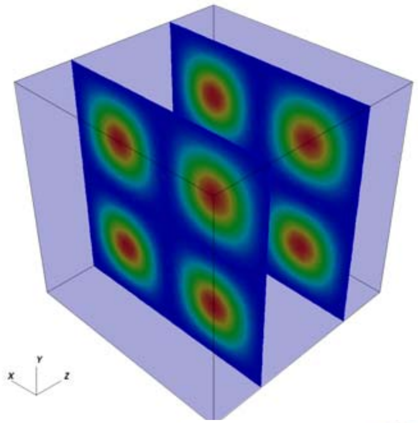
Effect of Viscous Scaling of Stabilization Matrix (P3)



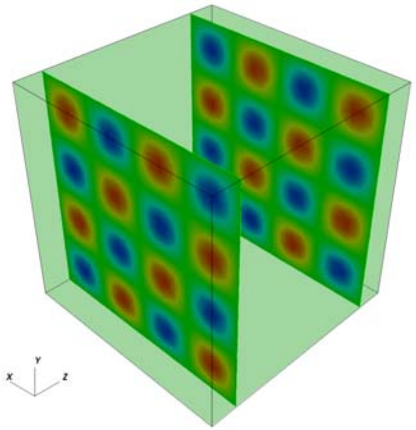
Manufactured Solutions for 3D



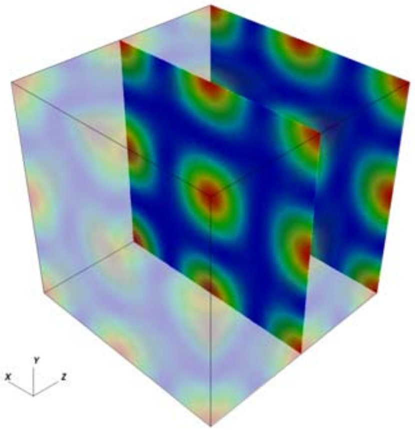
Density



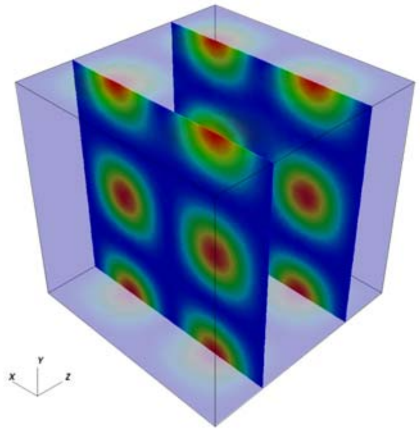
Temperature



u-velocity



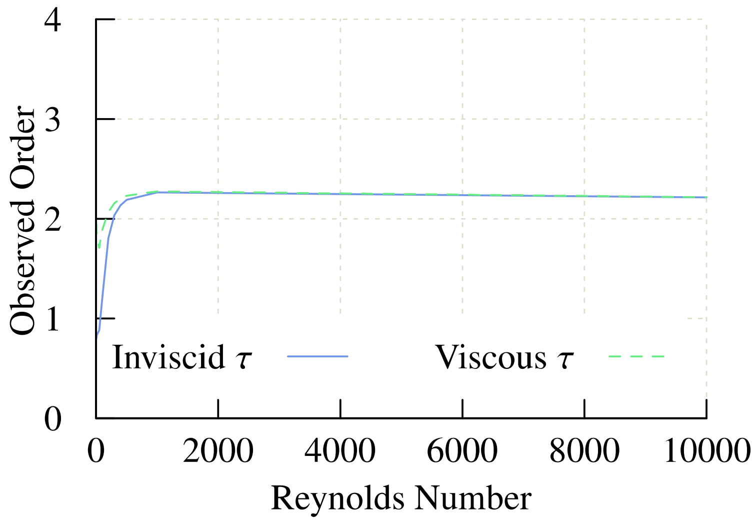
v-velocity



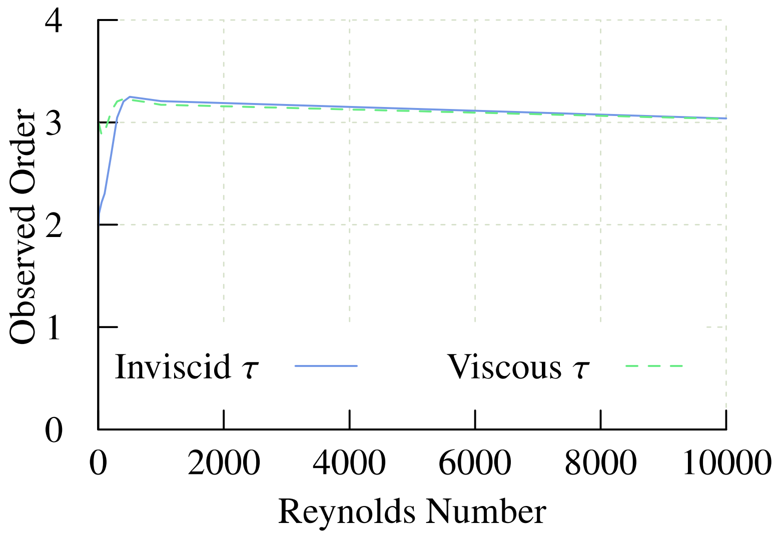
w-velocity

Order of Accuracy using Manufactured Solutions for 3D

Effect of Viscous Scaling of Stabilization Matrix



Linear elements

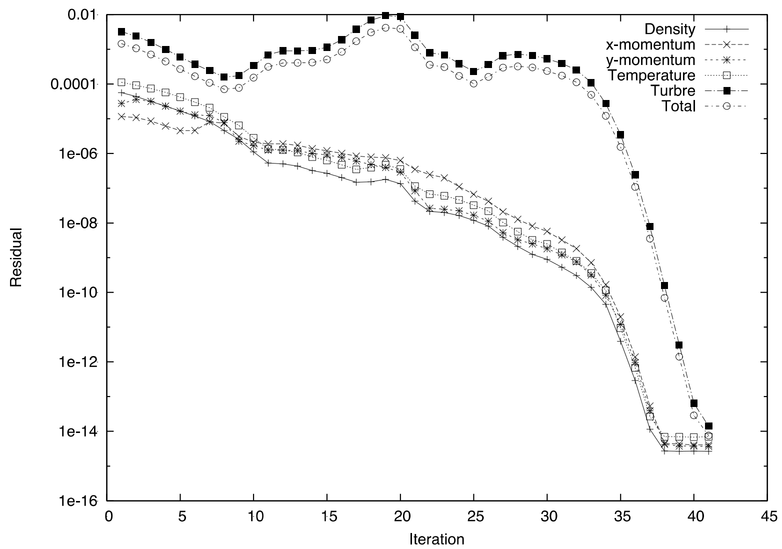


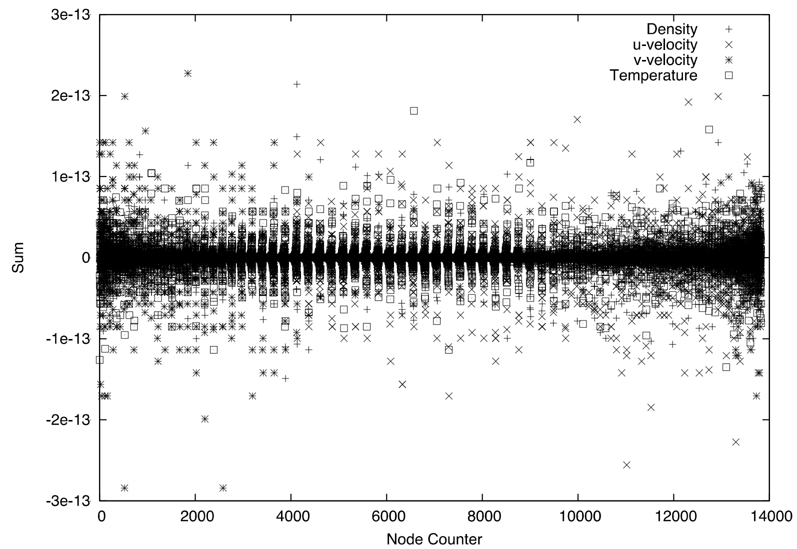
Quadratic elements

Conservation

- Conservation proven by Venkatakrishnan et al.
- At convergence global conservation can be checked by summing columns of linearization matrix
- Demonstration using third-order solution for turbulent flow

NACA 0012 Airfoil: $M_\infty = 0.15,\ \alpha = 10^o,\ \mathrm{Re}{=}6\mathrm{x}10^6$





Discontinuous Galerkin

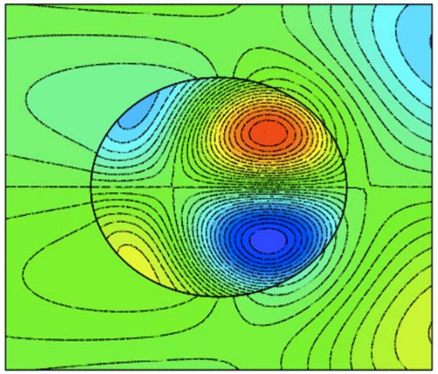
$$\iiint_{\Omega_k} w_i \frac{\partial Q}{\partial t} \partial\Omega_k - \iiint_{\Omega_k} \nabla w_i \cdot \left(F\left(Q\right) - F_v\left(Q\right)\right) \partial\Omega_k + \iint_{\Gamma_k} w_k \left(F\left(Q\right) - F_v\left(Q\right)\right) \cdot \hat{n}\, \partial\Gamma_k = 0$$

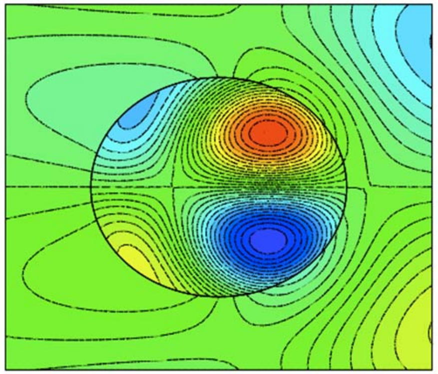
- Solution assumed discontinuous across element interfaces
- Surface integral evaluation using Riemann solver
- Viscous terms handled using symmetric interior penalty method

Accuracy Effects Caused by Inaccurate Geometry

- Second-order schemes model surface using linear segments
- Linear surface reduces order property for high-order elements

Exact Solution



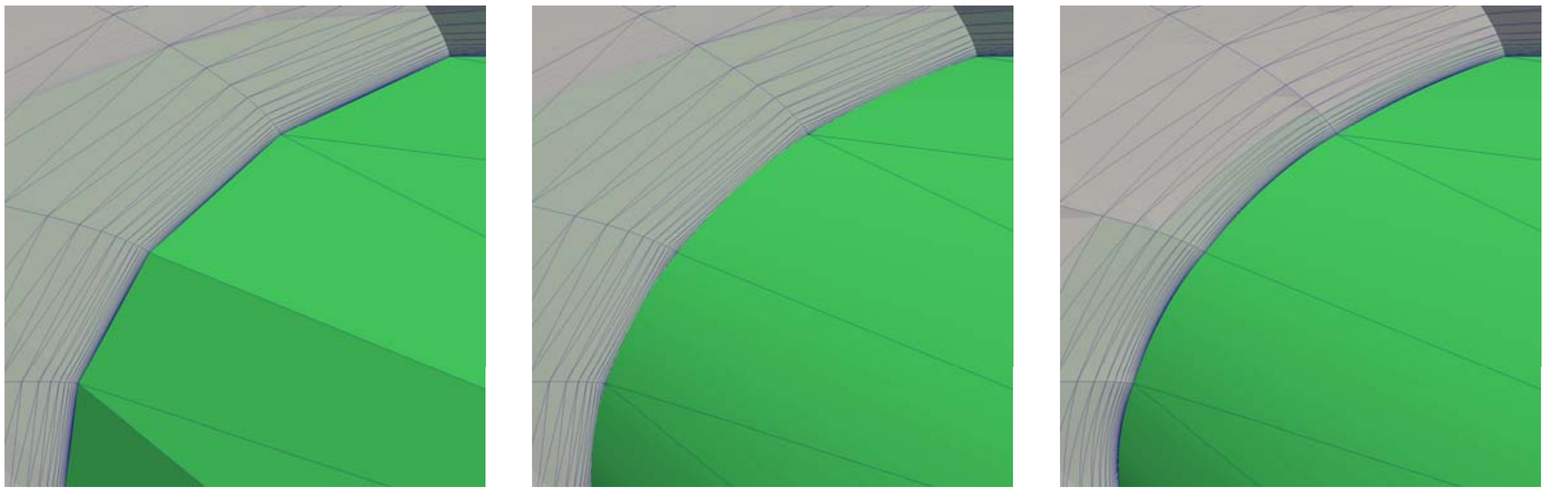


Solution from quadratic elements

	Linear	Quadratic (linear geometry)	Quadratic (quadratic geometry)
B_x	2.2181	2.0675	2.9667
B_y	2.3263	2.0546	3.0138
D_z	2.4441	2.0697	3.0666

