Emergent Hydrodynamics in a 1D Bose-Fermi Mixture

Xi-Wen Guan

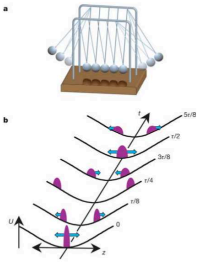
I. Generalized Hydrodynamics: elementary introduction

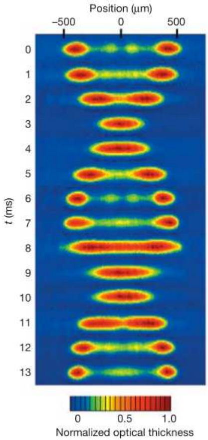
Key collaborators:

Sheng Wang, Xiangguo Yin, Yang Yang Chen and Yunbo Zhang

Beijing Computational Science Research Centre, June 2019

Although their individual motions are complex, their collective behaviour acquires qualitatively a new form of simplicity - collective motion of “particles”.

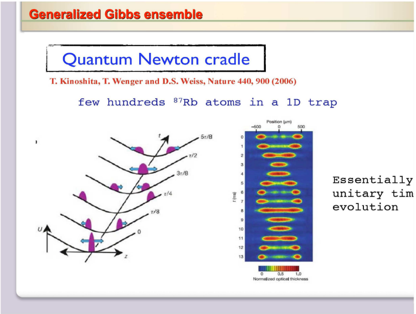




Kinoshita, Wenger, Weiss (Nature, 2006)
GHD theory: Caux, Doyon, Dubail, Konik, Yoshimura (2017)

Bragg pulse imparts the cloud into two oppositely-moved parts that do not thermalize

Non-equilibrium isolated quantum system in 1D does not thermalize.



$$\hat{\rho} = \frac{1}{Z} \exp\left(- \sum_m \lambda_m \hat{\mathcal{I}}_m \right) , \quad Z = Tr \exp\left(- \sum_m \lambda_m \hat{\mathcal{I}}_m \right)$$

“The absence of damping in 1D Bose gases may lead to potential applications in force sensing and atom interferometry”.

$$\partial_t \rho + \partial_x \left[v^{\text{eff}} \rho \right] = \left(\frac{\partial_x V}{m} \right) \partial_\theta \rho$$

Caux et al, arXiv1711.00873

toplasm causes dysfunction of NAC, leading to incorrect sorting of proteins to the ER and mitochondria. A link between cytosolic protein aggregation and ER stress is well established (*43*), and it will be interesting to investigate the role of NAC in this context.

REFERENCES AND NOTES

1. M. Halic *et al.*, *Nature* **427**, 808–814 (2004).
2. M. Halic *et al.*, *Science* **312**, 745–747 (2006).
3. Y. Nyathi, B. M. Wilkinson, M. R. Pool, *Biochim. Biophys. Acta* **1833**, 2392–2402 (2013).
4. T. A. Rapoport, *Nature* **450**, 663–669 (2007).
5. T. Schwartz, G. Blobel, *Cell* **112**, 793–803 (2003).
6. P. Walter, I. Ibrahimi, G. Blobel, *J. Cell Biol.* **91**, 545–550 (1981).
7. B. Wiedmann, H. Sakai, T. A. Davis, M. Wiedmann, *Nature* **370**, 434–440 (1994).
8. N. Borgese, W. Mok, G. Kreibich, D. D. Sabatini, *J. Mol. Biol.* **88**, 559–580 (1974).
9. B. Jungnickel, T. A. Rapoport, *Cell* **82**, 261–270 (1995).
10. K. U. Kalies, D. Görlich, T. A. Rapoport, *J. Cell Biol.* **126**, 925–934 (1994).
11. B. Lauring, H. Sakai, G. Kreibich, M. Wiedmann, *Proc. Natl. Acad. Sci. U.S.A.* **92**, 5411–5415 (1995).
12. A. Prinz, E. Hartmann, K. U. Kalies, *Biol. Chem.* **381**, 1025–1029 (2000).
13. T. A. Bloss, E. S. Witze, J. H. Rothman, *Nature* **424**, 1066–1071 (2003).
14. S. L. Rea, D. Wu, J. R. Cypser, J. W. Vaupel, T. E. Johnson, *Nat. Genet.* **37**, 894–898 (2005).
15. Materials and methods are available as supplementary materials on *Science* Online.
16. M. Calfon *et al.*, *Nature* **415**, 92–96 (2002).
17. T. Yoneda *et al.*, *J. Cell Sci.* **117**, 4055–4066 (2004).
18. J. Kirstein-Miles, A. Scior, E. Deuerling, R. I. Morimoto, *EMBO J.* **32**, 1451–1468 (2013).
19. M. R. Adelman, D. D. Sabatini, G. Blobel, *J. Cell Biol.* **56**, 206–229 (1973).
20. S. S. Vembar, J. L. Brodsky, *Nat. Rev. Mol. Cell Biol.* **9**, 944–957 (2008).
21. F. Urano *et al.*, *J. Cell Biol.* **158**, 639–646 (2002).
22. F. Schwarz, M. Aebi, *Curr. Opin. Struct. Biol.* **21**, 576–582 (2011).
23. S. Cabantous, T. C. Terwilliger, G. S. Waldo, *Nat. Biotechnol.* **23**, 102–107 (2005).
24. W. Neupert, J. M. Herrmann, *Annu. Rev. Biochem.* **76**, 723–749 (2007).
25. O. Yogev, O. Pines, *Biochim. Biophys. Acta* **1808**, 1012–1020 (2011).
26. I. Möller *et al.*, *Proc. Natl. Acad. Sci. U.S.A.* **95**, 13425–13430 (1998).
27. A. Neuhof, M. M. Rolls, B. Jungnickel, K. U. Kalies, T. A. Rapoport, *Mol. Biol. Cell* **9**, 103–115 (1998).
28. D. Raden, R. Gilmore, *Mol. Biol. Cell* **9**, 117–130 (1998).
29. M. del Alamo *et al.*, *PLOS Biol.* **9**, e1001100 (2011).
30. B. Reimann *et al.*, *Yeast* **15**, 397–407 (1999).
31. T. Ast, G. Cohen, M. Schuldiner, *Cell* **152**, 1134–1145 (2013).
32. S. C. Mutka, P. Walter, *Mol. Biol. Cell* **12**, 577–588 (2001).
33. R. M. Voorhees, I. S. Fernández, S. H. Scheres, R. S. Hegde, *Cell* **157**, 1632–1643 (2014).

REPORTS

QUANTUM GASES

Experimental observation of a generalized Gibbs ensemble

Tim Langen,[1] Sebastian Erne,[1,2,3] Remi Geiger,[1] Bernhard Rauer,[1] Thomas Schweigler,[1] Maximilian Kuhnert,[1] Wolfgang Rohringer,[1] Igor E. Mazets,[1,4,5] Thomas Gasenzer,[2,3] Jörg Schmiedmayer[1]*

The description of the non-equilibrium dynamics of isolated quantum many-body systems within the framework of statistical mechanics is a fundamental open question. Conventional thermodynamical ensembles fail to describe the large class of systems that exhibit nontrivial conserved quantities, and generalized ensembles have been predicted to maximize entropy in these systems. We show experimentally that a degenerate one-dimensional Bose gas relaxes to a state that can be described by such a generalized ensemble. This is verified through a detailed study of correlation functions up to 10th order. The applicability of the generalized ensemble description for isolated quantum many-body systems points to a natural emergence of classical statistical properties from the microscopic unitary quantum evolution.

Information theory provides powerful concepts for statistical mechanics and quantum many-body physics. In particular, the principle of entropy maximization (*1*–*3*) leads to the well-known thermodynamical ensembles, which are fundamentally constrained by conserved quantities such as energy or particle number (*4*). However, physical systems can contain many additional conserved quantities, which raises the question of whether there exists a more general statistical description for the steady states of quantum many-body systems (*5*).

Specifically, the presence of nontrivial conserved quantities puts constraints on the available phase space of a system, which strongly affects the dynamics (*6*–*9*) and inhibits thermalization (*10*–*12*). Instead of relaxing to steady states described by the usual thermodynamical ensembles, a generalized Gibbs ensemble (GGE) was proposed to describe the corresponding steady states via the many-body density matrix

$$\hat{\rho} = \frac{1}{Z}\exp\left(-\sum_m \lambda_m \hat{\mathcal{I}}_m\right) \qquad (1)$$

(*3*, *11*, *13*, *14*), where $\hat{\mathcal{I}}_m$ denotes a set of conserved quantities and $Z = \mathrm{Tr}[\exp(-\sum_m \lambda_m \hat{\mathcal{I}}_m)]$ is the partition function. The Lagrange multipliers λ_m associated with the conserved quantities are obtained by maximization of the entropy under the condition that the expectation values of the conserved quantities are fixed to their initial values. The emergence of such a maximum-entropy state does not contradict a unitary evolution according to quantum mechanics. Rather, it reflects that the true quantum state is indistinguishable from the maximum-entropy ensemble with respect to a set of sufficiently local observables (*5*).

