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Unique determination of a perfectly conducting ball by a finite number of electric far field data

Guanghui Hu, Xiaodong Liu, Bo Zhang *

LSEC and Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

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A B S T R A C T

In this paper,we prove uniqueness in determining a perfectly conducting ball in the inverse electromagnetic scattering problem by a finite number of electric far field patterns with a single incident direction and polarization. It is emphasized that we use only one electric far field pattern datum to uniquely determine the radius of a ball if it is centered at the origin with radius $R < \sqrt{2}/k$. Furthermore, if its center was not given as a prior information, three more measurement data must be added to uniquely determine its center. The main tool used here is some new results on zeros of spherical Bessel and spherical Neumann functions.

1. Introduction

The propagation of a time-harmonic electromagnetic wave (with the time variation of the form $e^{-i\omega t}$, $\omega > 0$) in a homogeneous, lossless, isotropic medium in $\mathbb{R}^3$ is modeled by the time-harmonic Maxwell equations:

$$\operatorname{curl} E - ikH = 0, \qquad \operatorname{curl} H + ikE = 0, \tag{1.1}$$

where $k = \omega\sqrt{\epsilon\mu}$ is the wavenumber given in terms of the wave frequency ω and the electric permittivity ε and the magnetic permeability μ of the medium. The scattering of a time-harmonic electromagnetic wave by a perfect conductor D in $\mathbb{R}^3$ leads to the following boundary condition:

$$\nu \times E = 0 \quad \text{on } \partial D. \tag{1.2}$$

The total fields E, H must satisfy (1.1) in $\mathbb{R}^3 \backslash \overline{D}$ and is decomposed as $E = E^i + E^s$, $H = H^i + H^s$, where E^i, H^i are the given incident fields and E^s, H^s are the unknown scattered fields which are required to satisfy the Silver–Müller radiation condition

$$\lim_{|x|\to\infty} \big(H^s(x) \times x - |x| E^s(x)\big) = 0 \tag{1.3}$$

uniformly with respect to all directions. Such a radiation condition ensures uniqueness of solutions to the exterior boundary value problem and leads to an asymptotic behavior of the form

$$E^s = \frac{e^{ik|x|}}{|x|}\left\{E_\infty\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|}\right)\right\}, \qquad H^s = \frac{e^{ik|x|}}{|x|}\left\{H_\infty\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|}\right)\right\} \tag{1.4}$$

* Corresponding author. Fax: +86 10 6254 1689.
E-mail addresses: huguanghui@amss.ac.cn (G. Hu), lxd230@163.com (X. Liu), b.zhang@amt.ac.cn (B. Zhang).

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as $|x| \to \infty$. Here, the vector fields E_∞ and H_∞, which are defined on the unit sphere Ω in $\mathbb{R}^3$, are known as the electric far field pattern and the magnetic far field pattern of the scattered fields, respectively. Throughout this paper, we write $E^s(\cdot; d, p, k)$, $H^s(\cdot; d, p, k)$ and $E_\infty(\cdot; d, p, k)$, $H_\infty(\cdot; d, p, k)$ to indicate the dependence on the direction d, the polarization p of the incident field and the wavenumber k. The inverse problem we are considering is to determine the shape of the scatterer D from the electric far field pattern $E_\infty(\cdot, d, p)$ for only one incident plane wave with the incident direction d and the polarization p. The question of uniqueness in the inverse scattering problem is of theoretical interest, and a positive answer is required in order to proceed to efficient numerical methods of solutions.

The first uniqueness result in inverse acoustic scattering was given by Schiffer [3,9]. In proving this uniqueness result he used properties of eigenvalues of the negative Laplacian with a Dirichlet boundary condition under the assumption that the far field patterns are completely known for an infinite number of incident plane vaves. A different method using singular sources was proposed in [6] to prove Schiffer's uniqueness result and a similar uniqueness result for the case of transmission conditions, which also requires an infinite number of incident waves. Given a priori information that the unknown scatterers lie inside a given ball, Colton and Sleeman [4] proved uniqueness for a finite number of incident plane waves. This result was later improved by using the Faber–Krahn inequality in [5]. In the case of a ball, Liu [10] established a uniqueness result for the Dirichlet boundary condition, whilst Yun [14] established a similar result for the Neumann boundary condition. In both [10] and [14], the far field pattern is assumed to be completely known for one incident plane wave. See also [2,11,13] for the case of polyhedral obstacles with one incident plane wave.

For the case of inverse electromagnetic scattering similar uniqueness results have been obtained (see, e.g. [3,7]). For example, a general obstacle and its boundary conditions can be uniquely determined by electric far field patterns for an infinite number of incident plane waves, and a ball can be uniquely determined for only one incident plane wave. However, the following questions are still unknown for the case of electromagnetic scattering:

1. Given a priori information on the size of the scatterer, whether or not the scatterer can be uniquely determined by a finite number of far field patterns depending on the diameter of the scatterer.
2. Whether or not a general scatterer can be uniquely determined by the far field pattern for only one incident plane wave.

The second question is an open problem for both the acoustic and the electromagnetic scattering problems. For the first question, it should be remarked that it was proved in [12] that the shape of a sound-soft/sound-hard ball in $\mathbb{R}^3$ or a sound-soft/sound-hard disk in $\mathbb{R}^2$ is uniquely determined by a single far field datum measured at one fixed observation for a single incident plane wave.