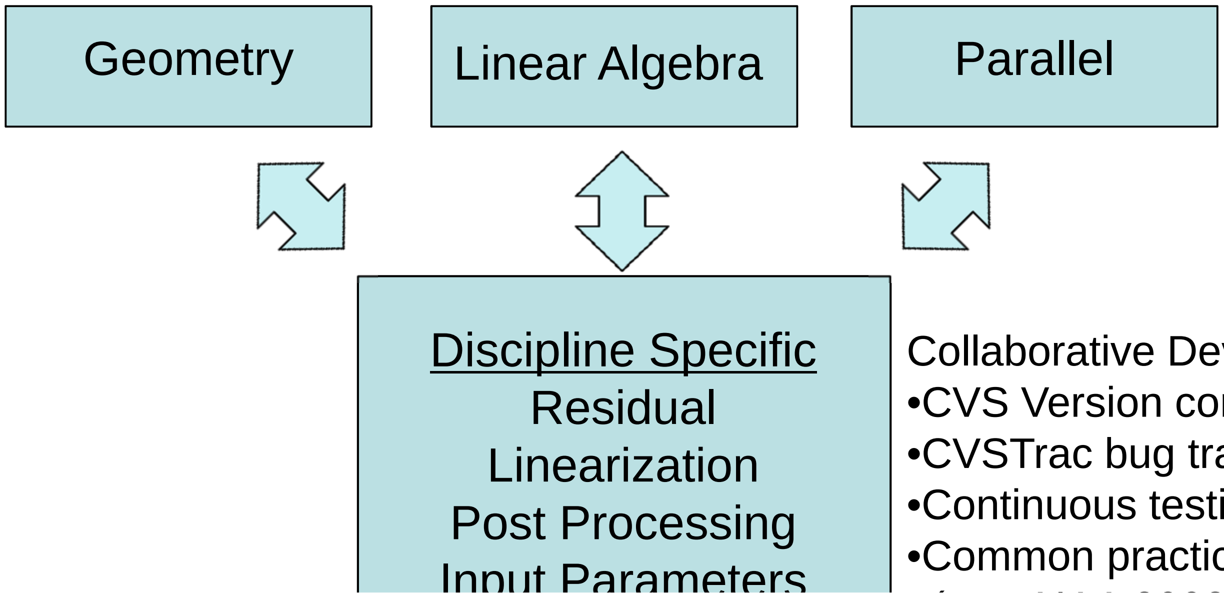
CAPRI Interface for CAD Geometry

- CAD – Watertight geometry definition is required
- Linear mesh – Initial mesh generated using CAD definition
- CAPRI – Higher-order points inserted into linear mesh and projected onto CAD definition via CAPRI interface
- Linear Elasticity – Surface displacements provided by CAPRI are propagated into interior



igh-Order Finite Element Framework

- Most computational simulation programs have similar structure and common components can be isolated into a single framework (code reuse)
- Discipline-specific applications (e.g. E&M + fluids) require new code in the form of residual routine and linearization (often just residual)
- Existing programs refactored to provide workable framework



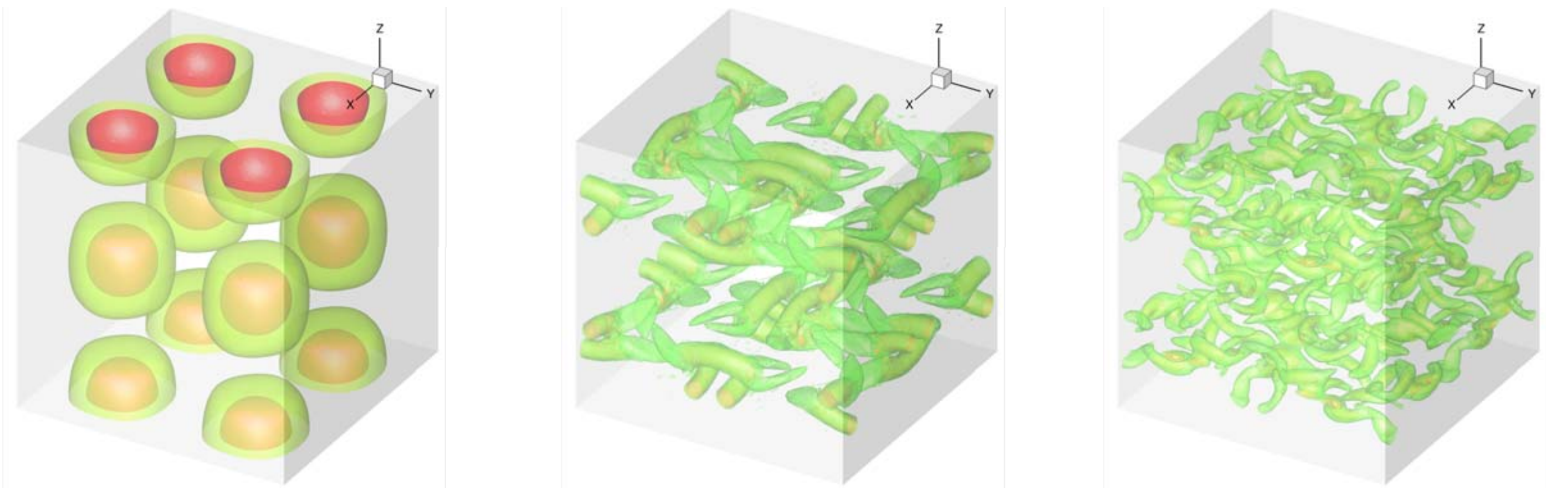
Collaborative Development (PG)
- CVS Version control
- CVSTrac bug tracking
- Continuous testing (future)
- Common practices

Engineering Disciplines

- Fluid dynamics
- Electromagnetics
- Structural Analysis
- Lithium-Ion Batteries
- Hydrogen Reforming (under development)

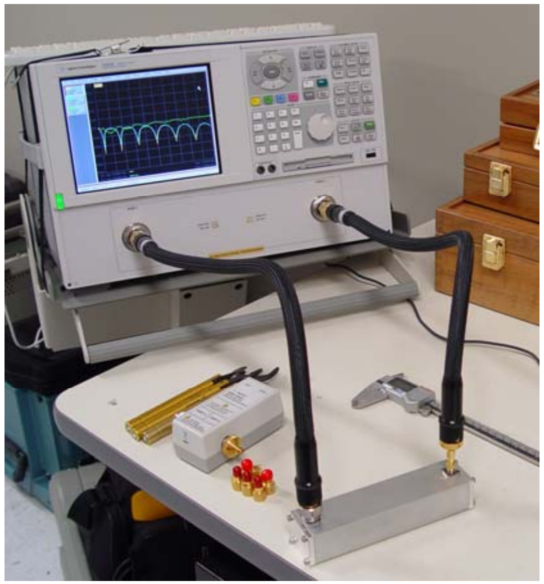
Fluid Dynamics

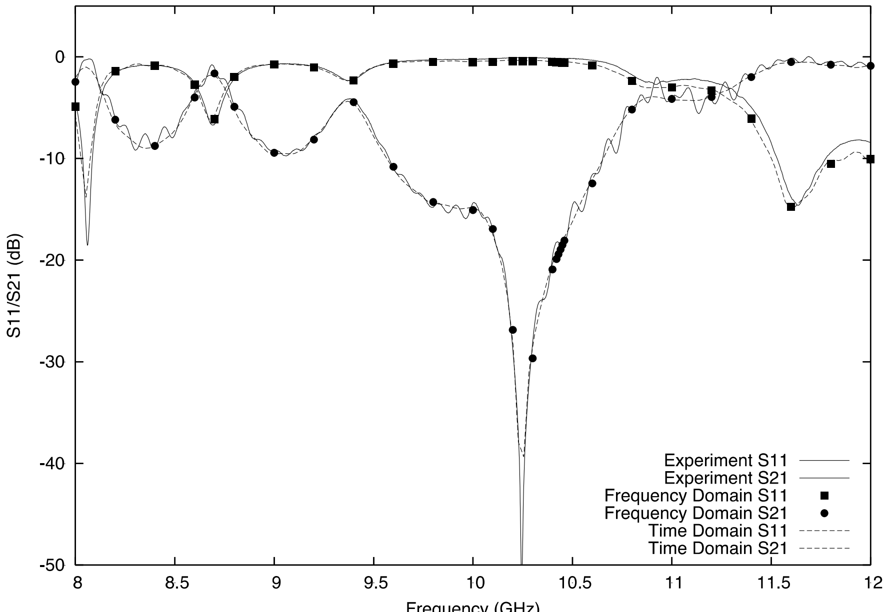
- mplicit time stepping
- Full Navier Stokes with Spalart-Allmaras turbulence model
- Petrov-Galerkin and discontinuous-Galerkin discretization



Electromagnetics

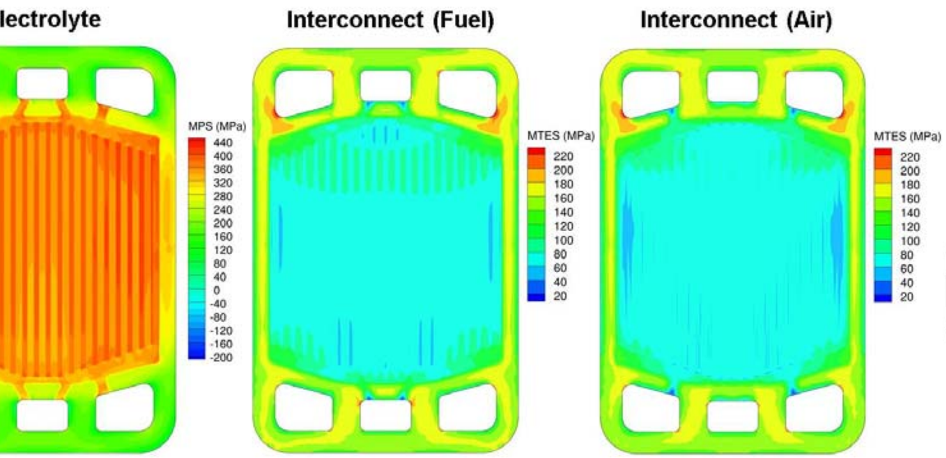
- Frequency domain and time-domain (implicit time stepping)
- Petrov-Galerkin and discontinuous-Galerkin discretization
- Frequency-dependent material properties

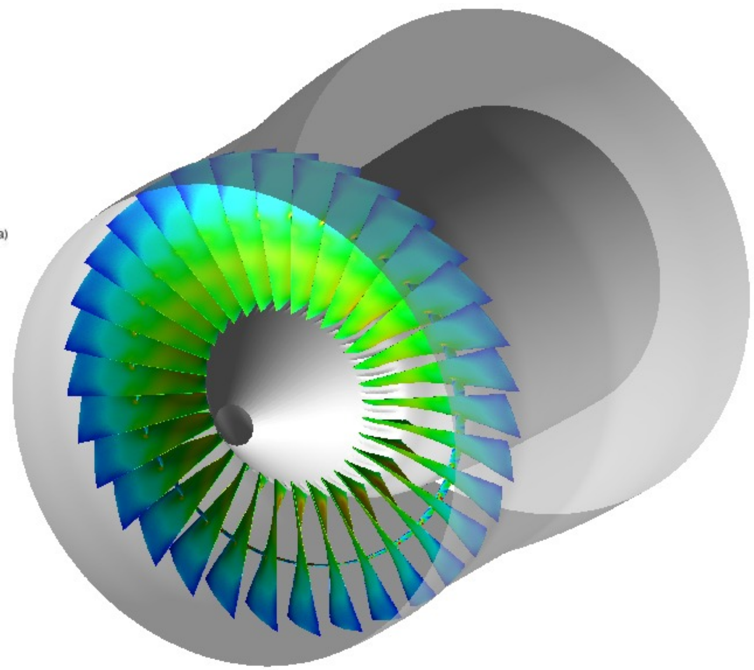




Structural Analysis

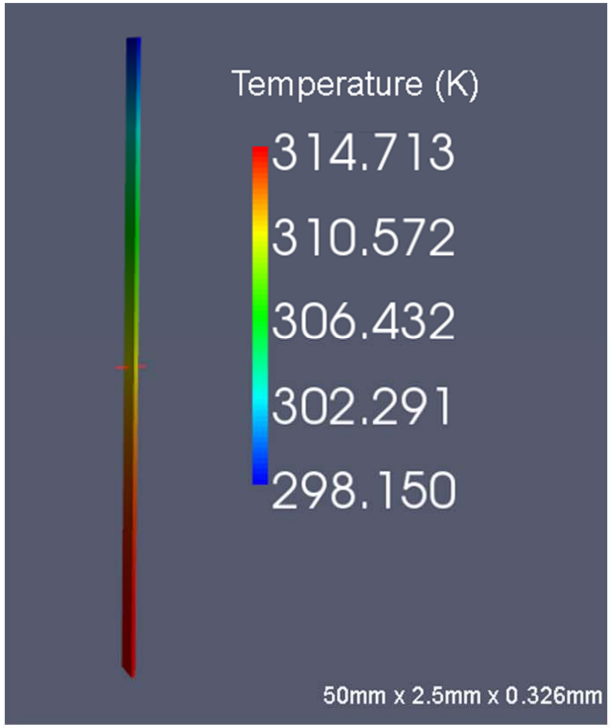
- Displacement-based structural dynamics
- Galerkin finite element
- Geometric and/or material nonlinearity
- Mechanical and thermal stresses