[1]Vienna Center for Quantum Science and Technology, Atominstitut, TU Wien, 1020 Vienna, Austria. [2]Institut für Theoretische Physik, Ruprecht-Karls-Universität Heidelberg, 69120 Heidelberg, Germany. [3]ExtreMe Matter Institute (EMMI), GSI Helmholtzzentrum für Schwerionenforschung GmbH, 64291 Darmstadt, Germany. [4]Wolfgang Pauli Institute, 1090 Vienna, Austria. [5]Ioffe Physico-Technical Institute, 194021 St. Petersburg, Russia.
***Corresponding author. E-mail: schmiedmayer@atomchip.org**

A degenerated 1D Bose gas relaxes to a state that can be described by such a generalized Gibbs ensemble. Langen, et al, Science 348, 207 (2015)

Contents

I. Generalized Hydrodynamics: elementary introduction

Lieb-Liniger model

$$H_{\mathrm{LL}} = \int dx \left(\frac{1}{2m} \partial_x \psi^\dagger \partial_x \psi + \lambda \psi^\dagger \psi^\dagger \psi \psi \right)$$

Two reservoirs:

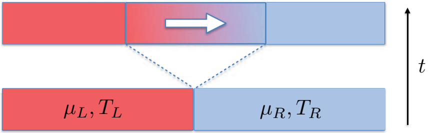


Figure 1. A geometry for far-from-equilibrium transport: two half-line reservoirs are prepared at one temperature or chemical potential for $x < 0$ and a different temperature or chemical potential for $x > 0$. At $t = 0$ the left and right reservoirs are connected in such a way that the final system is translation-invariant.

[1] Castro-Alvaredo et. al., Emergent hydrodynamics in integrable quantum systems out of equilibrium[J]. Physical Review X, 2016, 6(4): 041065
[2] Bertini et. al. Transport in out-of-equilibrium XXZ chain: Exact profiles of charges and currents, Phys. Rev. Lett. 117, 207201 (2016)

[3] Jiang, Chen, Guan, Chin. Phys. B 24, 050311 (2015)

Conformal field theory out of equilibrium: a review

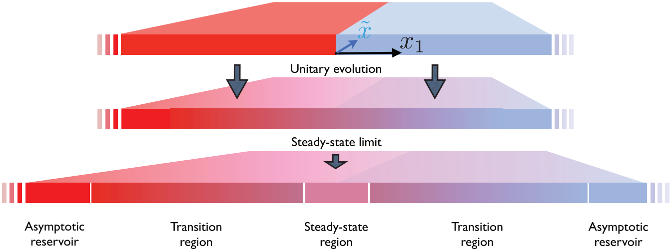


Figure 2. The partitioning approach. The direction of the flow is x_1, and $\tilde{x}$ represents the transverse coordinates. After any finite time, asymptotic reservoirs are still present and infinite in length. The central region, around $x_1 = 0$, is the steady-state region. At very large times, it is expected to be of very large extent.

Transient current $\langle j \rangle_{\rm stat} = \frac{\pi C}{12} \left(T_{\rm L}^2 - T_{\rm R}^2 \right)$

How to understand the emergent transient current in the Lieb-Liniger Bose gas?
What does Transient current behave like in the systems with two degrees of freedom?

- Do the non-equilibrium isolated quantum systems thermalize? open question
- The integrable systems has no thermalization because of the infinitely many conserved charges.
- For homogeneous systems, local observables are described by Generalized Gibbs Ensemble in the relaxed state:

$$\hat{\rho} = \frac{1}{Z} \exp\left(-\sum_m \lambda_m \hat{\mathcal{I}}_m \right), \quad Z = Tr \exp\left(-\sum_m \lambda_m \hat{\mathcal{I}}_m \right)$$

- Weakly inhomogeneous systems can be described by the Generalized Hydrodynamics at large space-time scales, i.e. continuity equations for particle density in each conserve charges

$$\partial_t \rho + \partial_x \left[v^{\mathrm{eff}} \rho \right] = 0$$

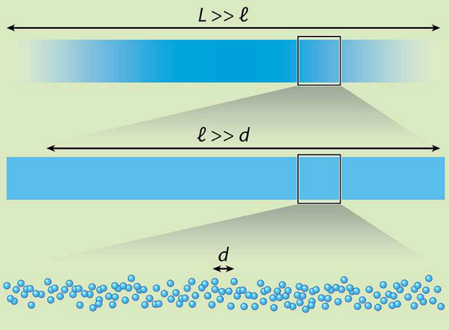


Figure 1: Castro-Alvaredo *et al.* [1] and Bertini *et al.* [2] used a hydrodynamics approach to describe interacting quantum particles in 1D (bottom). The approach takes a zoomed-out picture of the particles (middle), viewing it on a length scale l that is much longer than the average distance d between particles. In this way, the particles appear as a continuous medium, like a fluid. A description of the system on a very long length scale L can then be calculated, such as how its mass density varies in space (top) and how this quantity evolves in time. (APS/Carin Cain)

- Dynamical generalization of thermodynamic Bethe ansatz equations
- Generalized Hydrodynamic: $\partial_t q(x,t) + \partial_x j(x,t) = 0$

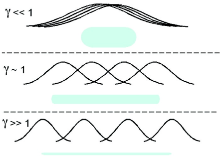
Hydrodynamic description

- I. The generalized Gibbs ensemble: density matrix $\rho = e^{-\sum \beta_i Q_i} / Tr\left(e^{-\sum \beta_i Q_i}\right)$, where $Q = (Q_1 \ldots)$ are all the conserved quantities and $\beta = (\beta_1 \ldots)$ are all the associated potentials. To determine an equilibrium of integrable systems, we need all associated potentials or all conserved quantities.
- II. Local time-space (equilibrium) approximation: densities $\hat{q}_i(x,t) \equiv \langle q_i(x,t) \rangle = \langle q_i \rangle_{\beta(x,t)}$, and currents $\hat{j}_i(x,t) \equiv \langle j_i(x,t) \rangle = \langle j_i \rangle_{\beta(x,t)}$. The intervals (cells) are locally homogenous enough, satisfying the GGE. In a large space and time scales, the cells present slow variations of quantum dynamics.
- III. Hydrodynamic equation: The densities $\hat{q}_i(x,t)$ and currents $\hat{j}_i(x,t)$ satisfy the Euler-type equation $\partial_t \hat{q}_i(x,t) + \partial_x \hat{j}(x,t) = 0$.
- IV. The steady state: The currents $\hat{j}_i(x,t) = \langle j_i \rangle_{\beta(x,t)} = F(q)$, then we have $\partial_t \hat{q}_i(x,t) + J(\hat{q}_i(x,t)) \partial_x \hat{q}(x,t) = 0$ with the Jacobian matrix of transformation from densities to currents $J(\hat{q}_i(x,t)) = \partial F_i(q)/\partial q_j$. For a proper choice the state coordinates, one can get $\partial_t \hat{n}_i(x,t) + v_i^{\mathrm{eff}}(\hat{n}(x,t)) \partial_x \hat{n}_i(x,t) = 0$.
- V. Rescaling: $\hat{q}(x,t) \to \hat{q}(\xi = x/t)$, $\beta(x,t) \to \beta(\xi = x/t)$. Then the Euler equation and the steady states read

$$[J(\hat{q}_i(\xi)) - \xi]\, \partial_\xi \hat{q}_i = 0$$
$$q^{\mathrm{stat}} := \hat{q}(\xi = 0), \qquad j^{\mathrm{stat}} := \hat{j}(\xi = 0)$$

that determine a critical ray.

Lieb-Liniger model: fundamental concepts and experiments



Lieb & Liniger in 1963 solved the 1D δ-function interacting Bose gas (Interaction range is much less than the mean distance between atoms)

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j}^{N} \delta(x_i - x_j)$$

$$c = -4\hbar^2/ma_{1\mathrm{D}}, \qquad \gamma = c/n$$

- Bethe anasatz wave function: quantum correlations

$$\Psi(x_1, x_2, \ldots, x_N) = \sum_p (-1)^p \Big[\prod_{1\le i<j\le N} \Big(1 + \frac{\mathrm{i}k_{pj} - \mathrm{i}k_{pi}}{c}\Big)\Big] \exp\Big(\sum_{j=1}^{N} \mathrm{i}k_{pj} x_j\Big)$$

- Energy spectrum: thermodynamics and collective nature

$$E = \frac{\hbar^2}{2m}\sum_{j=1}^{N} k_j^2, \;\; \exp(\mathrm{i}k_j L) = -\prod_{\ell=1}^{N} \frac{k_j - k_\ell + \mathrm{i}\,c}{k_j - k_\ell - \mathrm{i}\,c}, \;\; j = 1, \ldots N$$

- **Notations** k_j: wave number, c: interaction strength, N: particle number

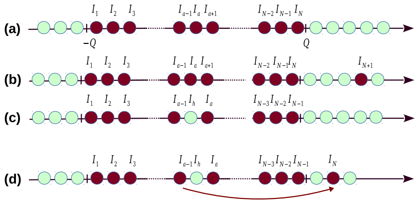
Table 1. Experiments of Lieb–Liniger gas.

quantum dynamics	^{87}Rb[61,71,74–76]
thermalization	^{87}Rb[61,70,74,75]
solitons	^{87}Rb[68,77]
fermionization	^{39}K[57,72]
YY thermodynamics	^{87}Rb[56,61,62,65–67,78]
strong coupling	^{87}Rb[24,25]
phase diagram	Cs[79]
3-body correlations	^{87}Rb,[58,63] Cs[60]
excited state	Cs[64]

Jiang, Chen, Guan, Chin. Phys. B 24, 050311 (2015)

However, the observation of the **Luttinger liquid and quantum criticality** of a 1D quantum system is a long-standing challenge.

Cazalilla, Citro, Giamarchi, Orignac, & Rigol, *Rev. Mod. Phys.* 83, 1405 (2011)
Guan, Batchelor, and Lee, *Rev. Mod. Phys.* 85, 1633 (2013)



Elementary excitations give the universal low energy physics

- (a) the quantum numbers for the ground state
- (b) adding a particle near right Fermi point (ΔN or backward scattering $2\Delta D$).
- (c) a hole excitation. The total number of particles is $N-1$.
- (d) a single particle-hole excitation ($N^{\pm}$).