In this paper we consider the uniqueness question of determining a perfectly conducting ball in $\mathbb{R}^3$ in inverse electromagnetic scattering problems by a finite number of electric far field patterns with a single incident direction and polarization. Precisely, we prove, in Section 3, that if a perfectly conducting ball of radius R and centered at the origin satisfies that $kR < \sqrt{2}$, then the ball can be uniquely determined by one electric far field datum $E_\infty(d; d, p, k)$ measured at the observation d (Theorem 3.1), which extends the result of [12] to the case of a perfectly conducting ball in inverse electromagnetic scattering, and in Section 4, that if the center of the ball is not given as a prior information then four electric far field pattern measurements are sufficient to uniquely determine the radius and the center of the ball (Theorem 4.1). It seems that four data are the least in the three-dimensional case since, in this case, there are totally four unknowns (the radius and three components of the center of the ball) to be determined.

A main tool used in the proof of Theorem 3.1 is some new results on zeros of spherical Bessel and spherical Neumann functions (Theorem 2.8), which are established in Section 2. Theorem 4.1 is proved using Theorem 3.1 in conjunction with a translation relation between the electric far field patterns for perfectly conducting balls. This idea can also be applied in the case of inverse acoustic scattering by a disk (in 2D) or a ball (in 3D) to determine both the radius and the center of the disk (or the ball) by using three (or four) far field patterns with a single incident direction, which generalizes the results in [12] for the case when the center is fixed at the origin (see Remark 4.2).

2. Zeros of spherical Bessel and spherical Neumann functions

Denote by $j_n(t)$ and $y_n(t)$ the spherical Bessel and Neumann functions of order n, respectively. Both functions satisfy the spherical Bessel differential equation

$$t^2 f''(t) + 2t f'(t) + \left[t^2 - n(n+1)\right] f(t) = 0, \tag{2.1}$$

and the Wronskian equality

$$j_n(t) y_n'(t) - j_n'(t) y_n(t) = \frac{1}{t^2}. \tag{2.2}$$

See [1,3,12] for more information on Bessel and Neumann functions. The following results can be found in [12].

Lemma 2.1. *For the spherical Bessel functions and their derivatives, we have that for each* $n = 0, 1, 2, \ldots,$

$$j_n(t) = \frac{t^n}{(2n+1)!}\left[1 + O\left(t^2\right)\right], \tag{2.3}$$

$$j_0'(t) = -\frac{t}{3}\left[1 + O\left(t^2\right)\right], \tag{2.4}$$

$$j_n'(t) = \frac{nt^{n-1}}{(2n+1)!}\left[1 + O\left(t^2\right)\right] \tag{2.5}$$

as $t \to +0$, whereas for the spherical Neumann functions and their derivatives, we have that for each $n = 0, 1, 2, \ldots,$

$$y_n(t) = -\frac{(2n)!}{2^n n!}\frac{1}{t^{n+1}}\left[1 + O\left(t^2\right)\right], \tag{2.6}$$

$$y_n'(t) = \frac{(2n)!}{2^n n!}\frac{n+1}{t^{n+2}}\left[1 + O\left(t^2\right)\right] \tag{2.7}$$

as $t \to +0$. Thus, for sufficiently small $t > 0$ and for each non-negative integer n, $j_n(t)$ and $j_n'(t)$ are positive with the only exception that $j_0'(t)$ is negative near the origin, whereas $y_n(t)$ is negative and $y_n'(t)$ is positive.

Now, denote by $\xi_{n,s}$, $\eta_{n,s}$, $\xi_{n,s}'$ and $\eta_{n,s}'$ the sth positive zeros of $j_n(t)$, $y_n(t)$, $j_n'(t)$ and $y_n'(t)$, respectively, for $n \in \mathbb{N}$. Then we have the following results which can be found in [12].

Lemma 2.2. *For $n \in \mathbb{N} \cup \{0\}$ the positive zeros of $j_n(t)$ are interlaced with those of $j_n'(t)$ in the following way:*

$$\sqrt{n(n+1)} \leqslant \xi_{n,1}' < \xi_{n,1} < \xi_{n,2}' < \xi_{n,2} < \xi_{n,3}' < \cdots,$$

where the equal sign can only be possible for the case $n = 0$ and $\xi_{0,1}'$ is defined to be zero. For $n \in \mathbb{N} \cup \{0\}$, the positive zeros of $y_n(t)$ are interlaced with those of $y_n'(t)$ as follows:

$$\sqrt{n(n+1)} < n + \frac{1}{2} < \eta_{n,1} < \eta_{n,1}' < \eta_{n,2} < \eta_{n,2}' < \eta_{n,3} < \cdots.$$

Corollary 2.3. *Let $n \in \mathbb{N} \cup \{0\}$. Then for each $s \in \mathbb{N}$ the sequences $\{\xi_{n,s}\}_{n=0}^{\infty}$ and $\{\eta_{n,s}\}_{n=0}^{\infty}$ are strictly monotonic increasing, that is*

$$\xi_{0,s} < \xi_{1,s} < \xi_{2,s} < \cdots < \xi_{n,s} < \xi_{n+1,s} < \cdots,$$
$$\eta_{0,s} < \eta_{1,s} < \eta_{2,s} < \cdots < \eta_{n,s} < \eta_{n+1,s} < \cdots.$$