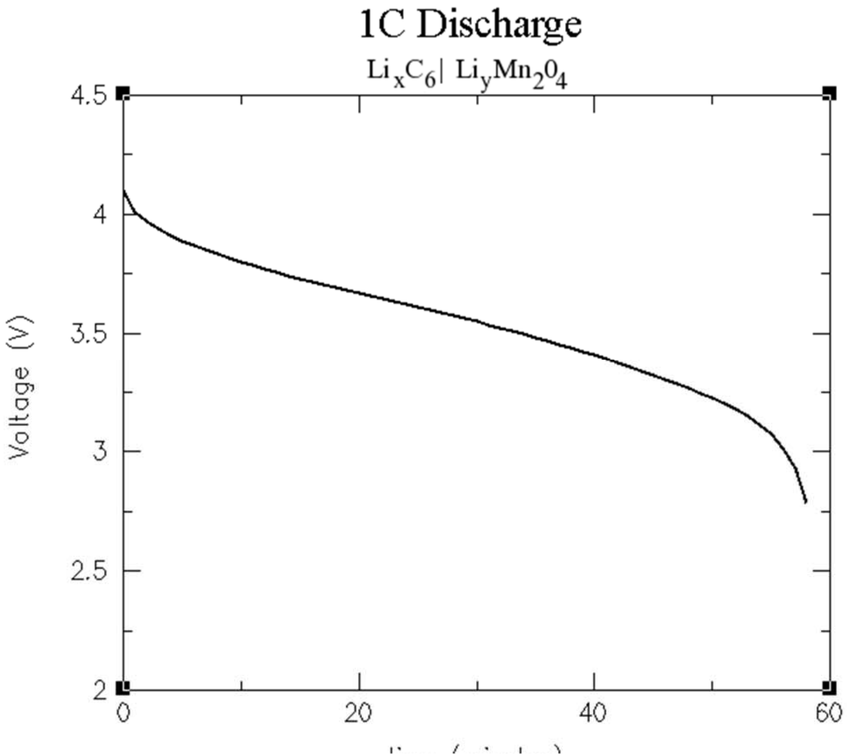




Lithium-Ion Batteries

- High-order Galerkin discretization
- Current collectors, electrodes, and separator all modeled





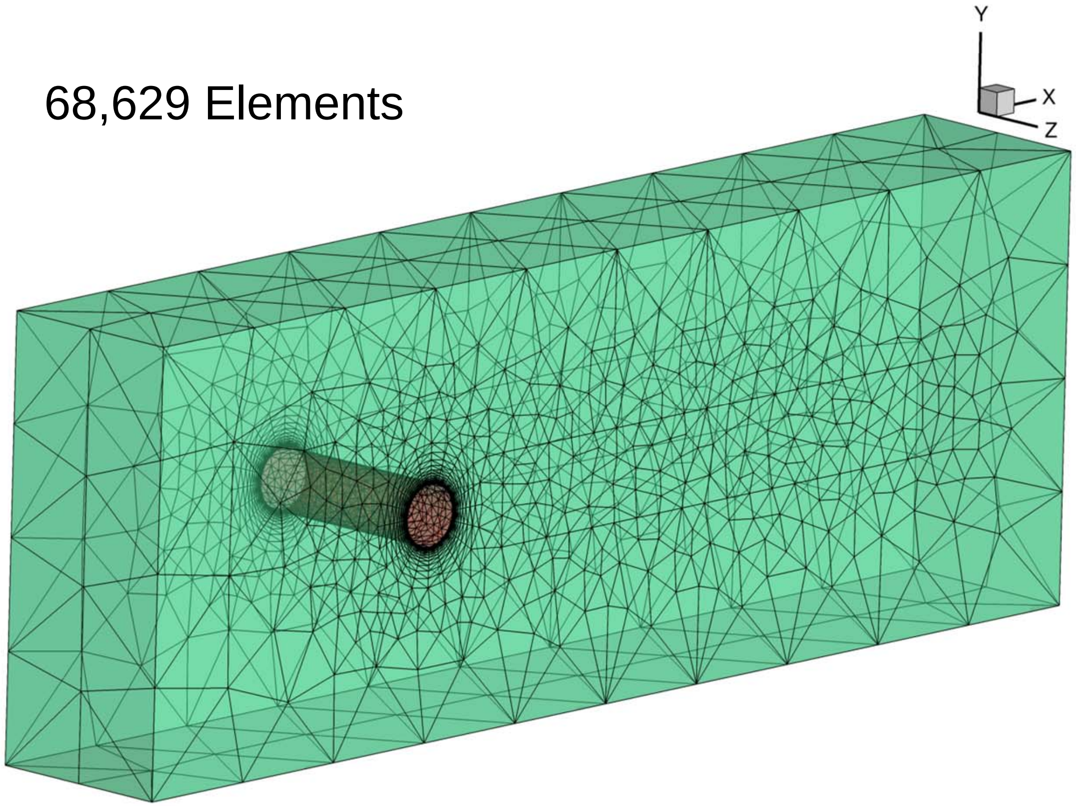
Example Fluid Dynamic Applications

- Three-dimensional cylinder
- Multielement airfoil
- Onera M6
- Trap wing
- Transonic airfoil

Three-Dimensional Cylinder

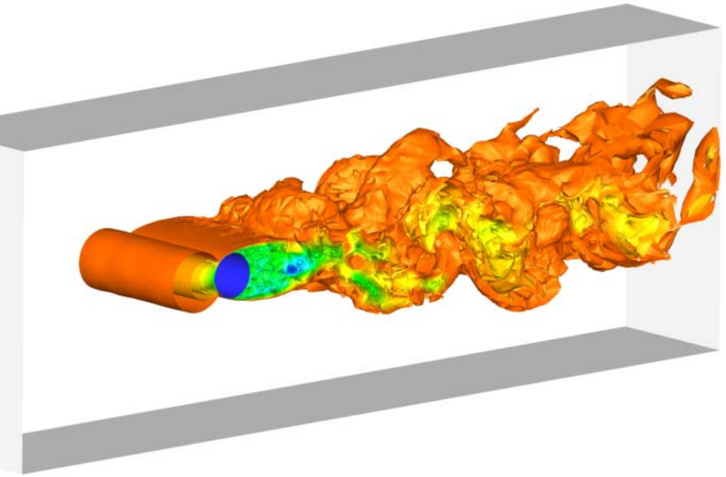
$$M_\infty = 0.2 \quad \mathrm{Re} = 2580$$