Total momentum and excitation energy

$$\begin{aligned}\Delta P &= \frac{2\pi}{L}\left[\Delta N \Delta D + N^{+} - N^{-}\right] + 2\Delta D k_{\mathrm{F}},\\ \Delta E &= \frac{2\pi v_s}{L}\left[\frac{1}{4}(\Delta N/Z)^2 + (\Delta D Z)^2 + N^{+} + N^{-}\right]\\ &= \frac{\pi}{2L}\left[v_s Z^{-2}(\Delta N)^2 + v_s Z^2 J^2 + 4 v_s N^{+} + 4 v_s N^{-}\right]\\ J &= 2\Delta D, \quad K = v_s/v_N, \qquad Z = 2\pi\rho(Q)\end{aligned}$$

Origin of Quantum liquid

$$\text{Bethe Ansatz result}\quad \Delta E = \frac{2\pi v_s}{L}(N^- + N^+) + \frac{\pi}{2L}\left(\frac{v_s}{Z^2}\Delta N^2 + v_s Z^2 (2D)^2\right)$$

$$\text{Haldane's Bosonization Hamiltonian}\quad H = v_s \sum_{q\neq 0} |q| \hat{b}_q^\dagger \hat{b}_q + \frac{\pi}{2L}\left(v_N\, \Delta N^2 + v_J\, J^2 \right)$$

$$H = \int dx \left(\frac{\pi v_s K}{2} \Pi^2 + \frac{v_s}{2\pi K} (\partial_x \phi)^2 \right)$$

where the canonical momenta Π conjugate to the phase ϕ obeying the standard Bose commutation relations $[\phi(x), \Pi(y)] = \mathrm{i}\delta(x - y)$.

Luttinger parameter $K = \sqrt{v_J / v_N} = Z^2$

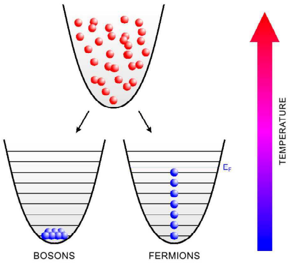
Sound velocity $v_s = \sqrt{v_N v_J} = \sqrt{\frac{L^2}{mN}\frac{\partial^2 E}{\partial L^2}}$

Density stiffness $v_N = \frac{v_s}{K} = \frac{L}{\pi\hbar}\left(\frac{\partial^2 E}{\partial N^2}\right)_{N=N_0}$

Phase stiffness $v_J = v_s K = \pi L \frac{\partial^2}{\partial \alpha} E$

e.g. K determines the leading order correlation $\langle \Psi^\dagger(x)\Psi(0)\rangle \sim 1/x^{1/2K}$

Bosons and Fermions



Quantum statistics:

1. quantum man-body systems
2. microscopic state energy E_i
3. partition function $Z = \sum_{i=1}^{\infty} W_i e^{-E_i/(k_B T)}$
4. free energy $F = -k_B T \ln Z$
5. challenge: finding new physics

- Yang-Yang equation (Gibbs ensemble): a brilliant method(J. Math. Phys. 10, 1115 (1969))

$$\varepsilon(k) = k^2 - \mu - \frac{Tc}{\pi} \int \frac{dq}{c^2 + (k-q)^2} \ln\left(1 + e^{-\varepsilon(q)/T}\right)$$

- Equation of state: per length pressure

$$p(\mu, T) = \frac{T}{2\pi} \int_{-\infty}^{\infty} \ln\left(1 + e^{-\varepsilon(k)/T}\right) dk$$

Dynamical generalization of the thermodynamics Bethe ansatz:
The generalized hydrodynamics of the 1D Bose gas

Dynamical generalization of the thermodynamics Bethe ansatz

Lieb-Liniger model

eigenstate $|\{\lambda_j\}\rangle$

number operator: $\hat{Q}_0 = \sum_{i=1}^{N} p_i^0$

momentum operator: $\hat{Q}_1 = \sum_{i=1}^{N} p_i$

Hamiltonian: $\hat{Q}_2 = \sum_{i=1}^{N} p_i^2$

$\hat{Q}_N = \sum_{i=1}^{N} p_i^N$

$$[H_0, \hat{Q}_n] = 0, \quad [\hat{Q}_n, \hat{Q}_m] = 0, \forall n, m.$$

$$\widehat{Q}_0 \,|\{\lambda\}\rangle = Q_0 \,|\{\lambda\}\rangle \,, \; Q_0 = N = \sum_j \lambda_j^0$$

$$\widehat{Q}_1 \,|\{\lambda\}\rangle = Q_1 \,|\{\lambda\}\rangle \,, \; Q_1 = \sum_j \lambda_j$$

$$\widehat{Q}_2 \,|\{\lambda\}\rangle = Q_2 \,|\{\lambda\}\rangle \,, \; Q_2 = \sum_j \lambda_j^2$$

$$\widehat{Q}_n \,|\{\lambda\}\rangle = Q_n \,|\{\lambda\}\rangle \,, \; Q_n = \sum_j \lambda_j^n$$

Bulding up the densities and currents of the conserved charges with the thermodynamic Bethe ansatz

Generalized Hamiltonian

$$H(\{\beta\}) = \sum_n \beta_n \hat{Q}_n$$

$$H(\{\beta\})|\{\lambda\}\rangle = E(\{\beta\})|\{\lambda\}\rangle, \quad E(\{\beta\}|\{\lambda\}) = \sum_{n=0}^{\infty} \sum_{j=1}^{N} \beta_n \lambda_j^n \equiv \sum_{j=1}^{N} \varepsilon_0(\lambda_j),$$

in which we have defined the function

$$\varepsilon_0(\lambda) \equiv \sum_{n=0}^{\infty} \beta_n \lambda^n$$

by interpreting the coefficients β_n as those of its power series.

Thermodynamic limit

Expectation values of the conserved charges: $Q_n = L \int_{-\infty}^{\infty} \mathrm{d}\lambda \lambda^n \rho(\lambda)$

generalized Gibbs "free energy" $G[\rho, \rho_h] = \sum_{n=0} \beta_n Q_n - S[\rho, \rho_h]$

Generalized TBA equation:

$$\varepsilon(\lambda) + a_2 * \ln(1 + \mathrm{e}^{-\varepsilon(\lambda)}) = \varepsilon_0(\lambda)$$

$h_i(\theta)$: the one-particle eigenvalue of conserved charge Q_i

θ is the velocity, $h_0(\theta) = 1, h_1(\theta) = p(\theta) \equiv m\theta, h_2(\theta) = E(\theta) \equiv m\theta^2/2$.

Average density

$$\mathcal{q}_i = \int d\theta \rho_p(\theta) h_i(\theta)$$

quasiparticle density: $\rho_p(\theta)$

hole density: $\rho_h(\theta)$

state density: $\rho_s(\theta) = \rho_p(\theta) + \rho_h(\theta)$

occupation number: $n(\theta) = \rho_p(\theta)/\rho_s(\theta)$

discrete BA eqn → $$2\pi\rho_s(\theta) = p'(\theta) + \int d\alpha \varphi(\theta - \alpha)\rho_p(\alpha)$$

"dressing" operation $h \mapsto h^{\mathrm{dr}}$

$$h^{\mathrm{dr}}(\theta) = h(\theta) + \int \frac{d\gamma}{2\pi} \varphi(\theta - \gamma) n(\gamma) h^{\mathrm{dr}}(\gamma)$$

$$2\pi\rho_p(\theta) = n(\theta)(p')^{\mathrm{dr}}(\theta)$$

$$h^{\mathrm{dr}} = (1 - \varphi\mathcal{N})^{-1} h$$

$$\mathcal{U} = \mathcal{N}(1 - \varphi\mathcal{N})^{-1}$$

$$\mathcal{q}_i = h_i \cdot \mathcal{U} p' = p' \cdot \mathcal{U} h_i$$

$$a \cdot b = \int d\theta/(2\pi) a(\theta) b(\theta)$$

$$\mathcal{q}_i = \int \frac{dp(\theta)}{2\pi} n(\theta) h_i^{\mathrm{dr}}(\theta)$$

Average current

$$j_0 = \int \frac{dE(\theta)}{2\pi} n(\theta)\, h_0^{\mathrm{dr}}(\theta)$$
$$= \mathrm{m} \int \frac{d\theta}{2\pi} \theta \mathrm{n}(\theta)\, h_0^{\mathrm{dr}}(\theta)$$
$$= \int \frac{d\theta}{2\pi} \theta n(\theta)\, (p')^{\mathrm{dr}}(\theta)$$
$$= \int d\theta \theta \rho_p(\theta)$$
$$= \frac{1}{m} \int d\theta \rho_p(\theta)\, p(\theta)$$
$$= \frac{1}{m} q_1$$

$$\mathtt{q}_i = \int \frac{dp(\theta)}{2\pi} n(\theta) h_i^{\mathrm{dr}}(\theta), \qquad \mathtt{j}_i = \int \frac{dE(\theta)}{2\pi} n(\theta) h_i^{\mathrm{dr}}(\theta)$$

$$m \int \frac{d\theta}{2\pi} n(\theta)\, h_i^{\mathrm{dr}}(\theta) \qquad \mathrm{m} \int \frac{d\theta}{2\pi} \theta \mathrm{n}(\theta)\, h_i^{\mathrm{dr}}(\theta)$$

$$j_i = \int \frac{dE(\theta)}{2\pi} n(\theta)\, h_i^{\mathrm{dr}}(\theta) = E' \cdot \mathcal{U} h_i = h_i \cdot \mathcal{U} E' = \int \frac{d\theta}{2\pi} h_i(\theta)\, n(\theta)\, (E')^{\mathrm{dr}}(\theta) \equiv \int d\theta \rho_c(\theta)\, h_i(\theta)$$

$$2\pi \rho_c(\theta) \equiv n(\theta)\, (E')^{\mathrm{dr}}(\theta) \equiv 2\pi v^{\mathrm{eff}}(\theta)\, \rho_p(\theta)$$

$$n(\theta) = \frac{1}{1+e^{\epsilon(\theta)}}, \qquad v^{\mathrm{eff}} \equiv: \frac{(E')^{\mathrm{dr}}(\theta)}{(p')^{\mathrm{dr}}(\theta)}$$
$$\epsilon(\theta) = \epsilon_0(\theta) - \int \frac{d\theta'}{2\pi} \phi(\theta-\theta') \ln\left(1+e^{-\epsilon(\theta')}\right)$$