Lemma 2.4. *For $n \in \mathbb{N} \cup \{0\}$, the positive zeros of $j_n'(t)$ are interlaced with those of $j_{n+1}'(t)$:*

$$\xi_{n,1}' < \xi_{n+1,1}' < \xi_{n,2}' < \xi_{n+1,2}' < \xi_{n,3}' < \cdots,$$

and the positive zeros of $y_n'(t)$ are interlaced with those of $y_{n+1}'(t)$:

$$\eta_{n,1}' < \eta_{n+1,1}' < \eta_{n,2}' < \eta_{n+1,2}' < \eta_{n,3}' < \cdots.$$

Corollary 2.5. *Let $n \in \mathbb{N} \cup \{0\}$. Then for each $s \in \mathbb{N}$ the sequences $\{\xi_{n,s}'\}_{n=0}^{\infty}$ and $\{\eta_{n,s}'\}_{n=0}^{\infty}$ are strictly monotonic increasing, that is*

$$\xi_{0,s}' < \xi_{1,s}' < \xi_{2,s}' < \cdots < \xi_{n,s}' < \xi_{n+1,s}' < \cdots,$$
$$\eta_{0,s}' < \eta_{1,s}' < \eta_{2,s}' < \cdots < \eta_{n,s}' < \eta_{n+1,s}' < \cdots.$$

Concerning the zeros of the cylinder functions $C_\nu(t) := \alpha J_\nu(t) + \beta Y_\nu(t)$, where α, β are real constants, t is positive and $J_\nu(t)$, $Y_\nu(t)$ are Bessel and Neumann functions of order ν defined for $t > 0$, respectively, we have the following result on their interlacing character due to Dixon.

Lemma 2.6. *Let a, b, c, d be constants such that $ad \neq bc$. Then the positive zeros of $aC_\nu(t) + btC_\nu'(t)$ that are larger than ν with $\nu \geqslant 0$ are interlaced with those of $cC_\nu(t) + dt\,C_\nu'(t)$ that are larger than ν. Moreover, all these zeros are not repeated.*

The above lemmas and corollaries can be found in [12]. The following corollary can be easily derived from Lemma 2.6.

Corollary 2.7. *Let a, b, c, d be constants such that $ad \neq bc$. Then the positive zeros of $ay_n(t) + bty_n'(t)$ that are larger than n are interlaced with those of $cy_n(t) + d\,ty_n'(t)$ that are larger than n. Moreover, all these zeros are not repeated.*

Theorem 2.8. *For the spherical Bessel function $j_n(t)$ and the spherical Neumann function $y_n(t)$, we have that for all $n \in \mathbb{N}$,*

$$j_n(t)y_n(t) < 0 \quad \text{for } t \in (0, 2.798386),$$
$$j_n(t) + tj_n'(t) > 0 \quad \text{for } t \in (0, 2.08157598),$$
$$y_n(t) + ty_n'(t) > 0 \quad \text{for } t \in (0, \sqrt{2}).$$

Proof. By Lemma 2.1, it is known that for sufficiently small $t > 0$ $j_n(t)$ is positive and $y_n(t)$ is negative. Thus for all $n \in \mathbb{N}$,

$$j_n(t) > 0 \quad \text{for } t \in (0, 4.493409),$$
$$y_n(t) < 0 \quad \text{for } t \in (0, 2.79836).$$

Since, by Corollary 2.3, $\xi_{1,1} = 4.493409$ is the smallest positive zero of $j_n(t)$ and $\eta_{1,1} = 2.798386$ is the smallest positive zero of $y_n(t)$ (see [1]), then

$$j_n(t)y_n(t) < 0 \quad \text{for } t \in (0, 2.798386)$$

for all $n \in \mathbb{N}$.

By Lemma 2.2 and Corollary 2.5, the smallest positive zeros of $j_n(t)$ and $j_n'(t)$ are given respectively by (see [1]):

$$\xi_{1,1} = 4.493409, \qquad \xi_{1,1}' = 2.08157598.$$

Thus we deduce from Lemma 2.1 that

$$j_n(t) + tj_n'(t) > 0 \quad \text{for } t \in (0, 2.08157598)$$

for all $n \in \mathbb{N}$.

By (2.6) and (2.7) it can be seen that $y_n(t) + ty_n'(t)$ is positive for sufficiently small $t > 0$. Let l_{n1} and l_{n2} be the first and second positive zeros of $y_n(t) + ty_n'(t)$. We may claim that $l_{n1} > \eta_{1,1}'$. In fact, if this were not true, that is $l_{n1} < \eta_{1,1}'$, then by Corollary 2.7 we would conclude that

$$y_n(t) + ty_n'(t) < 0 \quad \text{for } t \in (l_{n1}, l_{n2}) \tag{2.8}$$

for all $n \in \mathbb{N}$. Thus, by Lemmas 2.1 and 2.2, it is easy to see that

$$y_n(\eta_{1,1}') + ty_n'(\eta_{1,1}') > 0.$$

This, together with (2.8), implies that

$$\eta_{1,1}' > l_{n2}. \tag{2.9}$$

Now, by Corollary 2.7 with $a = 1$, $b = 1$, $c = 0$, $d = 1$, we deduce that the positive zeros of $y_n'(t)$ are interlaced with those of $y_n(t) + ty_n'(t)$. This contradicts (2.9). The claim is thus proved, so we conclude that

$$y_n(t) + ty_n'(t) > 0 \quad \text{for } t \in (0, \eta_{1,1}')$$

for all $n \in \mathbb{N}$. Since, by Lemma 2.2, $\eta_{1,1}' > \sqrt{2}$, we then have

$$y_n(t) + ty_n'(t) > 0 \quad \text{for } t \in (0, \sqrt{2})$$

for all $n \in \mathbb{N}$. The theorem is thus proved. □

3. Uniqueness for perfectly conducting balls centered at the origin

Theorem 3.1. *Given an incident direction $d \in S^2$, a polarization p $(p \perp d)$ and a wavenumber $k > 0$, let the incident plane wave be $E^i = pe^{ikx\cdot d}$. If a perfectly conducting ball of radius R and centered at the origin satisfies that $kR < \sqrt{2}$, then the ball is uniquely determined by one electric far field data $E_\infty(d; d, p, k, R)$.*

To prove the theorem we need the following lemmas of which the first one can be found in [3, Theorem 6.23].