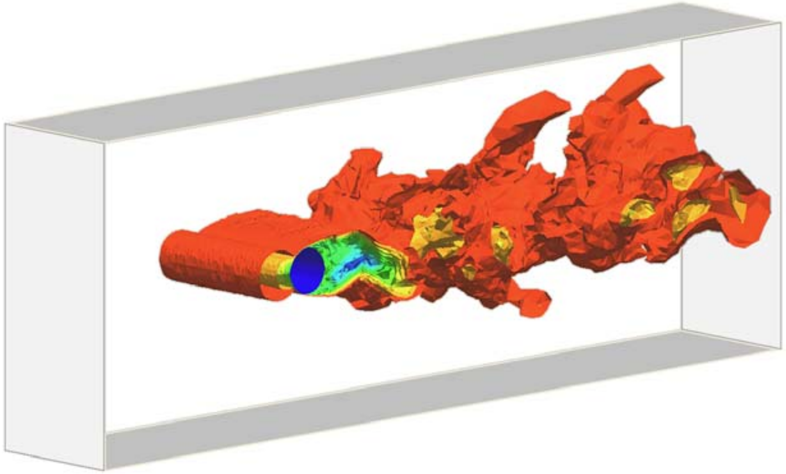
68,629 Elements



Three-Dimensional Cylinder

$$M_\infty = 0.2 \quad \mathrm{Re} = 2580$$



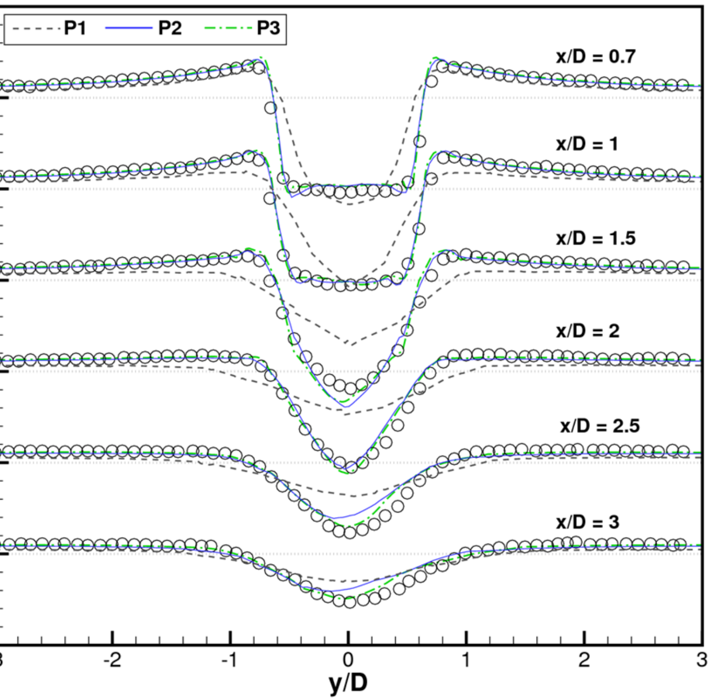


Discontinuous Galerkin P3

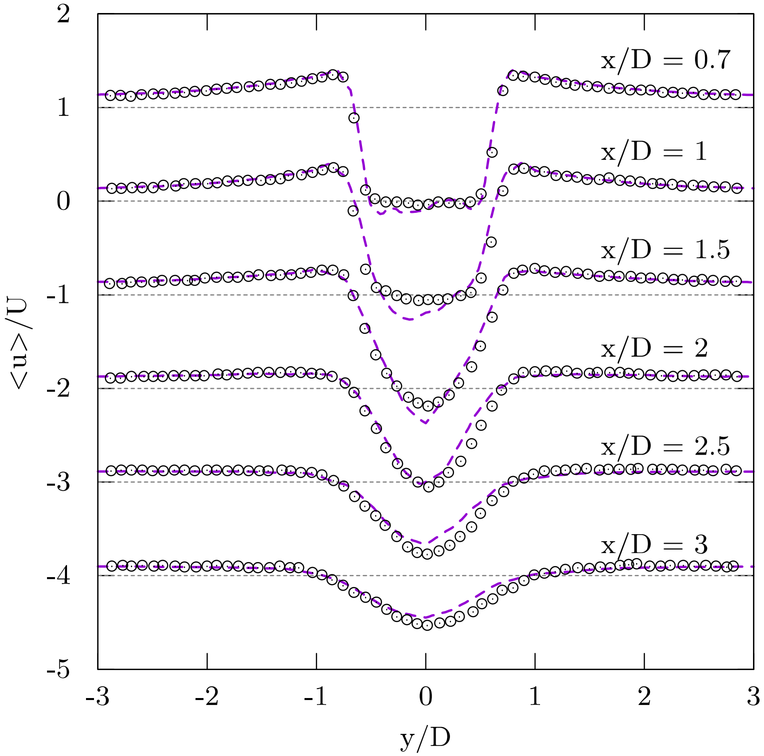
Petrov Galerkin P2

Three-Dimensional Cylinder
ime-Averaged U-Velocity Component

$$M_\infty = 0.2 \quad \mathrm{Re} = 2580$$



Discontinuous Galerkin

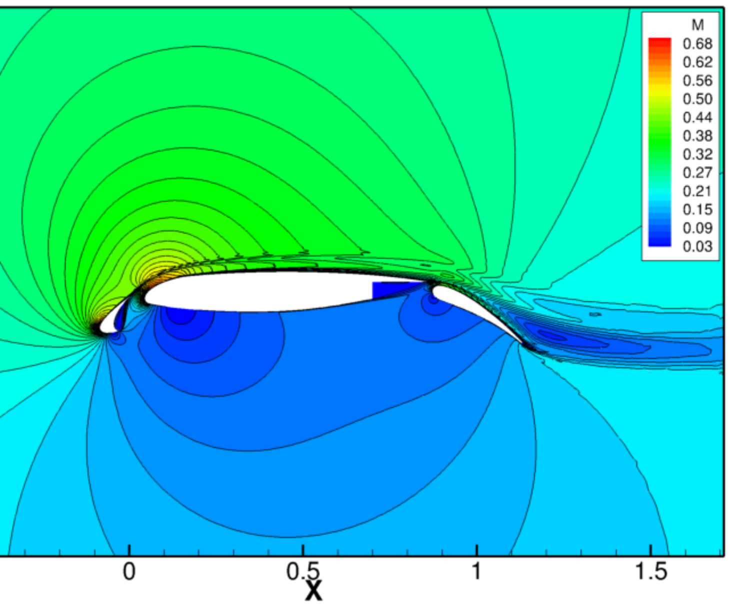


Petrov Galerkin

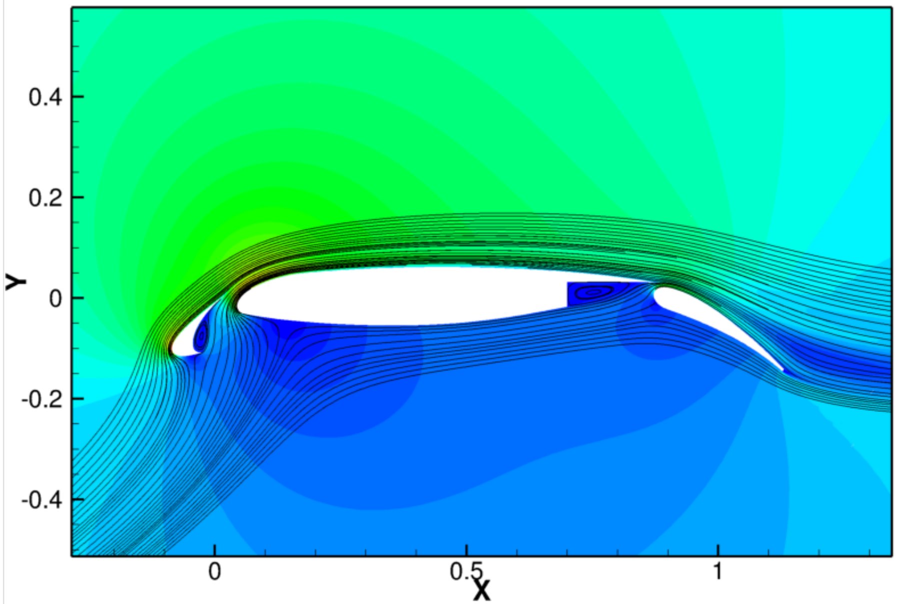
Multielement Airfoil

Douglas 30P-30N

$M_\infty = 0.2 \qquad \alpha = 16^o \qquad \mathrm{Re} = 9{,}000{,}000$



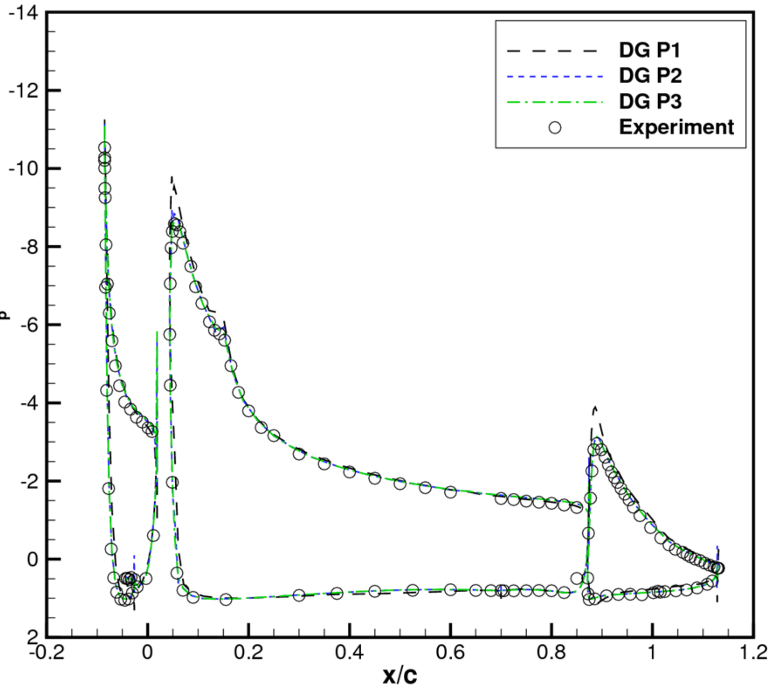
Mach Number Contours



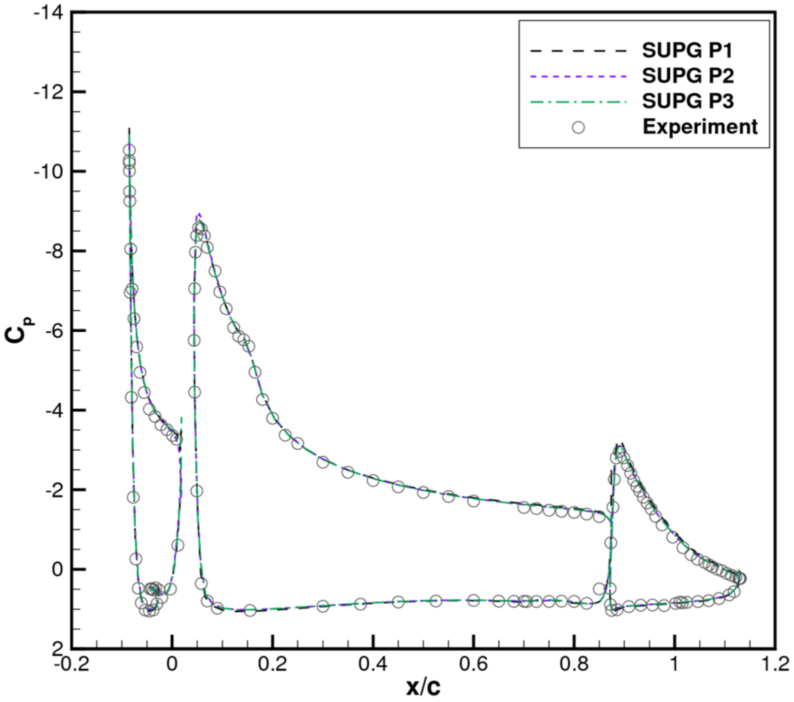
Streamlines

Multielement Airfoil

Pressure Distribution



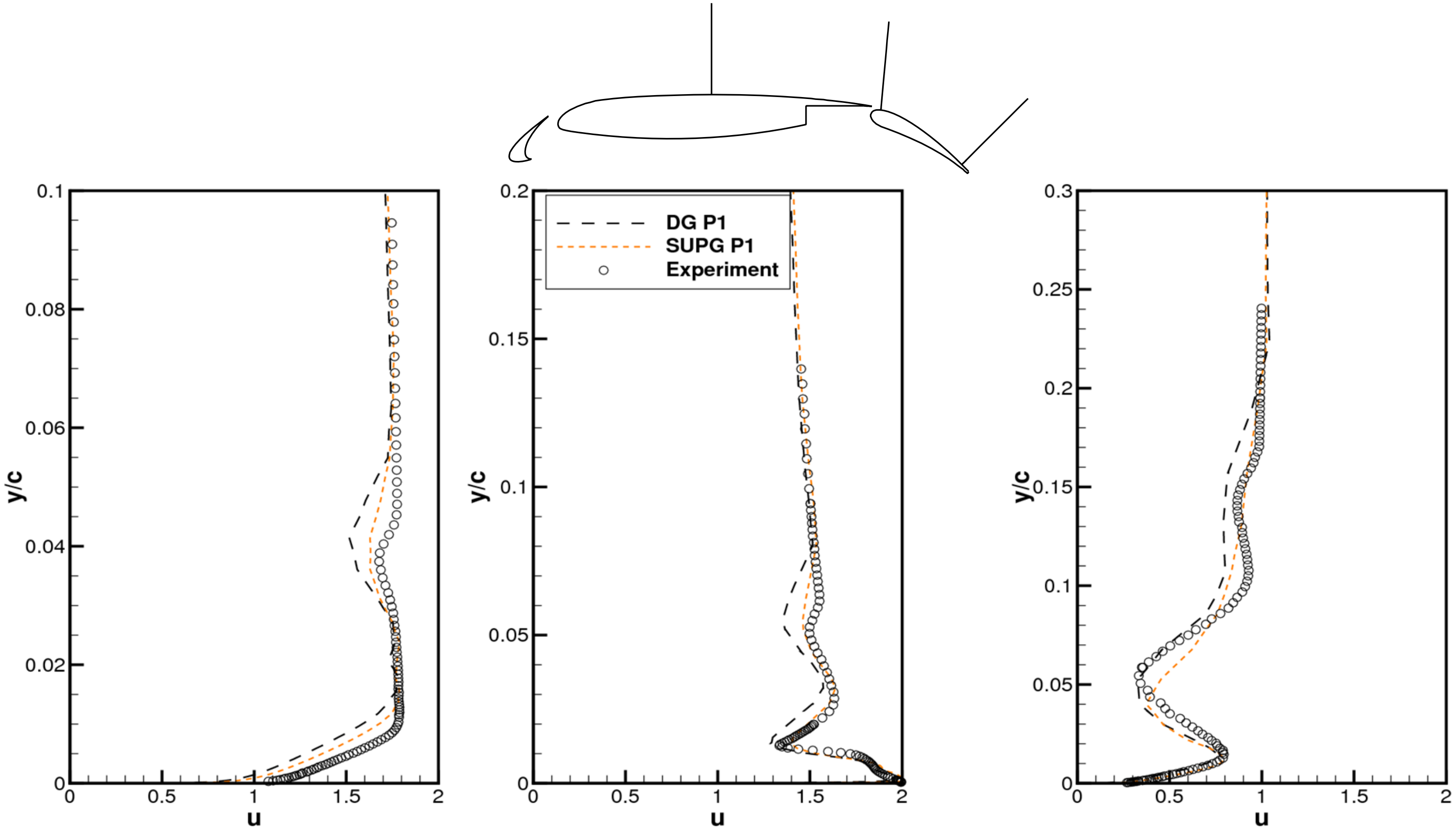
Discontinuous Galerkin



Petrov Galerkin

Multielement Airfoil

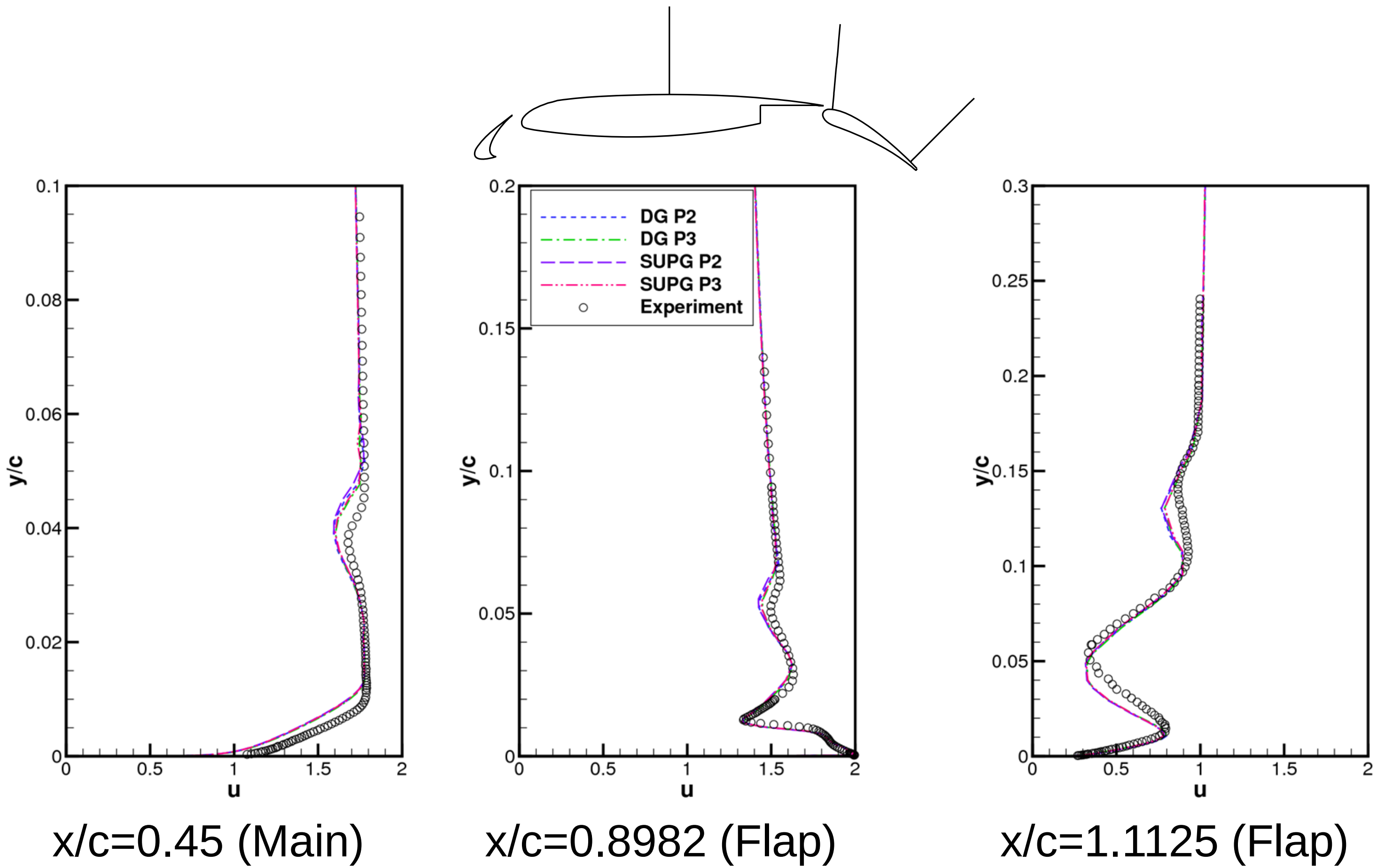
Velocity Profiles Linear Elements



x/c=0.45 (Main) x/c=0.8982 (Flap) x/c=1.1125 (Flap)