The densities and currents of charge in each cells depend on the states via occupation number $n(\theta)$, which can be obtained by the BA equation. v^{eff} is the effective velocity of quasiparticles.

method: hydrodynamic equation

fluid cell

$$\partial_t q(x,t) + \partial_x j(x,t) = 0$$

light-cone ansatz

$$q_i = \int d\theta \rho_p(\theta) h_i(\theta)$$

$$j_i = \int d\theta \rho_c(\theta) h_i(\theta)$$

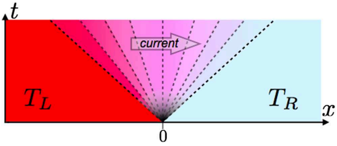


FIG. 1. The partitioning protocol. With ballistic transport, a current emerges after a transient period. Dotted lines represent different values of $\xi = x/t$. If a maximal velocity exists (e.g., due to the Lieb-Robinson bound), initial reservoirs are unaffected beyond it (light-cone effect). The steady state lies at $\xi = 0$.

$$\partial_t \rho_p(\theta) + \partial_x \rho_c(\theta) = 0$$

$$2\pi\rho_c(\theta) = n(\theta)(E')^{\mathrm{dr}}(\theta) = 2\pi v^{\mathrm{eff}}(\theta)\rho_p(\theta)$$

$$\partial_t \rho_p(\theta) + \partial_x [v^{\mathrm{eff}}(\theta)\rho_p(\theta)] = 0$$

$$\partial_t n(\theta) + v^{\mathrm{eff}}(\theta)\,\partial_x n(\theta) = 0$$

$$\xi = x/t$$

$$\left[v^{\mathrm{eff}}(\theta) - \xi\right] \partial_\xi n(\theta) = 0$$

initial condition

$$\lim_{\xi\to\pm\infty} n(\theta) = \lim_{x\to\pm\infty} n(\theta)\,|_{t=0} = n(\theta)\,|_{t=0}^{R,L}$$

$$\left[v^{\text{eff}}(\theta) - \xi\right]\partial_\xi n(\theta) = 0$$

solution

$$n(\theta) = n^L(\theta)\,\Theta(\theta-\theta_\star) + n^R(\theta)\,\Theta(\theta_\star-\theta)$$

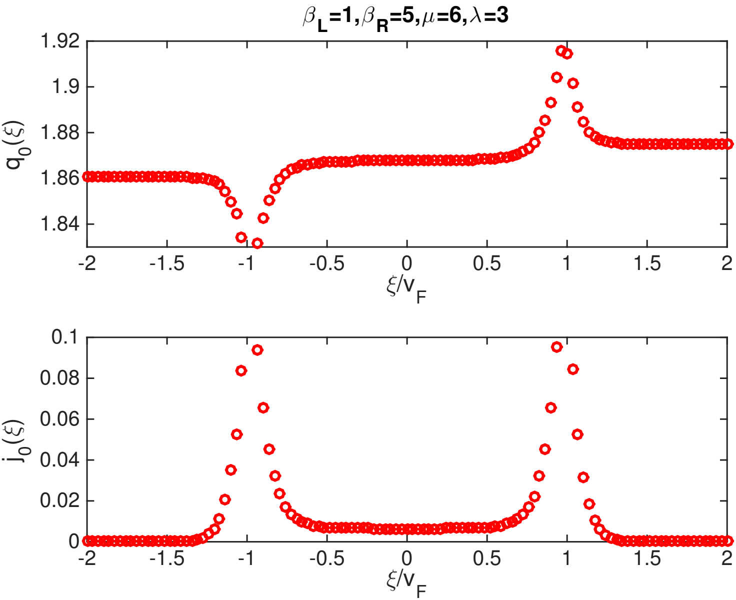
$$v^{\text{eff}}(\theta_\star) = \xi$$

$$p_\xi^{\text{dr}}(\theta_*) = 0$$

Conservation of entropy density is a fundamental propeerty of perfect fluids, as no viscosity effects are taken into account.

$$S(\theta) := \rho_s(\theta)\log\rho_s(\theta) - \rho_p(\theta)\ln\rho_p(\theta) - \rho_h(\theta)\ln\rho_h(\theta)$$

$$\partial_t S(\theta) + \partial_x\left[v^{\text{eff}}(\theta)\xi(\theta)\right] = 0$$


β_L=1,β_R=5,μ=6,λ=3
$q_0(\xi)$
1.92
1.9
1.88
1.86
1.84
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
ξ/v_F
$j_0(\xi)$
0.1
0.08
0.06
0.04
0.02
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
ξ/v_F

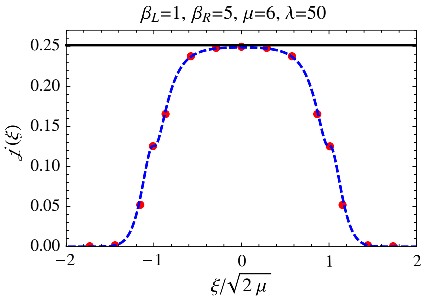


FIG. 7. Energy current in the Lieb-Liniger model for low temperatures, large coupling, and chemical potential $\mu = 6$ (circles). Local stationary points occur at $\alpha_{L,R} = 0$, that is, $\xi = \pm\sqrt{2\mu} = \pm 3.46$ (the Fermi velocity). The dashed curve represents the current [Eq. (47)] for the same temperatures and chemical potential, whose profile is not dissimilar to the plots shown in Fig. 5. As before, the bold horizontal line is the CFT value $\frac{\pi}{12}(1 - \frac{1}{25})$. The agreement is extremely good.

Emergence of generalized hydrodynamics determined by such a generalized Gibbs ensemble.

$$j^{\text{stat}} = \frac{\pi c k_B^2}{12\hbar}\left(T_L^2 - T_R^2\right)$$

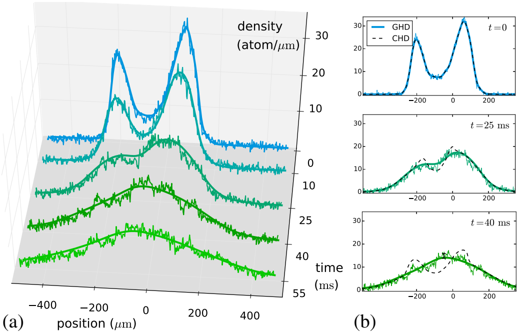


FIG. 3. (i) Longitudinal expansion of a cloud of $N = 6300 \pm 200$ atoms initially trapped in a double-well potential, compared with GHD. (ii) Even though the initial state is the same for GHD and CHD, both theories clearly differ at later times. CHD wrongly predicts the formation of two large density waves. The error bar shown at the center at $t = 40$ ms corresponds to a 68% confidence interval, and is representative for all data sets.

Emergence of generalized hydrodynamics determined by such a generalized Gibbs ensemble. Schemmer, et al, PRL 122, 090601(2019)

$$\partial_x \rho(x, v) + \partial_x \left[v^{\text{eff}} \rho(x, v) \right] = \left(\frac{\partial_x V(x)}{m} \right) \partial_v \rho(x, v), \quad v : \text{ rapidity}$$

Emergent Hydrodynamics in a 1D Bose-Fermi Mixture

Xi-Wen Guan

II. Ballistic transport of the Bose-Fermi mixures in 1D

Key collaborators:

Sheng Wang, Xiangguo Yin, Yang Yang Chen and Yunbo Zhang

Beijing Computational Science Research Centre, June 2019

- The integrable systems has no thermalization because of the infinitely many conserved charges.
- For homogeneous systems, local observables are described by Generalized Gibbs Ensemble $\langle \hat{O} \rangle = \mathrm{Tr} \hat{\rho}_{\mathrm{QQE}} \hat{O}$ in the relaxed state:

$$\hat{\rho} = \frac{1}{Z} \exp\left(-\sum_{m} \lambda_m \hat{\mathcal{I}}_m \right), \quad Z = Tr \exp\left(-\sum_{m} \lambda_m \hat{\mathcal{I}}_m \right)$$

- Weakly inhomogeneous systems can be described by the Generalized Hydrodynamics at large space-time scales, i.e. continuity equations for particle density in each conserve charges

$$\partial_t \rho + \partial_x \left[v^{\mathrm{eff}} \rho \right] = 0$$

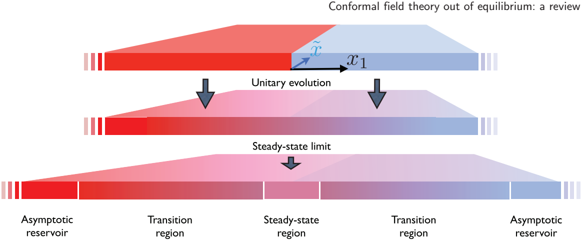


Figure 2. The partitioning approach. The direction of the flow is x_1, and $\tilde{x}$ represents the transverse coordinates. After any finite time, asymptotic reservoirs are still present and infinite in length. The central region, around $x_1 = 0$, is the steady-state region. At very large times, it is expected to be of very large extent.