Lemma 3.2. *For the orthonormal system Y_n^m, $m = -n, \dots, n$, of spherical harmonics of order $n > 0$, the vector spherical harmonics on the unit sphere Ω*

$$U_n^m = \frac{1}{\sqrt{n(n+1)}} \operatorname{Grad} Y_n^m, \qquad V_n^m = \hat{x} \times U_n^m$$

for $m = -n, \dots, n$, $n \in \mathbb{N}$ form a complete orthonormal system in $T^2(\Omega)$, where $\hat{x} = x/|x|$,

$$T^2(\Omega) := \{a : \Omega \to \mathbb{C}^3 \mid a \in L^2(\Omega),\ a \cdot \hat{x} = 0\},$$

and Grad ϕ *is the surface gradient of a continuously differentiable function ϕ on Ω defined by*

$$\operatorname{Grad}\phi = \frac{\partial \phi}{\partial \theta}\hat{\theta} + \frac{1}{\sin\theta}\frac{\partial \phi}{\partial \varphi}\hat{\varphi}.$$

Here $\hat{\theta}$, $\hat{\varphi}$ are the unit vectors in the directions of the spherical coordinates (θ, φ), respectively.

For $x \in \mathbb{R}^3 \backslash \{0\}$ write

$$M_n^m(x) = \operatorname{curl}\{x j_n(k|x|) Y_n^m(\hat{x})\}, \qquad N_n^m(x) = \operatorname{curl}\{x h_n^{(1)}(k|x|) Y_n^m(\hat{x})\}.$$

Then by a direct calculation we have

$$M_n^m(x) = j_n(k|x|) \operatorname{Grad} Y_n^m(\hat{x}) \times \hat{x}, \tag{3.1}$$

$$N_n^m(x) = h_n^{(1)}(k|x|) \operatorname{Grad} Y_n^m(\hat{x}) \times \hat{x}, \tag{3.2}$$

and

$$\hat{x} \times \operatorname{curl} M_n^m(x) = \frac{1}{|x|}\{j_n(k|x|) + k|x| j_n'(k|x|)\}\hat{x} \times \operatorname{Grad} Y_n^m(\hat{x}), \tag{3.3}$$

$$\hat{x} \times \operatorname{curl} N_n^m(x) = \frac{1}{|x|}\{h_n^{(1)}(k|x|) + k|x| h_n^{(1)\prime}(k|x|)\}\hat{x} \times \operatorname{Grad} Y_n^m(\hat{x}). \tag{3.4}$$

Lemma 3.3. *Let the incident plane wave E^i be given as in Theorem* 3.1 *and let $B(0, R)$ be a perfectly conducting ball. Then the scattered field E^s to the problem* (1.1)–(1.2) *has the following representation*:

$$E^s(x) = \sum_{n=1}^{+\infty} \sum_{m=-n}^{n} \left[a_n^m N_n^m(x) + b_n^m \operatorname{curl} N_n^m(x)\right], \quad |x| > R,$$

where

$$a_n^m = -\frac{i^n 4\pi R^2 j_n(kR)}{n(n+1) h_n^{(1)}(kR)} p \cdot \operatorname{Grad} \overline{Y_n^m(d)} \times d,$$

$$b_n^m = -\frac{i^{n-1} 4\pi R^2 [j_n(kR) + kR j_n'(kR)]}{kn(n+1)[h_n^{(1)}(kR) + kR h_n^{(1)\prime}(kR)]} p \cdot \operatorname{Grad} \overline{Y_n^m(d)}$$

and $i = \sqrt{-1}$.

Proof. By [3, Theorem 6.25] the scattered field E^s can be written as

$$E^s(x) = \sum_{n=1}^{+\infty} \sum_{m=-n}^{n} \left[a_n^m N_n^m(x) + b_n^m \operatorname{curl} N_n^m(x)\right], \quad |x| > R,$$

where the series (together with its derivatives) converges uniformly on compact subsets of $|x| > R$. Thus, on the sphere $|x| = R$ we have

$$\hat{x} \times E^s(x) = \sum_{n=1}^{+\infty} \sum_{m=-n}^{n} \left[a_n^m h_n^{(1)}(k|x|) \operatorname{Grad} Y_n^m(\hat{x}) + b_n^m \frac{1}{|x|}\{h_n^{(1)}(k|x|) + k|x| h_n^{(1)\prime}(k|x|)\}\hat{x} \times \operatorname{Grad} Y_n^m(\hat{x})\right]. \tag{3.5}$$

On the other hand, by Lemma 3.2 we have

$$\hat{x} \times p e^{ikx\cdot d} = \sum_{n=1}^{+\infty} \sum_{m=-n}^{n} \left[A_n^m \operatorname{Grad} Y_n^m(\hat{x}) + B_n^m \hat{x} \times \operatorname{Grad} Y_n^m(\hat{x})\right]. \tag{3.6}$$