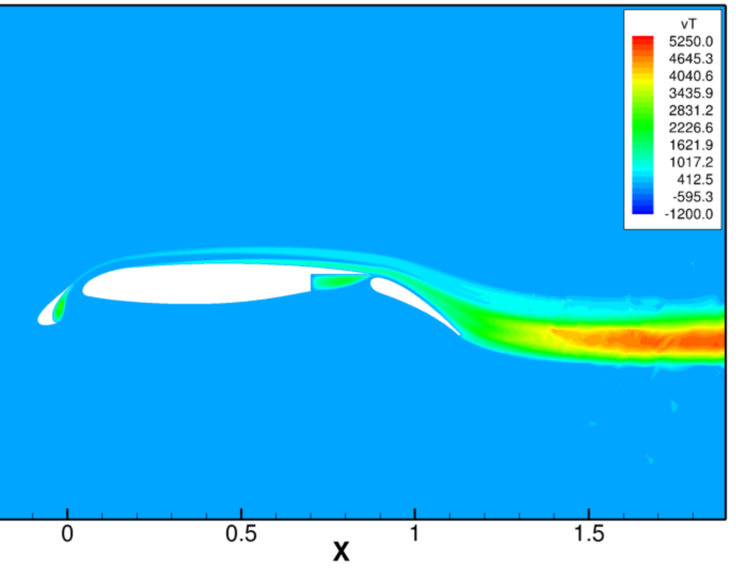
Multielement Airfoil

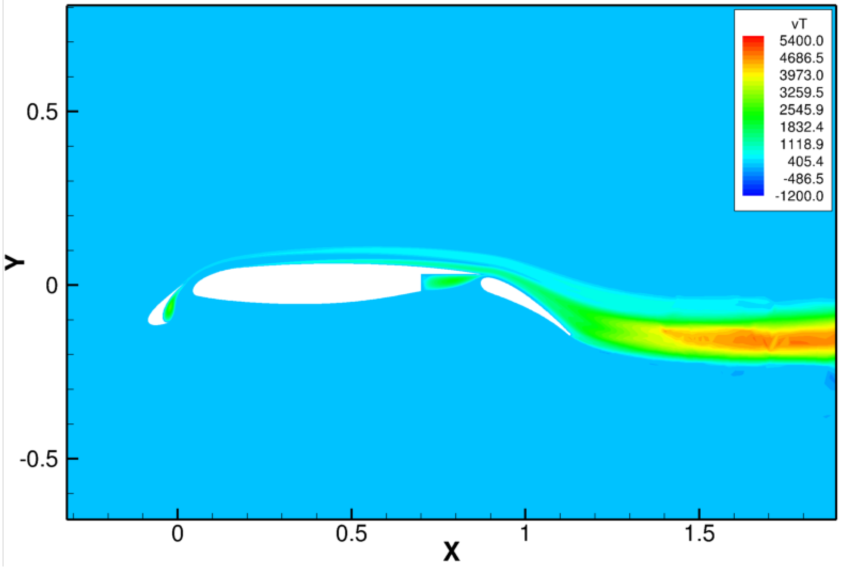
Velocity Profiles Quadratic and Cubic Elements



Multielement Airfoil

Turbulence Working Variable Fourth Order DG and PG



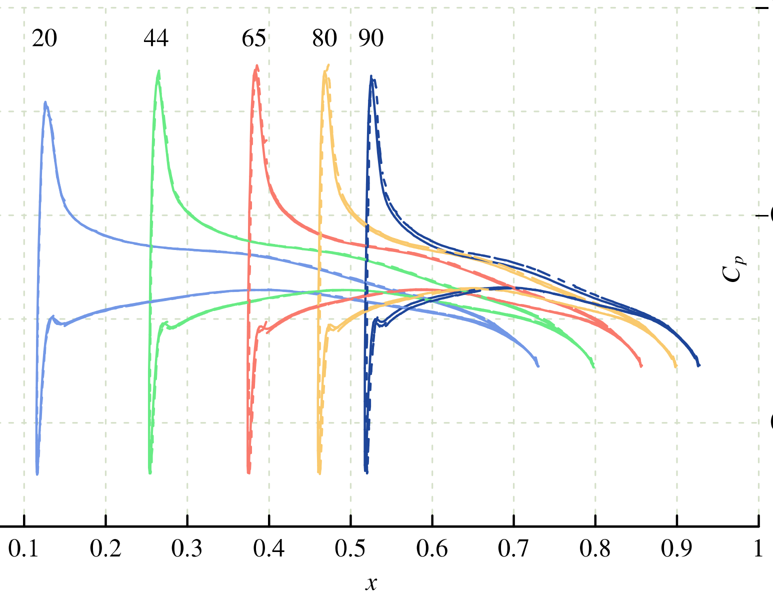


Discontinuous Galerkin

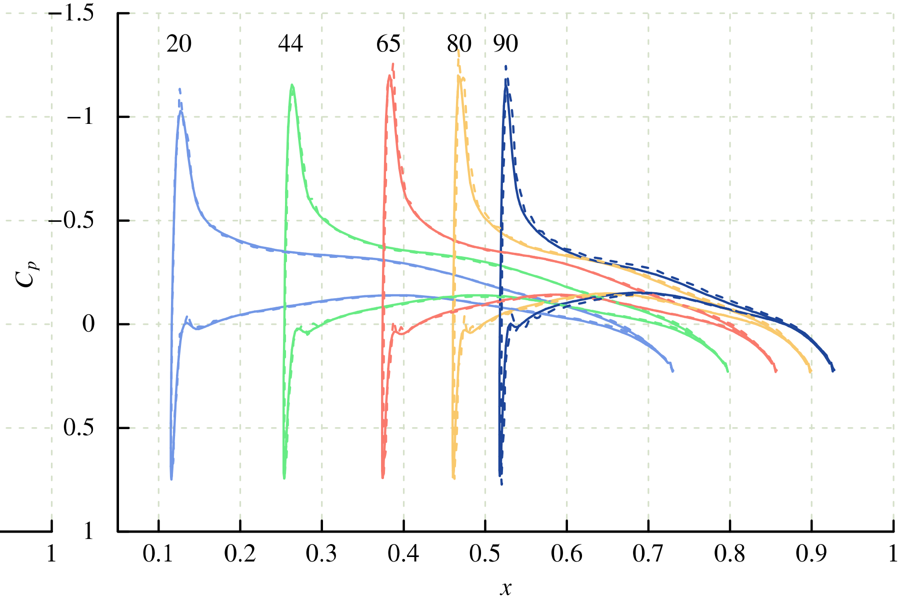
Petrov Galerkin

ONERA M6 Comparisons with CFL3D

$$M_\infty = 0.2 \quad \alpha = 3.02^o \quad \mathrm{Re} = 11,270,000$$



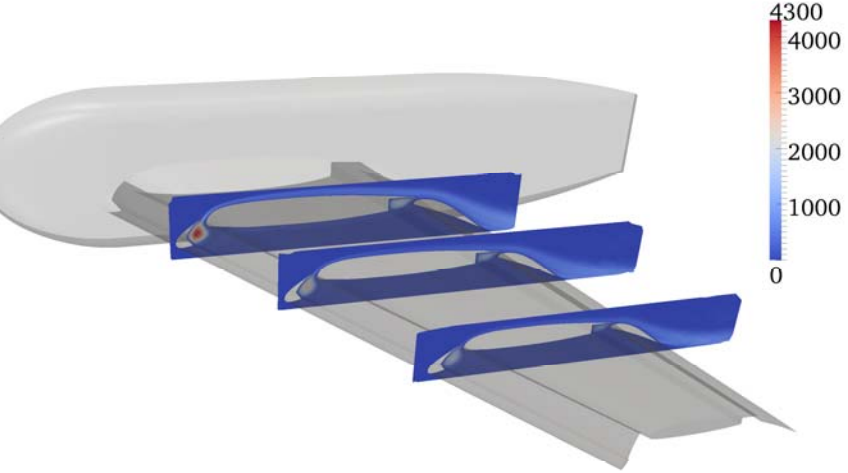
Discontinuous Galerkin P2



Petrov Galerkin P2

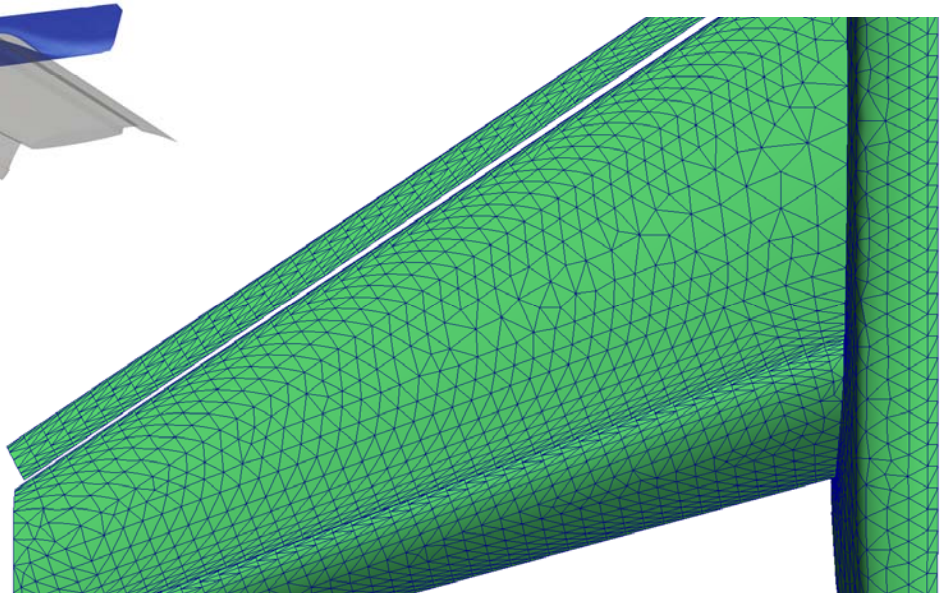
Trap Wing (Petrov-Galerkin Scheme)

$$M_\infty = 0.2 \quad \alpha = 12.99^o \quad \text{Re} = 4,300,000$$



Turbulence Working Variable

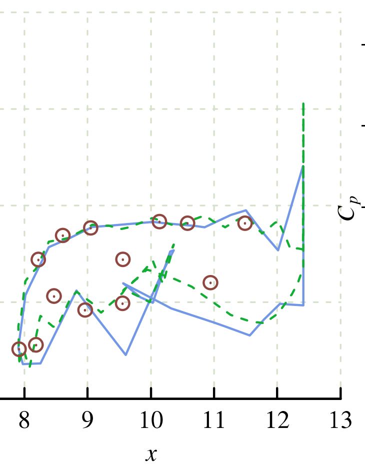
1,126,835 Elements
194,370 DOF P1
1,126,835 DOF P2