Transient current $\langle j \rangle_{\text{stat}} = \frac{\pi C}{12}\left(T_{\text{L}}^2 - T_{\text{R}}^2\right)$

How does this transient current occur in the Lieb-Liniger Bose gas? What does the transient current behave like in the systems with two degrees of freedom?

A Bose-Fermi mixture system (BF) is described by the Hamiltonian,

$$H = \int_0^L dx \left(\frac{\hbar^2}{2m_b} \partial_x \psi_b^+ \partial_x \psi_b + \frac{\hbar^2}{2m_f} \partial_x \psi_f^+ \partial_x \psi_f + \frac{g_{bb}}{2} \psi_b^+ \psi_b^+ \psi_b \psi_b + g_{bf} \psi_b^+ \psi_f^+ \psi_f \psi_b \right)$$

where ψ_b and ψ_f are boson and fermion field operators, m_b and m_f are boson and fermion masses, g_{bb} and g_{bf} are boson-boson and boson-fermion interaction strengths. here the fermions we consider are spinless, so $g_{ff} = 0$.

When $m_b = m_f = m$ and $g_{bb} = g_{bf} = g$, the system is integrable.

Lai, Yang, Phys. Rev. A 3, 393 (1973)
Imambekov, Demler, Phys. Rev. A 73, 021602 (2006); Ann. Phys. 321, 2390 (2006)
Batchelor, Bortz, Guan, Oelkers, Phys. Rev. A 72, 061603 (2005)
Guan, Batchelor, Lee, Phys. Rev. A 78, 023621 (2008)
Yin, Chen, Zhang, Phys. Rev. A 79, 053604 (2009)
Guan, Batchelor, and Lee, Rev. Mod. Phys. 85, 1633 (2013)

Considering a system with N total particles, M of which are bosons, and the rest are spinless fermions. The first-quantized form of the solvable Hamiltonian is

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j} \delta(x_i - x_j)$$

where $2m = \hbar = 1$, $c = mg/\hbar^2$.

Wavefunction of the system is supposed to be symmetric with respect to indices $i = 1, ..., M$(bosons) and antisymmetric with respect to $i = M + 1, ..., N$(fermions).
Coordinate Bethe wavefunction: $0 < x_{Q_1} < x_{Q_2} < ... < x_{Q_N} < L$

$$\Psi_\sigma(x) = \sum_P A_\sigma(P, Q) e^{i \sum_i k_{P_i} x_{Q_i}}$$

where $\sigma = (\sigma_1, \sigma_2, ..., \sigma_N)$, with σ_j denoting the $SU(1|1)$ component of the jth particles; P, Q are arbitrary permutations from S_N group.

By solving the Schrödinger equation and using the continuity of wavefunction, we get the two-body scattering relation

$$A_\sigma(P,Q) = \frac{(k_{P_a} - k_{P_b})P_{Q_aQ_b} + ic}{k_{P_a} - k_{P_b} - ic} A_\sigma(P',Q)$$

where

$$A_\sigma(P',Q') = P_{Q_aQ_b} A_\sigma(P',Q)$$

$$Q = (...Q_aQ_b...), Q' = (...Q_bQ_a...)$$

$$P = (...P_aP_b...), P' = (...P_bP_a...), b = a+1$$

Permutation operator $P_{\sigma_{Q_a}\sigma_{Q_b}}$ of two $\sigma_{Q_a}\sigma_{Q_b}$:

$$P_{\sigma_{Q_a}\sigma_{Q_b}} P_{Q_a Q_b} = 1 \text{ (identical principle)}$$

$$A_{\ldots\sigma_{Q_a}\ldots\sigma_{Q_b}\ldots}(P', Q) = P_{\sigma_{Q_a}\sigma_{Q_b}} A_{\ldots\sigma_{Q_a}\ldots\sigma_{Q_b}\ldots}(P', Q')$$

Two-body scattering operator:

$$A_{\sigma}(P, Q) = S_{Q_a Q_b}(k_{P_b} - k_{P_a}) A_{\sigma}(P', Q)$$

where

$$S_{Q_a Q_b}(k_{P_b} - k_{P_a}) = \frac{(k_{P_b} - k_{P_a}) - icP_{\sigma_{Q_a}\sigma_{Q_b}}}{k_{P_b} - k_{P_a} + ic}$$

Periodic boundary condition:

$$\Psi(x_1,...,x_i,...,x_N) = \Psi(x_1,...,x_i+L,...,x_N)$$

Then

$$A_\sigma(P_i,P_1,...P_N;Q_i,Q_1,...,Q_N) = e^{ik_iL}A_\sigma(P_1,...P_N,P_i;Q_1,...,Q_N,Q_i)$$

Finally, the following equation need to be diagonalize

$$S_{i+1,i}(k_{i+1}-k_i)...S_{N,i}(k_N-k_i)S_{1,i}(k_1-k_i)...S_{i-1,i}(k_{i-1}-k_i)A_\sigma(P,Q)$$
$$=e^{ik_iL}A_\sigma(P,Q)$$

Local state space as the Hilbert space: $h_n = C^2$.
Entire state space with N particles: $H_N = \prod_{n=1}^{N} \otimes h_n$.
Operator $R_{ij}(u)$:

$$R_{ij}(u) = \frac{uP_{ij} - ic}{u + ic}, P_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which acts on the space $V_i \otimes V_j$. $V$is auxiliary space, $V = C^2$, and P_{ij} is the graded permutation operator.
Yang-Baxter equation:

$$R_{12}(\lambda - \mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda - \mu)$$

Lax operator:

$$L_j(k_j - u) = \frac{u - icP_{j\tau}}{u + ic}$$

which acts in $h_j \otimes V$ and the auxiliary space V is C^2.

Monodromy operator:

$$T_N(u) = L_N(k_N - u)...L_1(k_1 - u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

Graded RTT relation

$$R(u - v)[T'_N(u)T'_N(v)] = [T'_N(v)T'_N(u)]R(u - v)$$

where

$$T'_N(u) = T_N(u) \otimes_s I$$
$$T'_N(v) = I \otimes_s T_N(v)$$

$\otimes_s$ is the graded tensor product, and the definition is

$$(A \otimes_s B)^{ab}_{cd} = (-1)^{(p(a)+p(c))p(b)} A^a_c B^b_d$$

Graded ransfer matrix:

$$\tau(u) = str_{\tau} T_N(u) = A(u) - D(u)$$

After diagonalize the following equation

$$\tau(k_j) A_{\sigma}(P, Q) = e^{ik_i L} A_{\sigma}(P, Q)$$

Bethe Ansatz equation is obtained

$$\prod_{\alpha=1}^{M} \frac{k_j - \Lambda_{\alpha} + \frac{ic}{2}}{k_j - \Lambda_{\alpha} - \frac{ic}{2}} = e^{ik_j L}, j = 1, 2, ..., N$$

$$\prod_{j=1}^{N} \frac{k_j - \Lambda_{\beta} + \frac{ic}{2}}{k_j - \Lambda_{\beta} - \frac{ic}{2}} = 1, \beta = 1, 2, ..., M$$

Eigenvalue of transfer matrix:

$$t(u)=\prod_{\alpha=1}^{M}\left(\frac{u-\Lambda_\alpha+ic/2}{u-\Lambda_\alpha-ic/2}\right)\left(1-\frac{\prod_{j=1}^{N}(u-k_j)}{\prod_{j=1}^{N}(u-k_j-ic)}\right)$$

The coefficients of the polynomial are the eigenvalue of conserved quantities. They can be expressed as the multiplication of two parts. One part involves only the $k_1,...,k_N$, and the other one involves only the $\Lambda_1,...,\Lambda_M$.
There are two independent sets of conserved quantities respectively in terms of $k_1,...,k_N$ and $\Lambda_1,...,\Lambda_M$.

In the thermodynamic limit, Bethe Ansatz equation

$$\rho(k)+\rho_h(k)=\frac{1}{2\pi}+\int_{-\infty}^{\infty}a(k,\Lambda)\sigma(\Lambda)d\Lambda$$
$$\sigma(\Lambda)+\sigma_h(\Lambda)=\int_{-\infty}^{\infty}a(\Lambda,k)\rho(k)dk$$

where

$$a(k,\Lambda)=\frac{1}{2\pi}\frac{4c}{c^2+4(k-\Lambda)^2}$$

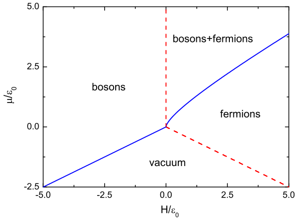
Denoting the two sets of single-particle conserved quantities as $h^i_\rho(k)$ and $h^i_\sigma(\Lambda)$. The conserved quantities of the whole system can be expressed as below.

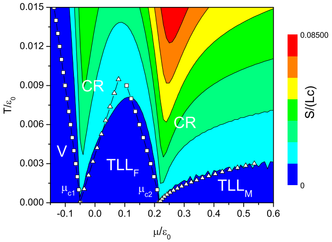
$$Q_\rho/L=\int h^i_\rho(k)\rho(k)dk$$
$$Q_\sigma/L=\int h^i_\sigma(\Lambda)\sigma(k)dk$$

With the exact expression of the Yang-Yang entropy, and minimizing the Gibbs free energy, the thermodynamic Bethe Ansatz equations for equilibrium states at finite temperature are obtained

$$\begin{aligned}\varepsilon(k) =& k^2 - \mu_f - T\int_{-\infty}^{\infty} a(k,\Lambda)\ln(1+e^{-\varphi(\Lambda)/T})d\Lambda\\ \varphi(\Lambda) =& \mu_f - \mu_b - T\int_{-\infty}^{\infty} a(\Lambda,k)\ln(1+e^{-\varepsilon(k)/T})dk,\end{aligned}$$

where μ_f and μ_b are chemical potentials of fermions and bosons.