From the boundary condition (1.2) it follows that $\hat{x} \times E^s = -\hat{x} \times p e^{ikx\cdot d}$. This together with (3.5) and (3.6) implies that

$$a_n^m = -\frac{A_n^m}{h_n^{(1)}(kR)}, \qquad b_n^m = -\frac{B_n^m R}{h_n^{(1)}(kR) + kR h_n^{(1)\prime}(kR)}. \tag{3.7}$$

We now compute A_n^m and B_n^m. By (3.6) and Lemma 3.2, it is seen that

$$A_n^m = \frac{1}{n(n+1)} \int_{\Omega} \hat{x} \times p e^{ikx\cdot d} \cdot \operatorname{Grad} \overline{Y_n^m(\hat{x})}\, ds, \tag{3.8}$$

$$B_n^m = \frac{1}{n(n+1)} \int_{\Omega} \hat{x} \times p e^{ikx\cdot d} \cdot \hat{x} \times \operatorname{Grad} \overline{Y_n^m(\hat{x})}\, ds = \frac{1}{n(n+1)} \int_{\Omega} e^{ikx\cdot d} p \cdot \operatorname{Grad} \overline{Y_n^m(\hat{x})}\, ds. \tag{3.9}$$

By [3, Theorem 6.24], the pair $E(x) = M_n^m(x)$, $H(x) = \frac{1}{ik} \operatorname{curl} M_n^m(x)$ is an entire solution to the Maxwell equations (1.1) in $\mathbb{R}^3$, and the pair $E(x) = N_n^m(x)$, $H(x) = \frac{1}{ik} \operatorname{curl} N_n^m(x)$ is a solution to the Maxwell equations (1.1) in $\mathbb{R}^3 \backslash \{0\}$ satisfying the Silver–Müller radiation condition (1.3). Thus, by the well-known Straton–Chu formula (see [3, Theorems 6.2 and 6.6]) together with the aid of (3.1)–(3.4) we obtain that for $x \in \mathbb{R}^3$ with $|x| = R$,

$$\begin{aligned} N_n^m(x) &= h_n^{(1)}(kR) \operatorname{curl}_x \int_{|y|=R} \operatorname{Grad} Y_n^m(\hat{y}) \Phi(x,y)\, ds(y) \\ &\quad + \frac{h_n^{(1)}(kR) + kR h_n^{(1)\prime}(kR)}{k^2 R} \operatorname{curl}_x \operatorname{curl}_x \int_{|y|=R} \hat{y} \times \operatorname{Grad} Y_n^m(\hat{y}) \Phi(x,y)\, ds(y), \end{aligned} \tag{3.10}$$

$$\begin{aligned} 0 &= j_n(kR) \operatorname{curl}_x \int_{|y|=R} \operatorname{Grad} Y_n^m(\hat{y}) \Phi(x,y)\, ds(y) \\ &\quad + \frac{j_n(kR) + kR j_n'(kR)}{k^2 R} \operatorname{curl}_x \operatorname{curl}_x \int_{|y|=R} \hat{y} \times \operatorname{Grad} Y_n^m(\hat{y}) \Phi(x,y)\, ds(y), \end{aligned} \tag{3.11}$$

where

$$\Phi(x,y) = \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|} \quad (x \neq y)$$

is the fundamental solution to the Helmholtz equation. By subtracting (3.11) multiplied with $h_n^{(1)}(kR)$ from (3.10) multiplied with $j_n(kR)$ and using the Wronskian equality (2.2), we obtain that

$$j_n(kR) N_n^m(x) = \frac{i}{k^3 R^2} \operatorname{curl}_x \operatorname{curl}_x \int_{|y|=R} \hat{y} \times \operatorname{Grad} Y_n^m(\hat{y}) \Phi(x,y)\, ds(y) \tag{3.12}$$

for $|x| > R$. By using (3.2) and the identity

$$p \cdot \operatorname{curl}_x \operatorname{curl}_x \big[C(y)\Phi(x,y)\big] = C(y) \cdot \operatorname{curl}_x \operatorname{curl}_x \big[p\Phi(x,y)\big],$$

it follows from (3.12) that

$$\int_{|y|=R} \hat{y} \times \operatorname{Grad} Y_n^m(\hat{y}) \cdot \operatorname{curl}_x \operatorname{curl}_x \big[p\Phi(x,y)\big]\, ds(y) = -ik^3 R^2 j_n(kR) h_n^{(1)}\big(k|x|\big) p \cdot \operatorname{Grad} Y_n^m(\hat{x}) \times \hat{x}$$

for $|x| > R$.