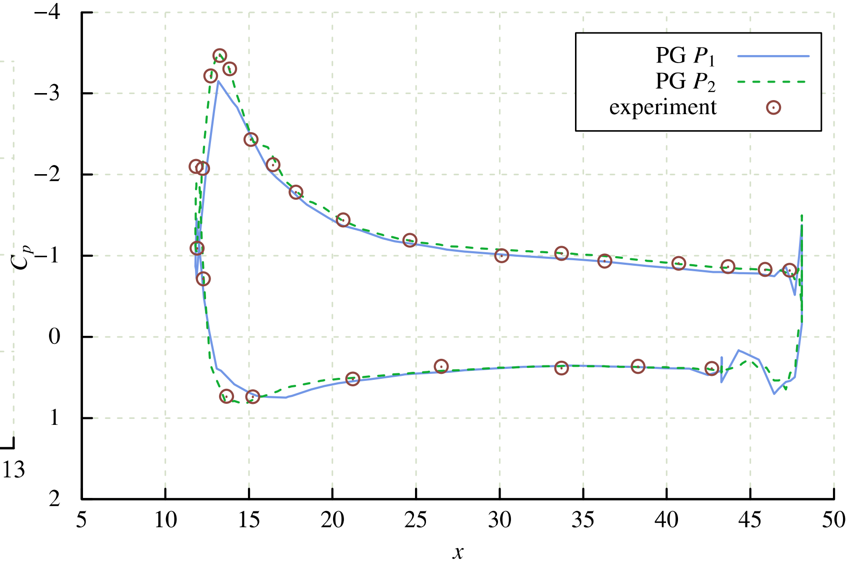
Trap Wing (Petrov Galerkin)

$$M_\infty = 0.2 \quad \alpha = 12.99^o \quad \mathrm{Re} = 4{,}300{,}000$$

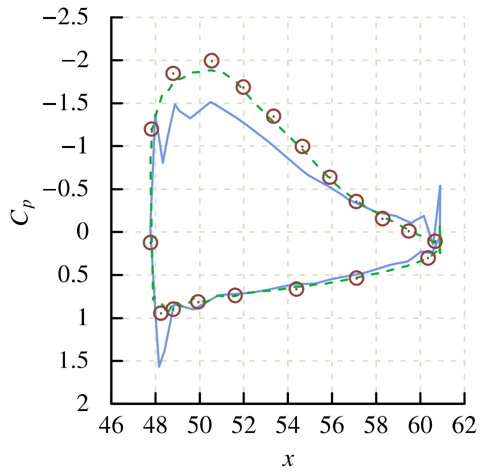
x/c=17%



Slat



Main Element

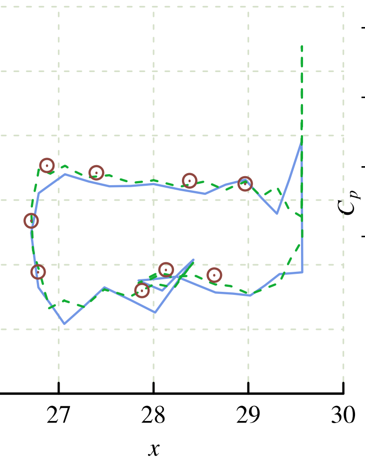


Flap

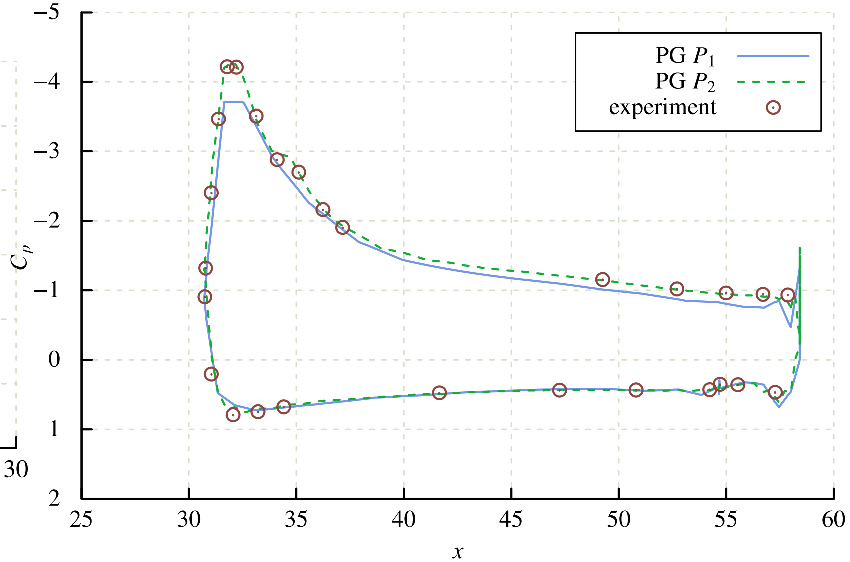
Trap Wing (Petrov Galerkin)

$$M_\infty = 0.2 \quad \alpha = 12.99^o \quad \mathrm{Re} = 4{,}300{,}000$$

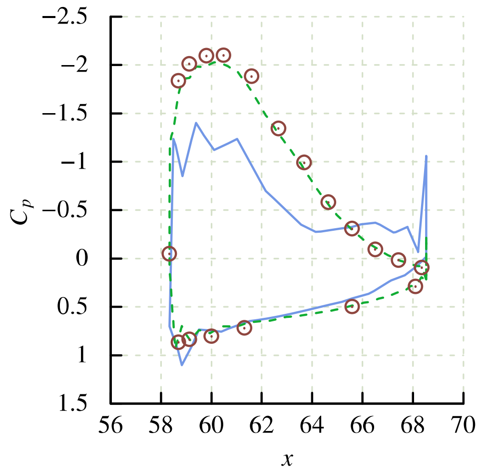
x/c=50%



Slat



Main Element

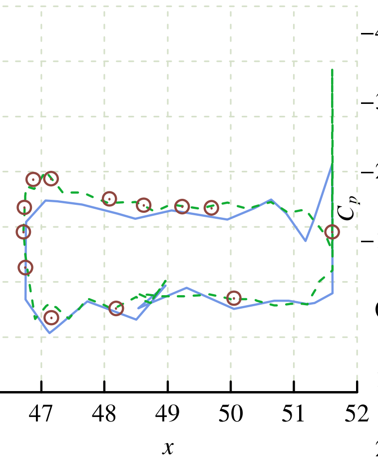


Flap

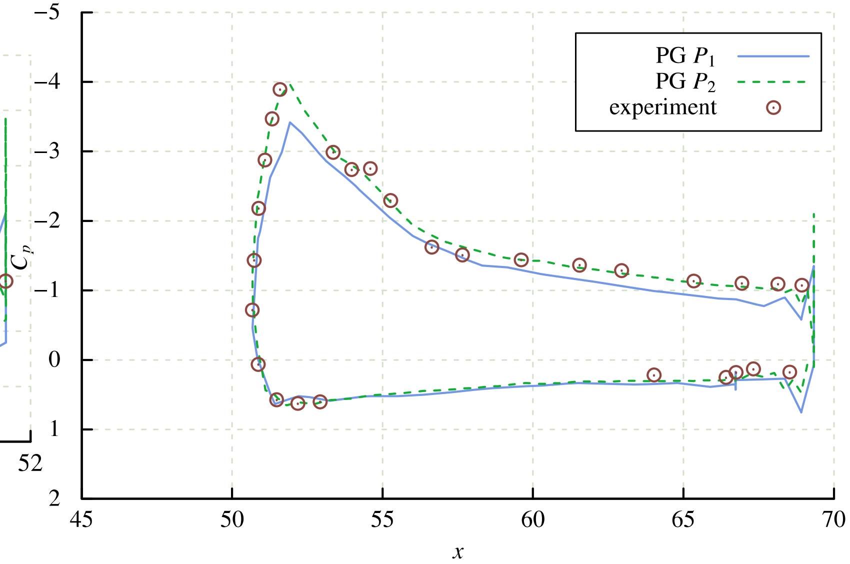
Trap Wing (Petrov Galerkin)

$M_\infty = 0.2 \quad \alpha = 12.99^o \quad \mathrm{Re} = 4{,}300{,}000$

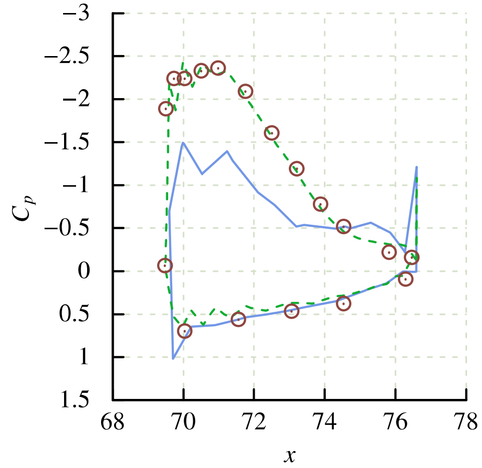
x/c=85%



Slat



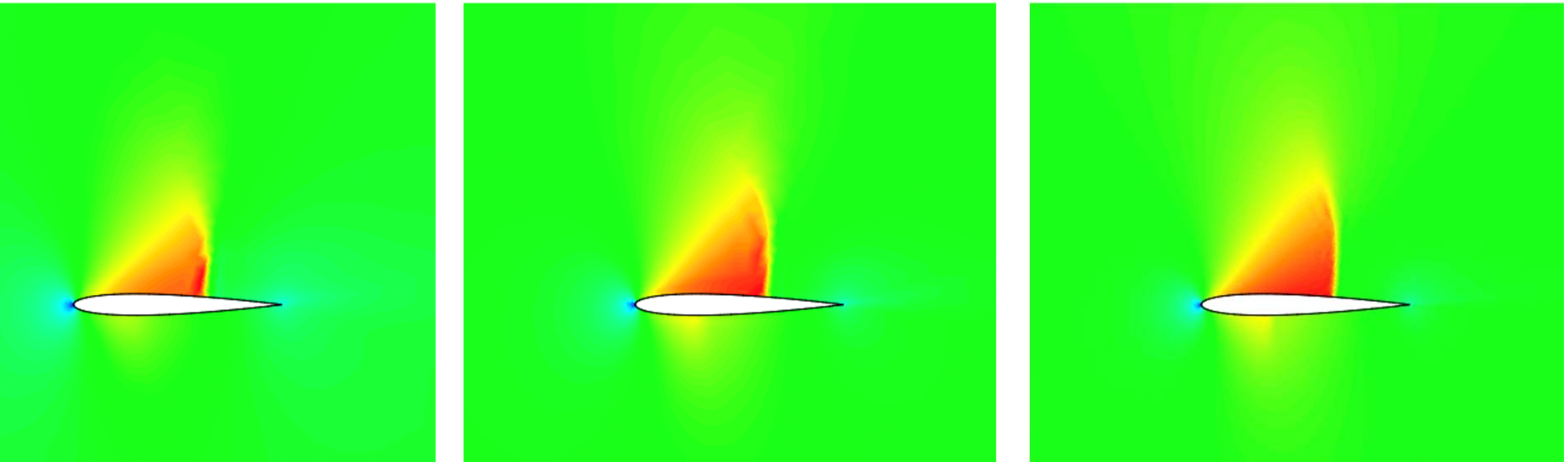
Main Element



Flap

Transonic NACA 0012

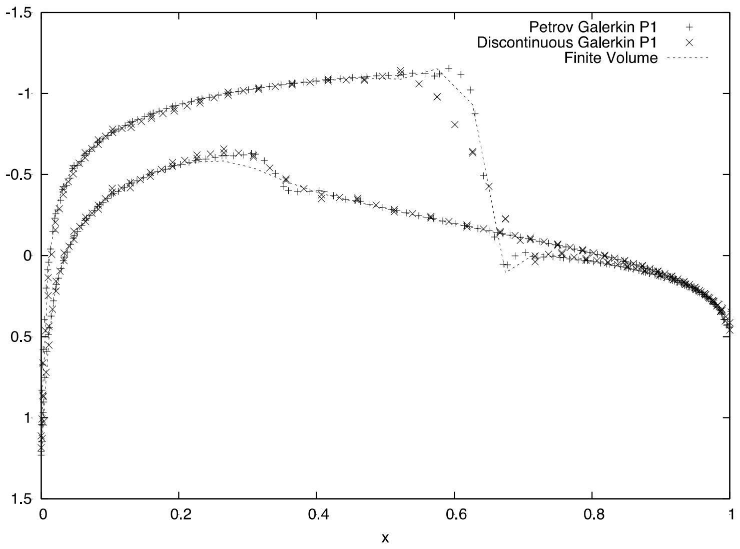
$M_\infty = 0.8 \qquad \alpha = 1.25^o$



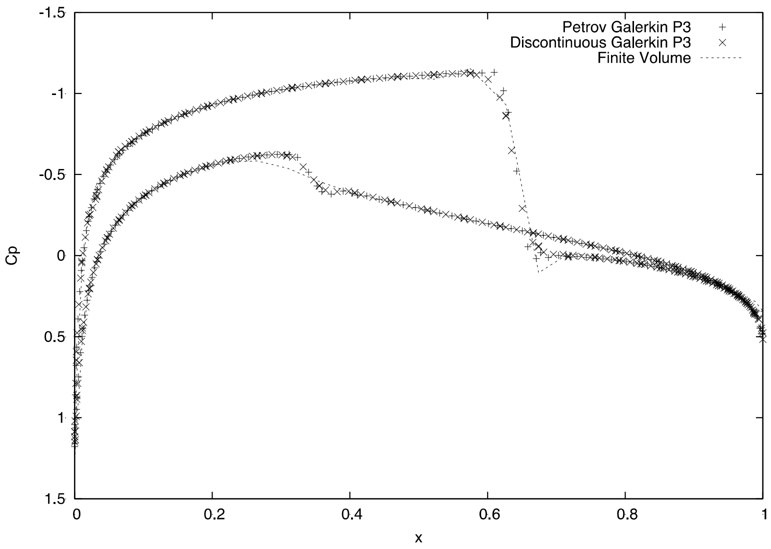
Finite Volume | Petrov Galerkin P1 | Petrov Galerkin P2