Phase diagram

Entropy at $H = 0.1\epsilon_b$

Luttinger liquid
$$F = E_0 - \frac{\pi C T^2}{6}\left(\frac{1}{v_b} + \frac{1}{v_f}\right)$$

Quantum criticality
$$n(T,\mu) = n_0 + T^{\frac{d}{z}+1-\frac{1}{\nu z}}\mathcal{G}\left(\frac{\mu-\mu_c}{T^{\frac{1}{\nu z}}}\right)$$
$$\kappa(T,\mu) = \kappa_0 + T^{\frac{d}{z}+1-\frac{2}{\nu z}}\mathcal{F}\left(\frac{\mu-\mu_c}{T^{\frac{1}{\nu z}}}\right)$$

Guan, Batchelor, and Lee, *Rev. Mod. Phys.* 85, 1633 (2013)

For the two independent sets of conserved quantities, continuity equations are

$$\partial_t q^i_\rho(x,t) + \partial_x j^i_\rho(x,t) = 0,$$
$$\partial_t q^i_\sigma(x,t) + \partial_x j^i_\sigma(x,t) = 0, i = 1, 2, ...$$

Local density approximation: For any time and any fuild cell Δx of the system, the Bethe Ansatz equations hold

$$\rho(k,x,t) + \rho_h(k,x,t) = \frac{1}{2\pi} + \int_{-\infty}^{\infty} a(k,\Lambda)\sigma(\Lambda,x,t)d\Lambda$$
$$\sigma(\Lambda,x,t) + \sigma_h(\Lambda,x,t) = \int_{-\infty}^{\infty} a(\Lambda,k)\rho(k,x,t)dk$$

Then

$$q^i_\rho(x,t) = \int dk\rho(k,x,t)h^i_\rho(k)$$
$$q^i_\sigma(x,t) = \int d\Lambda\sigma(\Lambda,x,t)h^i_\sigma(\Lambda)$$

The expectation value of the currents are very difficult to calculate, but for homogeneous stationary state, they are expressed as

$$j^i_\rho(x,t) = \int dk v_\rho(k,x,t)\rho(k,x,t)h^i_\rho(k)$$
$$j^i_\sigma(x,t) = \int d\Lambda v_\sigma(\Lambda,x,t)\sigma(\Lambda,x,t)h^i_\sigma(\Lambda)$$

where v_ρ and v_σ are the sound velocities of the excitations.

$$v_\rho(k) = \frac{\partial\varepsilon(k)}{\partial p^{dr}_\rho(k)} = \frac{\varepsilon'(k)}{p'^{dr}_\rho(k)}$$
$$v_\sigma(\Lambda) = \frac{\partial\varphi(\Lambda)}{\partial p^{dr}_\sigma(\Lambda)} = \frac{\varphi'(\Lambda)}{p'^{dr}_\sigma(\Lambda)}$$

where $\varepsilon(k)$ and $\varphi(\Lambda)$ are the energies of the excitations, $p^{dr}_\rho(k)$ and $p^{dr}_\sigma(\Lambda)$ are the momenta of the excitations.

From the exact expressions of the densities and currents, the continuity equations can be expressed as

$$\int dk \left(\partial_t \rho(k,x,t) + \partial_x(v_\rho(k,x,t)\rho(k,x,t))\right) h^i_\rho(k) = 0$$
$$\int d\Lambda \left(\partial_t \sigma(\Lambda,x,t) + \partial_x(v_\sigma(\Lambda,x,t)\sigma(\Lambda,x,t))\right) h^i_\sigma(\Lambda) = 0, i = 1, 2, ...$$

From the completeness of the function spaces $\{h^i_\rho(k)\}$ and $\{h^i_\sigma(\Lambda)\}$

$$\partial_t \rho(k,x,t) + \partial_x(v_\rho(k,x,t)\rho(k,x,t)) = 0$$
$$\partial_t \sigma(\Lambda,x,t) + \partial_x(v_\sigma(\Lambda,x,t)\sigma(\Lambda,x,t)) = 0$$

For sound velocities,

$$p_\rho'^{dr}(k) = 2\pi[\rho(k) + \rho_h(k)] = 1 + \int_{-\infty}^{\infty} d\Lambda a(k,\Lambda) n_\sigma(\Lambda) p_\sigma'^{dr}(\Lambda)$$

$$p_\sigma'^{dr}(\Lambda) = 2\pi[\sigma(\Lambda) + \sigma_h(\Lambda)] = \int_{-\infty}^{\infty} dk a(\Lambda,k) n_\rho(k) p_\rho'^{dr}(k)$$

$$\varepsilon'(k) = e'(k) + \int_{-\infty}^{\infty} d\Lambda a(k,\Lambda) n_\sigma(\Lambda) \varphi'(\Lambda)$$

$$\varphi'(\Lambda) = \int_{-\infty}^{\infty} dk a(\Lambda,k) n_\rho(k) \varepsilon'(k)$$

where

$$n_\rho(k) = \rho(k)/[\rho(k) + \rho_h(k)]$$

$$n_\sigma(\Lambda) = \sigma(\Lambda)/[\sigma(\Lambda) + \sigma_h(\Lambda)]$$

Then, the continuity equations can be simplified

$$\partial_t n_\rho(k,x,t) + v_\rho(k,x,t)\partial_x n_\rho(k,x,t) = 0$$

$$\partial_t n_\sigma(\Lambda,x,t) + v_\sigma(\Lambda,x,t)\partial_x n_\sigma(\Lambda,x,t) = 0$$

Scaling invariant

- Scaling transformation

$$\begin{cases} x' = \lambda^{\alpha} x \\ t' = \lambda^{\beta} t \\ (n'_{\rho}, n'_{\sigma}) = \lambda^{\gamma}(n_{\rho}, n_{\sigma}) \end{cases}$$

- After this transformation, the continuity equations and $\xi = xt^{a}$ and $V(\xi) = (n_{\rho}, n_{\sigma})t^{b}$ should keep unchanged.

$$\begin{cases} \gamma = 0 \\ \alpha = \beta \\ a = -1, b = 0 \end{cases}$$

- n_{ρ} and n_{σ} are functions of ξ.

$$\xi = \frac{x}{t}, (n_{\rho}, n_{\sigma}) = V(\xi) = V(\frac{x}{t})$$

- The continuity equations become

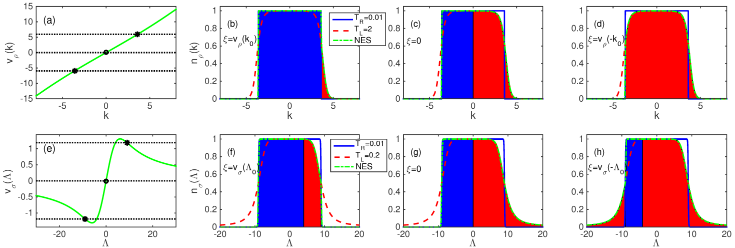
$$(v_\rho(k,\xi)-\xi)\partial_\xi n_\rho(k,\xi)=0$$
$$(v_\sigma(\Lambda,\xi)-\xi)\partial_\xi n_\sigma(\Lambda,\xi)=0$$

- Boundary conditions

$$\begin{cases} \lim_{\xi\to-\infty} n_\rho(k,\xi)=n^L_\rho(k), \lim_{\xi\to+\infty} n_\rho(k,\xi)=n^R_\rho(k) \\ \lim_{\xi\to-\infty} n_\sigma(\Lambda,\xi)=n^L_\sigma(k), \lim_{\xi\to+\infty} n_\sigma(\Lambda,\xi)=n^R_\sigma(k) \end{cases}$$

- Solutions

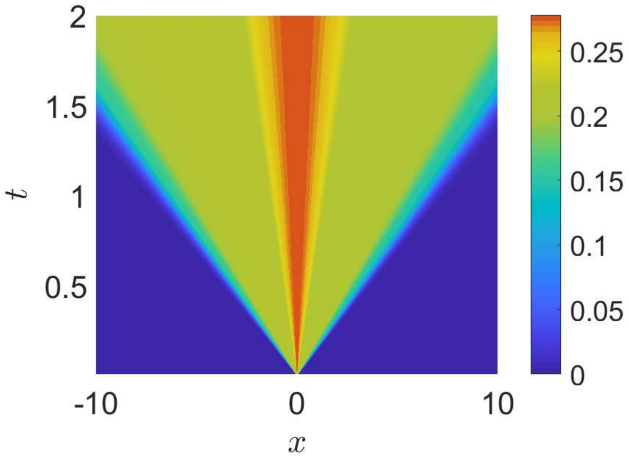
$$\begin{aligned} n_\rho(k,\xi) &= n^L_\rho(k)H(v_\rho(k,\xi)-\xi)+n^R_\rho(k)H(\xi-v_\rho(k,\xi)) \\ n_\sigma(\Lambda,\xi) &= n^L_\sigma(\Lambda)H(v_\sigma(\Lambda,\xi)-\xi)+n^R_\sigma(\Lambda)H(\xi-v_\sigma(\Lambda,\xi)) \end{aligned}$$



The sound velocities at zero temperature and the occupation number solutions of quasi-momentum (a-d) and rapidity (e-h) with $\mu_f = 12$, $\mu_b = 11$,$c = 10$. The Asterisks mark $k = 0, \pm k_0$(and/or $\Lambda = 0, \pm\Lambda_0$)

$$(m_\rho^*)^{-1} = \frac{\partial^2 \varepsilon_0(k)}{\partial p_\rho^{dr2}(k)}|_{k=k_0} = \frac{v_\rho^{0\prime}(k_0) v_\rho^0(k_0)}{\varepsilon_0{}'(k_0)}$$

$$\begin{aligned} n_\rho(k,\xi) &= n_\rho^L(k) H(v_\rho(k,\xi)-\xi) + n_\rho^R(k) H(\xi - v_\rho(k,\xi)) \\ n_\sigma(\Lambda,\xi) &= n_\sigma^L(\Lambda) H(v_\sigma(\Lambda,\xi)-\xi) + n_\sigma^R(\Lambda) H(\xi - v_\sigma(\Lambda,\xi)) \end{aligned}$$



The light-core diagram of the energy current. $T_L = 1$, $T_R = 0.5$, $\mu_b = 11$, $\mu_f = 12$, $c = 10$ which is in Bose-Fermi mixture phase.