Now, by the following asymptotic behavior

$$\operatorname{curl}_x \operatorname{curl}_x \left(p \frac{e^{ik|x-y|}}{|x-y|} \right) = k^2 \frac{e^{ik|x|}}{|x|} \left\{ e^{-ik\hat{x}\cdot y} \hat{x} \times (p \times \hat{x}) + O\left(\frac{|p|}{|x|} \right) \right\}, \quad |x| \to \infty,$$

$$h_n^{(1)}(t) = \frac{1}{t} e^{i(t - \frac{n}{2}\pi - \frac{\pi}{2})} \left\{ 1 + O\left(\frac{1}{t} \right) \right\}, \quad t \to +\infty,$$

we see that

$$\int_{|y|=R} e^{ik\hat{x}\cdot y} (\hat{x} \times p \times \hat{x}) \cdot \hat{y} \times \operatorname{Grad} \overline{Y_n^m(\hat{y})}\, ds(y) = -i^n 4\pi R^2 j_n(kR) p \cdot \operatorname{Grad} \overline{Y_n^m(\hat{x})} \times \hat{x}.$$

It thus follows that

$$\int_{|x|=R} e^{ikx\cdot d} p \cdot \hat{x} \times \operatorname{Grad} \overline{Y_n^m(\hat{x})}\, ds(x) = -i^n 4\pi R^2 j_n(kR) p \cdot \operatorname{Grad} \overline{Y_n^m(d)} \times d.$$

This together with (3.8) implies that

$$A_n^m = \frac{i^n 4\pi R^2 j_n(kR)}{n(n+1)} p \cdot \operatorname{Grad} \overline{Y_n^m(d)} \times d.$$

To compute B_n^m, we first derive from (3.10) and (3.11) that

$$\left[j_n(kR) + kR j_n'(kR)\right] N_n^m(x) = -\frac{i}{kR} \operatorname{curl}_x \int\limits_{|y|=R} \operatorname{Grad} Y_n^m(\hat{y}) \Phi(x, y)\, ds(y),$$

where use has been made of the Wronskian equality (2.2). By (3.2), the asymptotic behavior of the spherical Hankel function and the equality

$$\nabla_x \left(\frac{e^{ik|x-y|}}{|x-y|}\right) = ik \frac{e^{ik|x|}}{|x|} \left\{ e^{-ik\hat{x}\cdot y} \hat{x} + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \to \infty,$$

it follows that for $|x| > R$,

$$\int\limits_{|y|=R} e^{-ik\hat{x}\cdot y} \hat{x} \times \operatorname{Grad} Y_n^m(\hat{y})\, ds(y) = \frac{4\pi R[j_n(kR) + kR j_n'(kR)](-i)^{n+1}}{k} \operatorname{Grad} Y_n^m(\hat{x}) \times \hat{x}.$$

Thus,

$$\int\limits_{|y|=R} e^{ik\hat{x}\cdot y} \operatorname{Grad} \overline{Y_n^m(\hat{y})}\, ds(y) = -\frac{4\pi R[j_n(kR) + kR j_n'(kR)] i^{n+1}}{k} \operatorname{Grad} \overline{Y_n^m(\hat{x})} + C(\hat{x})\hat{x}$$

for some function $C(\hat{x})$. This, together with (3.9) and the fact that $p \cdot d = 0$, implies that

$$B_n^m = \frac{i^{n-1} 4\pi R[j_n(kR) + kR j_n'(kR)]}{kn(n+1)} p \cdot \operatorname{Grad} \overline{Y_n^m(d)}.$$

Combining (3.5) and (3.7) completes the proof of the lemma. □

Remark 3.4. By [3, Theorem 6.26] and Lemma 3.3, the electric far field pattern E_∞ of the scattered field E^s is given by

$$\begin{aligned} E_\infty(\hat{x}; d, p, k, R) &= \frac{1}{k} \sum_{n=1}^{+\infty} \frac{1}{i^{n+1}} \sum_{m=-n}^{n} \left\{ ik b_n^m \operatorname{Grad} Y_n^m(\hat{x}) - a_n^m \hat{x} \times \operatorname{Grad} Y_n^m(\hat{x}) \right\} \\ &= \sum_{n=1}^{+\infty} \frac{4\pi i R^2}{kn(n+1)} \left\{ \frac{j_n(kR) + kR j_n'(kR)}{h_n^{(1)}(kR) + kR h_n^{(1)\prime}(kR)} \sum_{m=-n}^{n} \left[p \cdot \operatorname{Grad} \overline{Y_n^m(d)}\right] \operatorname{Grad} Y_n^m(\hat{x}) \right. \\ &\quad \left. + \frac{j_n(kR)}{h_n^{(1)}(kR)} \sum_{m=-n}^{n} \left[p \cdot \operatorname{Grad} \overline{Y_n^m(d)} \times d\right] \operatorname{Grad} Y_n^m(\hat{x}) \times \hat{x} \right\}. \end{aligned} \tag{3.13}$$

By using [3, Theorem 2.8] it can be easily shown that the above electric far field pattern satisfies the well-known theorem of Karp (see [3, p. 197]):

$$E_\infty(Q\hat{x}; Qd, Qp, k, R) = Q E_\infty(\hat{x}; d, p, k, R) \tag{3.14}$$

for all $\hat{x}, d \in \Omega$, all $p \in \mathbb{R}^3$ and all rotations Q, i.e., for all real orthogonal matrices Q with $\det Q = 1$.

In order to prove our main theorem, we need the following result.

Lemma 3.5. *For $n = 1, 2, \ldots,$ we have*

$$\sum_{m=-n}^{n} \left[p \cdot \operatorname{Grad} \overline{Y_n^m(d)}\right] \operatorname{Grad} Y_n^m(d) = \frac{1}{8\pi} n(n+1)(2n+1) C(d, p),$$

$$\sum_{m=-n}^{n} \left[p \cdot \operatorname{Grad} \overline{Y_n^m(d)} \times d\right] \operatorname{Grad} Y_n^m(d) \times d = \frac{1}{8\pi} n(n+1)(2n+1) C(d, p),$$

where

$$C(d, p) = (\hat{\theta} \cdot p)\hat{\theta} + (\hat{\varphi} \cdot p)\hat{\varphi}$$

is a constant vector depending on d and p and $\hat{\theta}, \hat{\varphi}$ are the unit vectors in the directions of the spherical coordinates (θ, φ), respectively.