Transonic NACA 0012

$M_\infty = 0.8 \qquad \alpha = 1.25^o$



Linear Elements

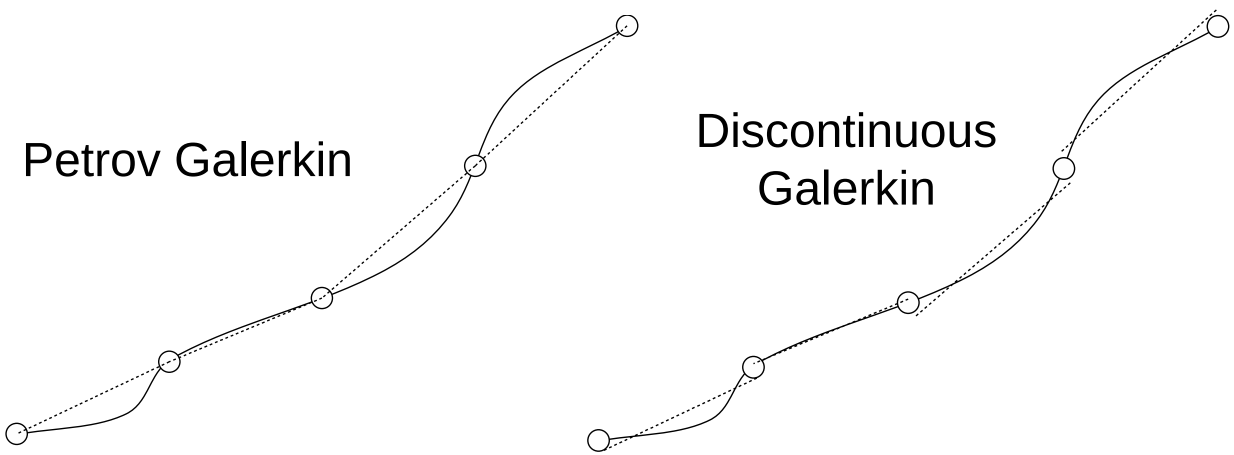


Cubic Elements

- Preliminary results adding switched viscous-like term
- Discontinuous Galerkin and Petrov-Galerkin terms not the same

Which Scheme to Use?

ntuition would indicate that there is an accuracy advantage on a given mesh for discontinuous Galerkin

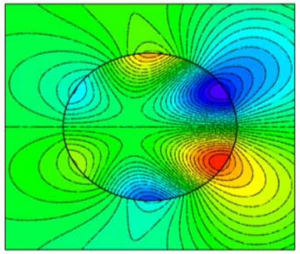
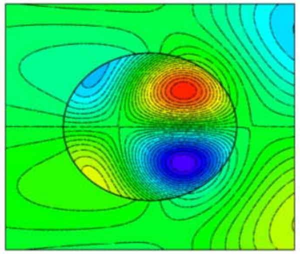
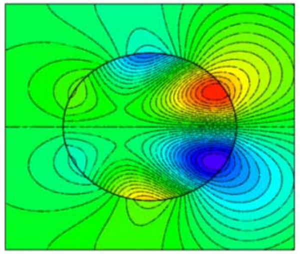
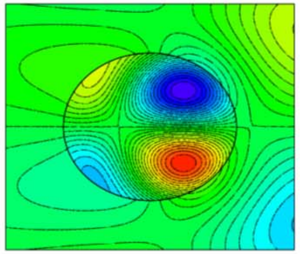
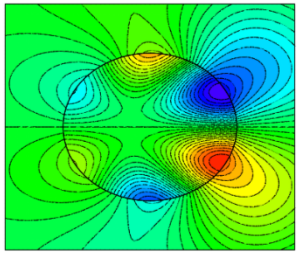
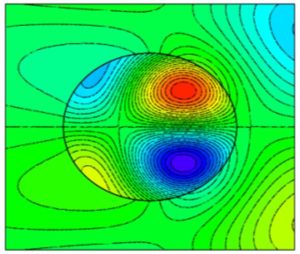
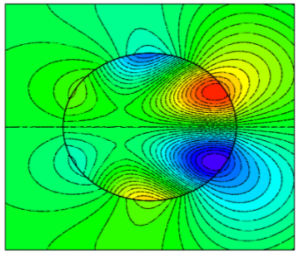
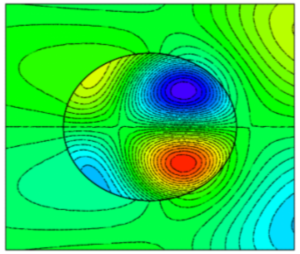


However, new degrees of freedom are created with discontinuities between elements

Do the benefits outweigh the cost?

2D Time-Domain Scattering from Dielectric Cylinder

- For fluid dynamics PG and DG codes solve different variables
- This causes confusing comparisons using MMS
- Electromagnetic application eliminates these effects

			
Exact T=0	Exact T=Pi/2	Exact T = Pi	Exact T = 3Pi/4
			
Petrov Galerkin			

2D Time-Domain Scattering from Dielectric Cylinder (P1 Elements)

DOF	L1 Error	L1 Slope	L2 Error	L2 Slope
369	2.52E-01		2.37E-01	
1348	6.00E-02	2.22	5.60E-02	2.23
5153	1.49E-2	2.08	1.39E-02	2.07

Petrov Galerkin

DOF	L1 Error	L1 Slope	L2 Error	L2 Slope
1824	2.52E-01		1.42E-01	
7314	6.00E-02	2.22	3.35E-02	2.08
29,376	1.49E-2	2.08	8.30E-03	2.01

Discontinuous Galerkin

2D Time-Domain Scattering from Dielectric Cylinder (P2 Elements)

DOF	L1 Error	L1 Slope	L2 Error	L2 Slope
1345	1.03E-02		1.05E-02	
5133	1.23E-03	3.28	1.21E-03	3.34
20,097	1.50E-4	3.13	1.51E-04	3.10

Petrov Galerkin

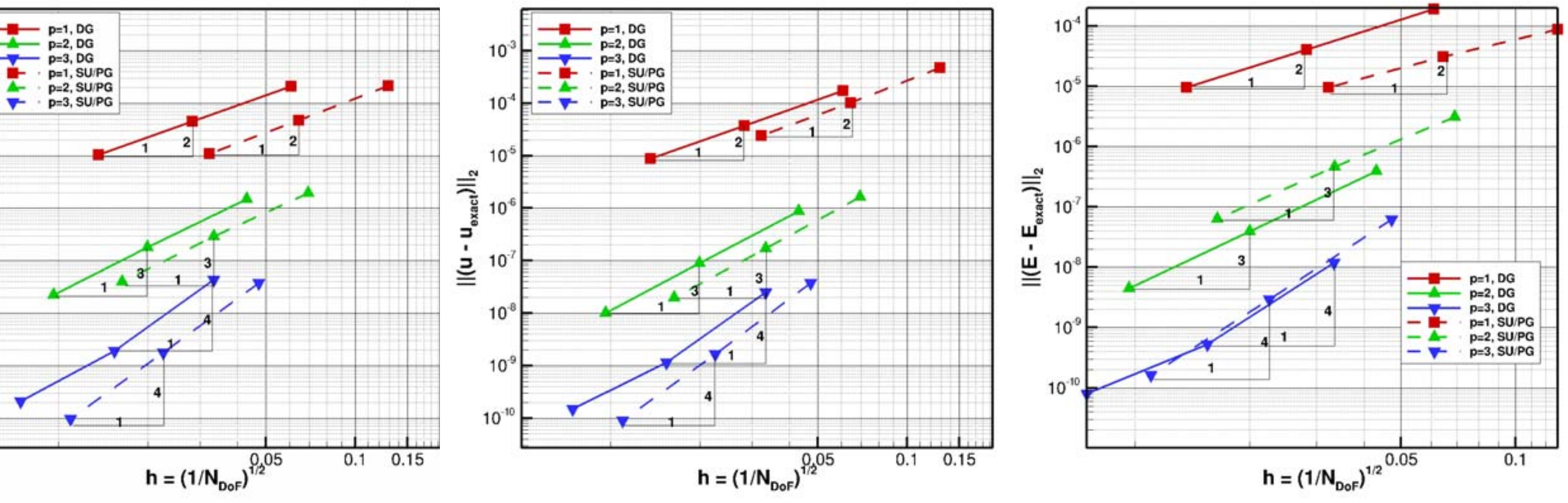
DOF	L1 Error	L1 Slope	L2 Error	L2 Slope
3648	1.00E-02		5.83E-03	
14,628	1.20E-03	3.06	6.69E-04	3.12
58,752	1.48E-4	3.01	8.42E-05	2.98

Discontinuous Galerkin

Which Scheme to Use?

Error in Manufactured Solution Per DOF

(Glasby et al. AIAA 2013-0692)

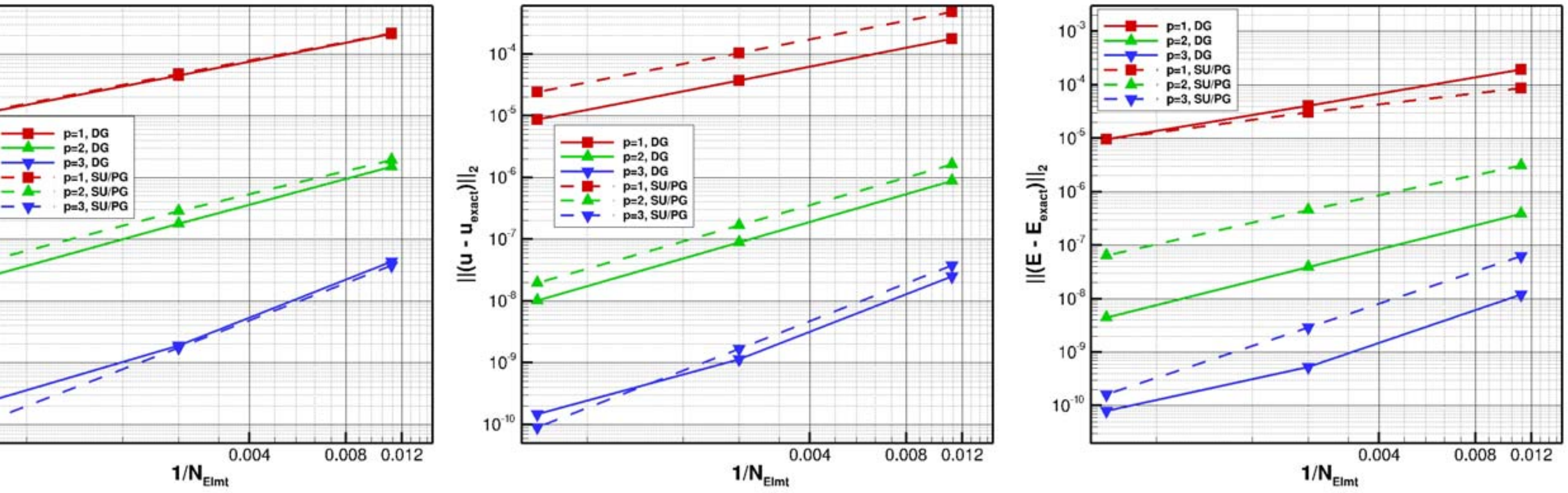


Petrov Galerkin exhibits lower error per degree of freedom

Which Scheme to Use?