Key observation: there exist steady states and transition regions

$$J^E = \frac{\pi^2}{12}\left(T_L^2 - T_R^2\right)\sum_{i=c,b} H\left(v_i - |\xi|\right)$$

- Dressed charges $f_q(k)$ and $g_q(\Lambda)$ are the charges carried by the excitations, and $f(k)$ and $g(\Lambda)$ are the charges carried by single particle.

$$
\begin{aligned}
f_q(k) =& f(k) + \int_{-\Lambda_0}^{\Lambda_0} a(k-\Lambda) g_q(\Lambda) d\Lambda \\
g_q(\Lambda) =& g(\Lambda) + \int_{-k_0}^{k_0} a(\Lambda - k) f_q(k) dk
\end{aligned}
$$

- Energy

$$
f(k) = k^2 - \mu_f, \quad g(\Lambda) = \mu_f - \mu_b
$$

- For total number of particles $f(k) = 1, \ g(\Lambda) = 0$
- For number of bosons $f(k) = 0, \ g(\Lambda) = 1$

- Densities

$$q(\xi) = \int_{-\infty}^{\infty} f(k)\rho(k,\xi)dk + \int_{-\infty}^{\infty} g(\Lambda)\sigma(\Lambda,\xi)d\Lambda$$

- Currents

$$j(\xi) = \int_{-\infty}^{\infty} f(k)v_\rho(k,\xi)\rho(k,\xi)dk + \int_{-\infty}^{\infty} g(\Lambda)v_\sigma(\Lambda,\xi)\sigma(\Lambda,\xi)d\Lambda$$

- Regime $\lim_{T\to 0}|\xi \pm v_\rho^0(k_0)| \neq 0$ and $\lim_{T\to 0}|\xi \pm v_\sigma^0(\Lambda_0)| \neq 0$.

$$
\begin{aligned}
q(\xi) = & q_0 + T^2\Omega_g(q,r)H(v_\sigma^0(\Lambda_0)-|\xi|) + T^2\Omega_g(q,1)H(-v_\sigma^0(\Lambda_0)-\xi) \\
& + r^2T^2\Omega_g(q,1)H(\xi - v_\sigma^0(\Lambda_0)) + T^2\Omega_f(q,r)H(v_\rho^0(k_0)-|\xi|) \\
& + T^2\Omega_f(q,1)H(-v_\rho^0(k_0)-\xi) + r^2T^2\Omega_f(q,1)H(\xi - v_\rho^0(k_0)) + O(T^3)
\end{aligned}
$$

- Definitions of $\Omega_g(q,r)$ and $\Omega_f(q,r)$

$$
\begin{aligned}
\Omega_g(q,r) = & \frac{\pi}{12\varphi_0'(\Lambda_0)v_\sigma^0(\Lambda_0)}\Big[g_q'(\Lambda_0) - r^2 g_q'(-\Lambda_0) \\
& - (g_q(\Lambda_0) + r^2 g_q(-\Lambda_0))\left(B(\Lambda_0) + \frac{v_\sigma^{0\,\prime}(\Lambda_0)}{v_\sigma^0(\Lambda_0)}\right)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\Omega_f(q,r) = & \frac{\pi}{12\varepsilon_0'(k_0)v_\rho^0(k_0)}\Big[f_q'(k_0) - r^2 f_q'(-k_0) \\
& - (f_q(k_0) + r^2 f_q(-k_0))\left(A(k_0) + \frac{v_\rho^{0\,\prime}(k_0)}{v_\rho^0(k_0)}\right)\Big]
\end{aligned}
$$

- Regime $\lim_{T\to 0}|\xi \pm v_\rho^0(k_0)| \neq 0$ and $\lim_{T\to 0}|\xi \pm v_\sigma^0(\Lambda_0)| \neq 0$.

$$\begin{aligned} j(\xi) = & T^2\Pi_g(q,r)H(v_\sigma^0(\Lambda_0)-|\xi|) + T^2\Pi_g(q,1)H(-v_\sigma^0(\Lambda_0)-\xi) \\ & + r^2T^2\Pi_g(q,1)H(\xi - v_\sigma^0(\Lambda_0)) + T^2\Pi_f(q,r)H(v_\rho^0(k_0)-|\xi|) \\ & + T^2\Pi_f(q,1)H(-v_\rho^0(k_0)-\xi) + r^2T^2\Pi_f(q,1)H(\xi - v_\rho^0(k_0)) \\ & + O(T^3) \end{aligned}$$

- Definitions of $\Omega_g(q,r)$ and $\Omega_f(q,r)$

$$\Pi_g(q,r) = \frac{\pi}{12\varphi_0'(\Lambda_0)}\Big[g_q'(\Lambda_0)+r^2g_q'(-\Lambda_0)-\big(g_q(\Lambda_0)-r^2g_q(-\Lambda_0)\big)B(\Lambda_0)\Big]$$

$$\Pi_f(q,r) = \frac{\pi}{12\varepsilon_0'(k_0)}\Big[f_q'(k_0) - r^2f_q'(-k_0) - \big(f_q(k_0) + r^2f_q(-k_0)\big)A(k_0)\Big]$$

The energy current indicates a ballistic transport of quasiparticles

$$J^E = \frac{\pi^2}{12}\left(T_L^2 - T_R^2\right)\sum_{i=c,b} H\left(v_i - |\xi|\right)$$

It is well established that a linear Luttinger liquid approximates the dynamics of a real critical Hamiltonian up to contributions that are irrelevant in the renormalization group sense. While this guarantees that such description gives the most relevant contribution to the large distance behavior of correlation functions on the ground state, one could wonder whether or not the irrelevant terms affect the characterization of low-temperature transport, i.e. $\epsilon_r(k) = v|k| + \frac{r}{2m*}|k|k + O(k^3)$.
(PRL 120, 176801 (2018)).

- The transition regime $\xi \pm v_\rho^0(k_0) \sim O(T)$

$$q(\xi) = q_0 + T\Omega_f^-(q,r) D_r\left(\varepsilon_0'(k_0)\frac{\xi - v_\rho^0(k_0)}{T v_\rho^{0\prime}(k_0)}\right) - T\Omega_f^+(q,r) D_r\left(\varepsilon_0'(k_0)\frac{\xi + v_\rho^0(k_0)}{T v_\rho^{0\prime}(k_0)}\right) + O(T^2)$$

$$j(\xi) = T\Pi_f^-(q,r) D_r\left(\varepsilon_0'(k_0)\frac{\xi - v_\rho^0(k_0)}{T v_\rho^{0\prime}(k_0)}\right) + T\Pi_f^+(q,r) D_r\left(\varepsilon_0'(k_0)\frac{\xi + v_\rho^0(k_0)}{T v_\rho^{0\prime}(k_0)}\right) + O(T^2)$$

- Definition of $D_r(z)$

$$D_r(z) = \ln(1 + e^z) - r\ln(1 + e^z)$$

- The transition regime $\xi \pm v_\sigma^0(\Lambda_0) \sim O(T)$

$$q(\xi) = q_0 + T\Omega_g^-(q,r)D_r\left(\varphi_0'(\Lambda_0)\frac{\xi - v_\sigma^0(\Lambda_0)}{T{v_\sigma^0}'(\Lambda_0)}\right) - T\Omega_g^+(q,r)D_r\left(\varphi_0'(\Lambda_0)\frac{\xi + v_\sigma^0(\Lambda_0)}{T{v_\sigma^0}'(\Lambda_0)}\right) + O(T^2)$$

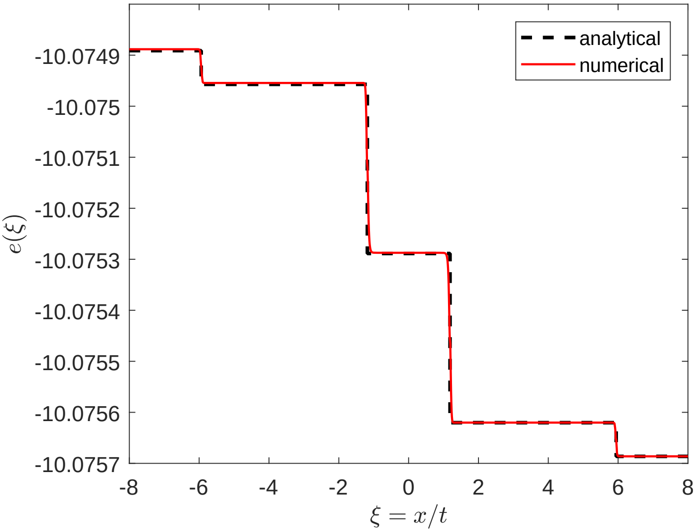
$$j(\xi) = T\Pi_g^-(q,r)D_r\left(\varphi_0'(\Lambda_0)\frac{\xi - v_\sigma^0(\Lambda_0)}{T{v_\sigma^0}'(\Lambda_0)}\right) + T\Pi_g^+(q,r)D_r\left(\varphi_0'(\Lambda_0)\frac{\xi + v_\sigma^0(\Lambda_0)}{T{v_\sigma^0}'(\Lambda_0)}\right) + O(T^2)$$