Proof. From the additional theorem of the spherical harmonic functions:

$$\sum_{m=-n}^{n} \overline{Y_n^m(\hat{x})} Y_n^m(\hat{y}) = \frac{2n+1}{4\pi} P_n(\cos\omega)$$

where ω denotes the angle between $\hat{x}$ and $\hat{y}$ and P_n denote the Legendre polynomial, it follows that

$$\sum_{m=-n}^{n} \big[p \cdot \mathrm{Grad}\, \overline{Y_n^m(\hat{x})}\big] \mathrm{Grad}\, Y_n^m(\hat{y}) = \frac{2n+1}{4\pi} \mathrm{Grad}_{\hat{y}}\big[p \cdot \mathrm{Grad}_{\hat{x}} P_n(\cos\omega)\big].$$

Now, fix $d \in S^2$ and $p \in \mathbb{R}^3$ with $p \perp d$. We may choose a proper coordinate system $ox_1x_2x_3$ such that the spherical coordinate representation $d = d(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ satisfies $\theta \neq 0$. Let $\hat{x} = \hat{x}(\theta_1,\varphi_1)$, $\hat{y} = \hat{y}(\theta_2,\varphi_2)$ lie in a small coordinate neighborhood of d so that

$$\cos\omega = \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2) + \cos\theta_1 \cos\theta_2 =: f(\theta_1,\theta_2,\varphi_1,\varphi_2).$$

Then

$$\mathrm{Grad}_{\hat{x}} P_n(\cos\omega) = P_n' f_{\theta_1} \hat{\theta}_1 + \frac{1}{\sin\theta_1} P_n' f_{\varphi_1} \hat{\varphi}_1,$$

where

$$P_n' = \frac{dP_n(t)}{dt}, \qquad f_{\theta_i} = \frac{\partial f}{\partial \theta_i}, \qquad f_{\varphi_i} = \frac{\partial f}{\partial \varphi_i}, \quad i = 1, 2.$$

Thus, we have

$$\begin{aligned} \mathrm{Grad}_{\hat{y}}\big[p \cdot \mathrm{Grad}_{\hat{x}} P_n(\cos\omega)\big] = {} & \big(P_n'' f_{\theta_2} f_{\theta_1} + P_n' f_{\theta_1\theta_2}\big)(\hat{\theta}_1 \cdot p)\hat{\theta}_2 + \big(P_n'' f_{\theta_2} f_{\varphi_1} + P_n' f_{\varphi_1\theta_2}\big)\frac{1}{\sin\theta_1}(\hat{\varphi}_1 \cdot p)\hat{\theta}_2 \\ & + \big(P_n'' f_{\varphi_2} f_{\theta_1} + P_n' f_{\theta_1\varphi_2}\big)\frac{1}{\sin\theta_2}(\hat{\theta}_1 \cdot p)\hat{\varphi}_2 + \big(P_n'' f_{\varphi_2} f_{\varphi_1} + P_n' f_{\varphi_1\varphi_2}\big)\frac{1}{\sin\theta_2}\frac{1}{\sin\theta_1}(\hat{\varphi}_1 \cdot p)\hat{\varphi}_2, \end{aligned}$$

where $P_n'' = d^2P_n(t)/dt^2$, $g_{\varphi\theta} = \partial^2 g/\partial\varphi\partial\theta$ for a function $g(\varphi,\theta)$. Let $\hat{x} = \hat{y} = d(\theta,\varphi)$ in the above equation, that is $\theta_i = \theta$, $\varphi_i = \varphi$ $(i = 1, 2)$. Then $\omega = 0$ and

$$f_{\theta_i} = f_{\varphi_i} = f_{\varphi_i\theta_j} = 0, \qquad f_{\theta_i\theta_j} = 1, \qquad f_{\varphi_i\varphi_j} = \sin^2\theta, \quad i, j = 1, 2, \ i \neq j,$$

so that

$$\mathrm{Grad}_{\hat{y}}\big[p \cdot \mathrm{Grad}_{\hat{x}} P_n(\cos\theta)\big]\big|_{\hat{x}=\hat{y}=d} = P_n'(1)\big[(\hat{\theta} \cdot p)\hat{\theta} + (\hat{\varphi} \cdot p)\hat{\varphi}\big].$$

Since $P_n(t)$ satisfies the Legendre differential equation

$$\big(1 - t^2\big)P_n'' - 2tP_n'(t) + n(n+1)P_n(t) = 0, \quad n = 0, 1, 2, \ldots, \ -1 \leqslant t \leqslant 1,$$

we have

$$P_n'(1) = \frac{n(n+1)}{2}, \quad n = 0, 1, 2, \ldots.$$

Consequently, we have

$$\sum_{m=-n}^{n} \big[p \cdot \mathrm{Grad}\, \overline{Y_n^m(d)}\big] \mathrm{Grad}\, Y_n^m(d) = \frac{1}{8\pi} n(n+1)(2n+1)C(d,p),$$

where

$$C(d,p) = (\hat{\theta} \cdot p)\hat{\theta} + (\hat{\varphi} \cdot p)\hat{\varphi}$$

is a tangential vector depending on p and d.