Error in Manufactured Solution Per Element

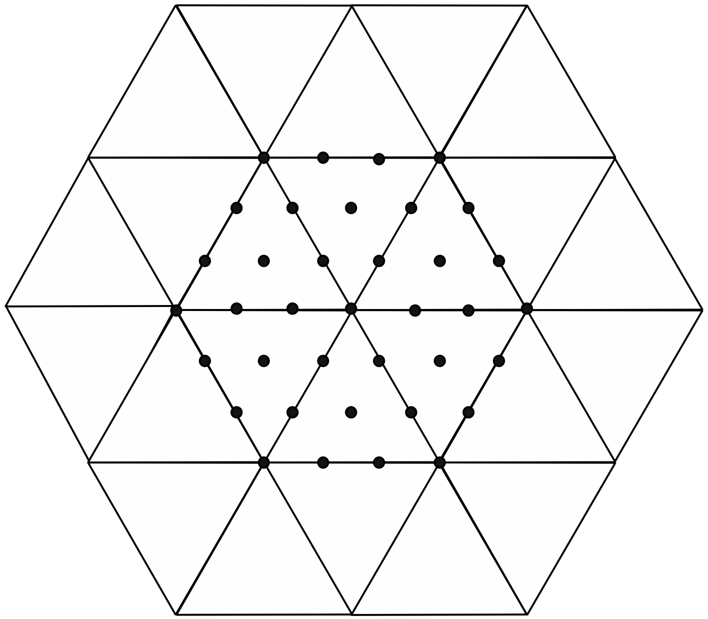
(Glasby et al. AIAA 2013-0692)



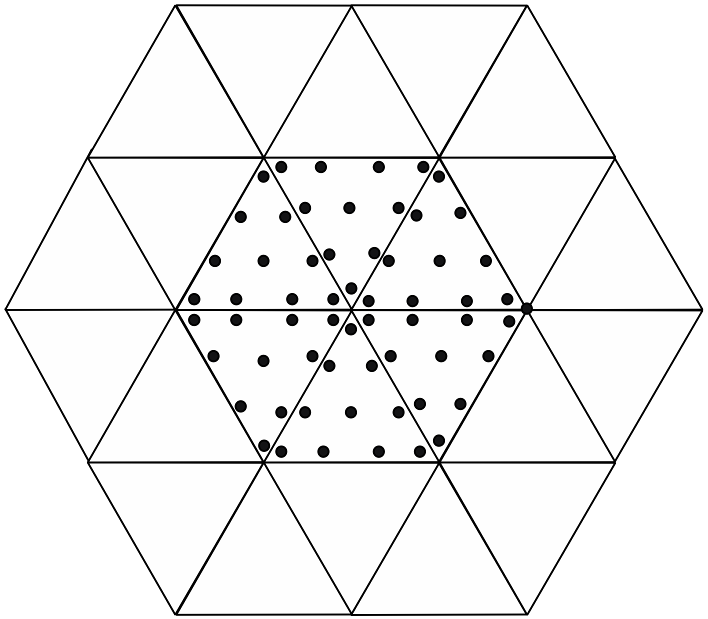
- Discontinuous Galerkin exhibits lower error per element
- Results are for low Reynolds number MMS but typical for Euler, Navier Stokes, and Electromagnetic application

Which Scheme to Use?

Estimating DOF and Number of Non Zero Entries in Matrix



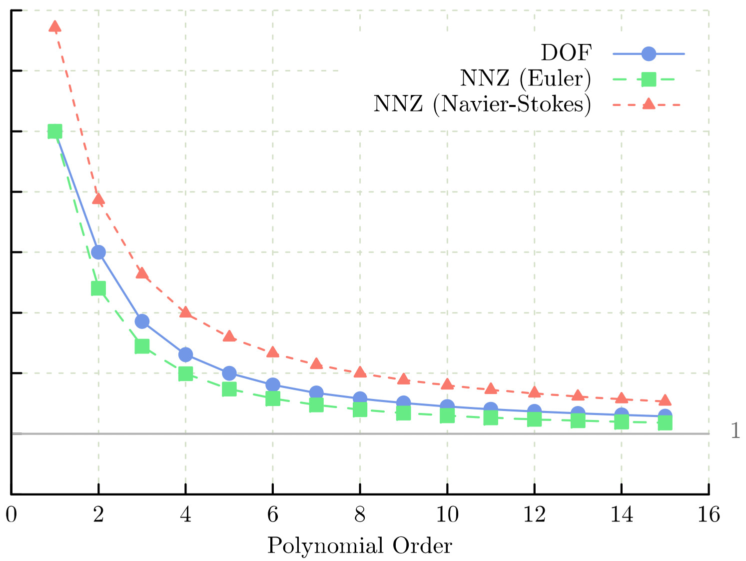
Petrov Galerkin



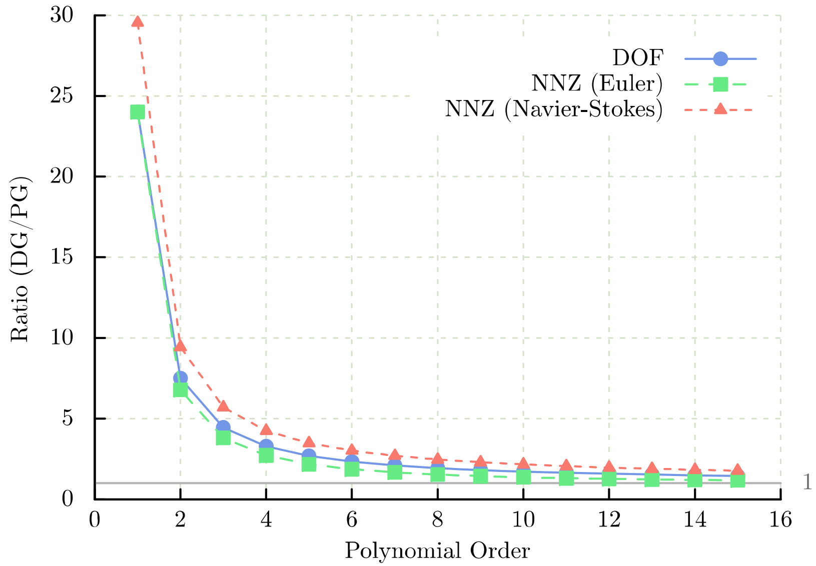
Discontinuous Galerkin

Which Scheme to Use?

Estimating Ratio of DOF and Number of Non Zero Entries in Matrix Between PG and DG



Two Dimensions (Triangles)



Three Dimensions (Tetrahedrons)

Which Scheme to Use?

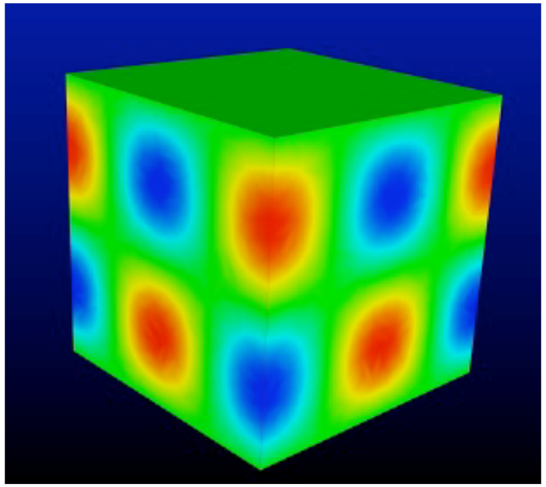
DOF and Number of Non Zero Entries in Matrix
Cubic Volume Subdivided into Elements

	Tetrahedron		Hexahedron		Prismatic	
	DOF	NNZ	DOF	NNZ	DOF	NNZ
P1	22.16	19.8	7.53	5.74	11.35	9.42
P2	7.19	6.20	2.92	2.14	4.02	3.15

- Discontinuous Galerkin compares more favorably for hexahedrons, worst case is for tetrahedrons
- Higher DOF and NNZ translates into more memory, more work per iteration, and generally more iterations (search directions for GMRES)
- At low-to-moderate orders, Petrov Galerkin appears to have advantages over discontinuous Galerkin
- Higher orders may favor discontinuous Galerkin

Which Scheme to Use?

Resonant Cavity: 1.85 GHz
Magnetic Field Intensity



- Advancing fixed number of time steps to compare efficiencies
- Independent of equation set

Ratio of time for fixed number of time steps		
	DOF Ratio	Actual Time Ratio
Linear	22.16	27
Quadratic	7.19	12

(DG required more search directions)

Which Scheme to Use?

Many factors effect the accuracy of a given scheme so it is difficult, if not impossible, to make a broad conclusion

- Boundary condition type / order / weak v. strong
- Basis functions and quadrature rules
- Solution and comparison variables
- Flux function / stabilization matrix

While number of stabilization matrices for PG is approximately the same as the number of flux evaluations for DG, stabilization matrix more expensive for Euler

Higher DOF translates to more search directions

Very high order is unclear but work advantages for PG at low-to-moderate orders are difficult for DG to overcome

Which Scheme to Use for Explicit Schemes?

- Previous discussion is for implicit schemes typically used for turbulent flows
- For inviscid flows with explicit time advancement, DG should be less expensive because residual is computed on an element-by-element basis and it is less expensive than PG
- For viscous flows this conclusion is unclear because symmetric interior penalty method adds a significant number of terms

Summary

- Described Petrov-Galerkin scheme in moderate detail
  - Stabilization matrix for inviscid and viscous flows
  - Confirmed accuracy
  - Conservation
- Discontinuous-Galerkin and Petrov-Galerkin methods work well for inviscid, laminar, and turbulent flows
- Petrov-Galerkin method appears overlooked method for implicit schemes with low-to-moderate orders of accuracy
- Efficiency comparisons for explicit schemes ongoing
- Developing framework for high-order finite element solutions to multidisciplinary problems

Suggested Reading

Brooks, A.N., and Hughes, T.J.R., “Streamline Upwind/Petrov-Galerkin Formulations for Convection Dominated Flows with Particular Emphasis on the Incompressible Navier-Stokes Equations,” Comp. Methods Appl. Mech. Eng., Vol. 32, No. 103, 1982, pp. 199-259.

Franca, L.P., Frey, S.L., and Hughes, “Stabilized Finite-Element Methods I. Application to the Advective-Diffusive Model,” Comp. Methods Appl. Mech. Eng., Vol. 95, No. 2, 1992, pp. 253-276.

Shakib, F., Hughes, T.J.R., and Johan, Z., “A New Finite Element Formulation for Computational Fluid Dynamics X. The Compressible Euler and Navier-Stokes Equations,” Comp. Meth. Appl. Mech. Eng., Vol. 89, No. 1-3 1991, pp. 141-219.

Suggested Reading

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Anderson, W.K., Wang, L., Kapadia, S., Tanis, C., and Hilbert, B., "Petrov-Galerkin and Discontinuous-Galerkin Methods for Time-Domain and Frequency-Domain Electromagnetic Simulations," J. Comp. Physics, Vol. 230, No. 23, 2011, pp. 8360-8385.

Erwin, J.T., Anderson, W.K., Kapadia, S., and Wang, L., "Three-Dimensional Stabilized Finite Elements for Compressible Navier-Stokes," AIAA Journal, Vol. 51, No. 6, 2013, pp. 1404-1419.

Suggested Reading

- Fries, T.P., and Matthies, H.G., “A Review of Petrov-Galerkin Stabilization Approaches and an Extension to Mesh Free Methods,” Inst. Of Scientific Comp., Univ. of Braunschweig Inst. Of Tech., TR-2004-01, Brunschick, Germany, 2004.
- Wang, L., Anderson, W.K., Erwin, J.T., and Kapadia, S., “High-Order Methods for Solutions of Three-Dimensional Turbulent Flows,” AIAA 2013-2856, 2013.
- Glasby, R., Burgess, N., Anderson, W.K., Wang, L., Allmaras, S.R., and Mavriplis, D.J., “Comparison of SUPG and DG Finite-Element Techniques for Compressible Navier-Stokes Equations on Anisotropic Unstructured Meshes,” AIAA 2013-0691

Suggested Reading

Erwin, J.T., Anderson, W.K., Wang, L., and Kapadia, S., “High-Order Finite-Element Method for Three-Dimensional Turbulent Navier-Stokes,” AIAA 2013-2571.

Salari, K., and Knupp, P., “Code Verification by the Method of Manufactured Solutions,” Sandia National Lab., TR-SAND2000-1444, June 2000.

Stabilization Matrix

Scaling of stabilization necessary to maintain order property between inviscid and viscous limit

$$a\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right) = 0$$

Weighted residual

$$\iiint_{\Omega}\left[N + \left(\frac{\partial N}{\partial x}a\right)\overbrace{\left[\tau\right]}^{O\left(h^{?}\right)}\right]\left[\overbrace{a\frac{\partial u}{\partial x}}^{O\left(h^{p}\right)} - \overbrace{\frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right)}^{O\left(h^{p-1}\right)}\right]d\Omega = 0$$

First bracketed term indicates that τ must be at least $O\left(h\right)$

If solution converges as $O\left(h^{p+1}\right)$ then derivatives converge with order $O\left(h^{p}\right)$ and second derivatives as $O\left(h^{p-1}\right)$

For viscous flow, τ needs to scale as $O\left(h^{2}\right)$ instead of $O\left(h\right)$