- Definitions of $\Omega_f^{\pm}(q,r)$ and$\Omega_g^{\pm}(q,r)$

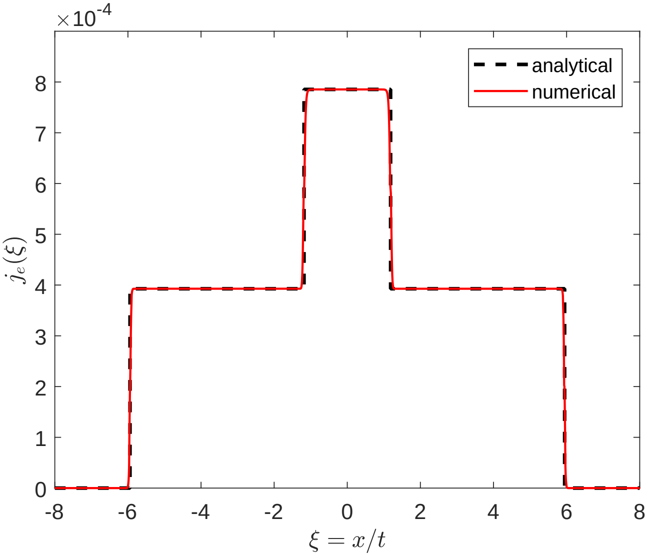
$$\Omega_f^{\pm}(q,r) = \frac{sign(v_\rho^{0\prime}(k_0))}{2\pi v_\rho^0(k_0))} f_q(\mp k_0)$$
$$\Omega_g^{\pm}(q,r) = \frac{sign(v_\sigma^{0\prime}(\Lambda_0))}{2\pi v_\sigma^0(\Lambda_0))} g_q(\mp \Lambda_0)$$

- Definitions of $\Pi_f^{\pm}(q,r)$ and $\Pi_g^{\pm}(q,r)$

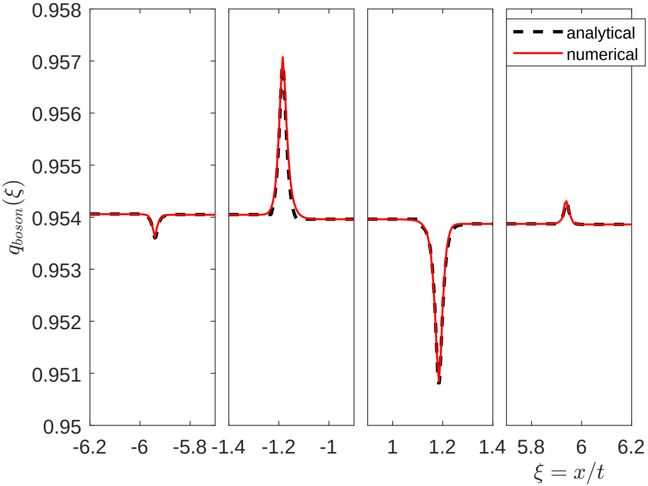
$$\Pi_f^{\pm}(q,r) = v_\rho^0(k_0))\Omega_f^{\pm}(q,r)$$
$$\Pi_g^{\pm}(q,r) = v_\sigma^0(\Lambda_0))\Omega_g^{\pm}(q,r)$$



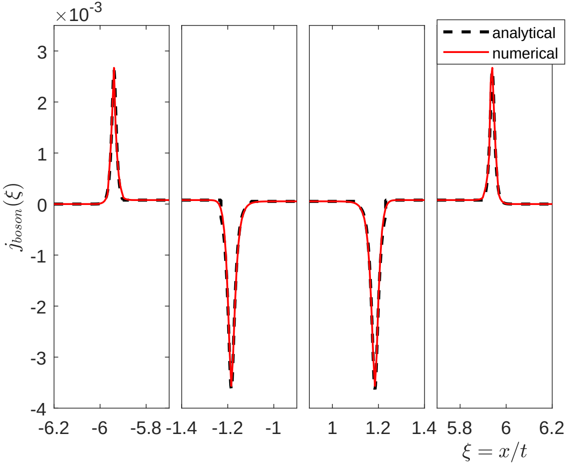
Profiles of energy density at low temperature. $T_L = 0.04$, $T_R = 0.01$, $\mu_b = 11$, $\mu_f = 12$, $c = 10$ which is in Bose-Fermi mixture phase.



Profiles of energy current at low temperature. $T_L = 0.04$, $T_R = 0.01$, $\mu_b = 11$, $\mu_f = 12$, $c = 10$ which is in Bose-Fermi mixture phase.

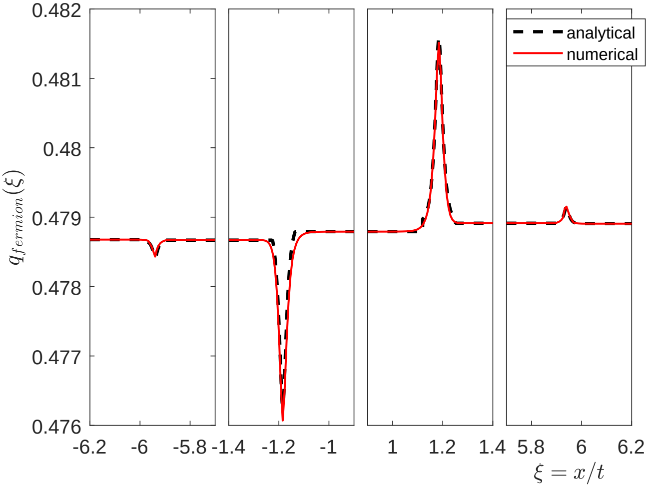


Profiles of Bose density at low temperature. $T_L = 0.04$, $T_R = 0.01$, $\mu_b = 11$, $\mu_f = 12$, $c = 10$ which is in Bose-Fermi mixture phase.

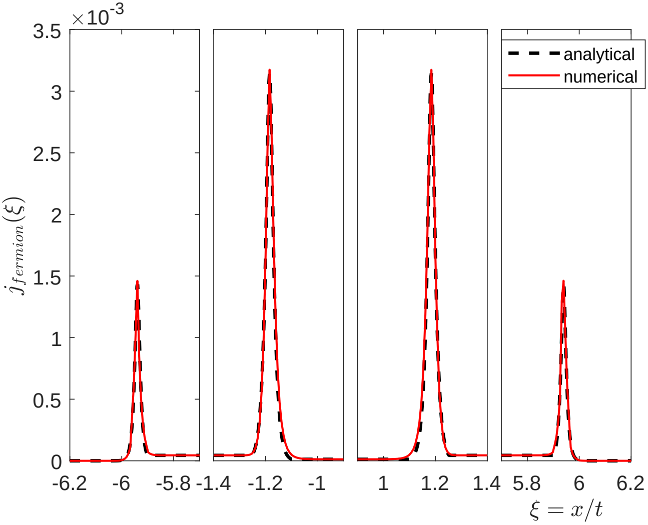


Profiles of Bose current at low temperature. $T_L = 0.04$, $T_R = 0.01$, $\mu_b = 11$, $\mu_f = 12$, $c = 10$ which is in Bose-Fermi mixture phase.

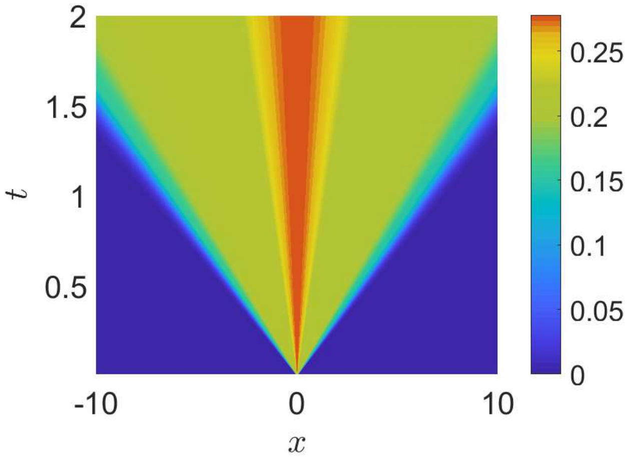
The backward currents of bosons due to the negative mass of effective excitations $\epsilon_r(k) = v|k| + \frac{r}{2m*}|k|k + O(k^3)$ and $p_r(k) = r|k|$.



Profiles of Fermi density at low temperature. $T_L = 0.04$, $T_R = 0.01$, $\mu_b = 11$, $\mu_f = 12$, $c = 10$ which is in Bose-Fermi mixture phase.



Profiles of Fermi current at low temperature. $T_L = 0.04$, $T_R = 0.01$, $\mu_b = 11$, $\mu_f = 12$, $c = 10$ which is in Bose-Fermi mixture phase.



The light-core diagram of the energy current. $T_L = 1$, $T_R = 0.5$, $\mu_b = 11$, $\mu_f = 12$, $c = 10$ which is in Bose-Fermi mixture phase.

Key observation: there exist steady states and transition regions

$$J^E = \frac{\pi^2}{12}\left(T_L^2 - T_R^2\right)\sum_{i=c,b} H\left(v_i - |\xi|\right)$$

Conclusion

I. Universal ballistic transports of integrable systems has been introduced, i.e. Lieb-Linger Bose gas and the Bose-Fermi mixtures.

II. A close connection to the Luttinger liquid theory has been discussed.

On-going research topics

- Emergent Hydrodynamics of quansi-1D ultracold atoms and hydrodynamic diffusion in integrable systems

$$\partial_t q_i(x,t) + \partial_x J_i(x,t) = 0$$
$$\langle J_i(x,t)\rangle = \mathcal{F}_i(x,t) + \sum_j \mathcal{F}_{i,j}(x,t)\partial_x\langle q_j(x,t)\rangle + \ldots$$