Arguing similarly as above gives that

$$\sum_{m=-n}^{n} \big[p \cdot \mathrm{Grad}\, \overline{Y_n^m(d)} \times d\big] \mathrm{Grad}\, Y_n^m(d) \times d = \frac{1}{8\pi} n(n+1)(2n+1)C'(d,p),$$

where

$$C'(d,p) = \big[(\hat{\theta}\times d)\cdot p\big]\hat{\theta}\times d + \big[(\hat{\varphi}\times d)\cdot p\big]\hat{\varphi}\times d$$

is also a tangential vector depending on p and d. Since

$$\hat{\varphi}\times d = \hat{\theta}, \qquad \hat{\theta}\times d = -\hat{\varphi},$$

then $C'(d,p) = C(d,p)$. The proof is thus completed. □

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Choose the coordinate system used in the proof of Lemma 3.5. From (3.13) and Lemma 3.5 it follows that

$$E_\infty(d;k,d,p,R) = \frac{iR^2}{2k}\sum_{n=1}^{\infty}(2n+1)\left\{\frac{j_n(kR)+kRj_n'(kR)}{h_n^{(1)}(kR)+kRh_n^{(1)\prime}(kR)} + \frac{j_n(kR)}{h_n^{(1)}(kR)}\right\}C(d,p).$$

Suppose there are two balls with different radius R_1 and R_2 ($< R_1$) generating the same electric far field pattern on the unit ball at $\hat{x} = d$, that is,

$$E_\infty(d;k,d,p,R_1) = E_\infty(d;k,d,p,R_2).$$

Then we have

$$\begin{aligned} &R_1^2\sum_{n=1}^{\infty}(2n+1)\left\{\frac{j_n(kR_1)+kR_1j_n'(kR_1)}{h_n^{(1)}(kR_1)+kR_1h_n^{(1)\prime}(kR_1)} + \frac{j_n(kR_1)}{h_n^{(1)}(kR_1)}\right\} \\ &\quad = R_2^2\sum_{n=1}^{\infty}(2n+1)\left\{\frac{j_n(kR_2)+kR_2j_n'(kR_2)}{h_n^{(1)}(kR_2)+kR_2h_n^{(1)\prime}(kR_2)} + \frac{j_n(kR_2)}{h_n^{(1)}(kR_2)}\right\}. \end{aligned} \tag{3.15}$$

Set

$$f_n(t) := \frac{j_n(t)+tj_n'(t)}{h_n^{(1)}(t)+th_n^{(1)\prime}(t)} = \alpha_n(t) + i\widetilde{\alpha}_n(t),$$

$$g_n(t) := \frac{j_n(t)}{h_n^{(1)}(t)} = \beta_n(t) + i\widetilde{\beta}_n(t)$$

for $t\in(0,\infty)$ and $n = 1,2,\ldots$. Then it is easy to see that

$$\alpha_n(t) = \frac{x^2(t)}{x^2(t)+y^2(t)}, \qquad \beta_n(t) = \frac{j_n^2(t)}{j_n^2(t)+y_n^2(t)},$$

where $x(t) = j_n(t)+tj_n'(t)$, $y(t) = y_n(t)+ty_n'(t)$. From (3.15) it follows that

$$t_1^2\sum_{n=1}^{\infty}(2n+1)\big[\alpha_n(t_1)+\beta_n(t_1)\big] = t_2^2\sum_{n=1}^{\infty}(2n+1)\big[\alpha_n(t_2)+\beta_n(t_2)\big], \tag{3.16}$$

where $t_2 = kR_2 < kR_1 = t_1$. Now, making use of the definition of $\alpha_n(t)$, the Bessel differential equation (2.1) and the Wronskian equality (2.2), it is derived that

$$\alpha_n'(t) = \frac{2x(t)y(t)}{[x^2(t)+y^2(t)]^2}\,\frac{n(n+1)-t^2}{t^2}.$$

By Theorem 2.8, it is seen that for $n = 1,2,\ldots$, $\alpha_n'(t) > 0$ for $t\in(0,\sqrt{2})$, which implies that $\alpha_n(t)$ is strictly monotonic increasing for $t\in(0,\sqrt{2})$ uniformly for $n = 1,2,\ldots$. Similarly, it can be shown that $\beta_n(t)$ is strictly monotonic increasing for $t\in(0,\sqrt{2})$ uniformly for $n = 1,2,\ldots$. Thus we have, on noting that $0 < t_2 < t_1 < \sqrt{2}$, that $t_1^2[\alpha_n(t_1)+\beta_n(t_1)] > t_2^2[\alpha_n(t_2)+\beta_n(t_2)] > 0$ for $n = 1,2,\ldots$. This contradicts the equality (3.16). The theorem is thus proved. □

4. Uniqueness for perfectly conducting balls without a prior information on the center

Theorem 4.1. *Let the incident plane wave E^i be given as in Theorem 3.1 and let $B(x,R)$ be a perfectly conducting ball centered at x with radius R. Let $d_1 = d$ and let $d_j\in S^2$ ($j = 2,3,4$) be such that d_2-d, d_3-d, d_4-d are three linearly independent vectors in $\mathbb{R}^3$. For $R_1,R_2\in\mathbb{R}_+$ and $x_1,x_2\in\mathbb{R}^3$, if $kR_1,kR_2 < \sqrt{2}$ and*

$$E_\infty\big(d_j;d,p,B(x_1,R_1)\big) = E_\infty\big(d_j;d,p,B(x_2,R_2)\big), \quad j = 1,2,3,4,$$

then

$$R_1 = R_2, \qquad x_1 = x_2.$$

Proof. We first establish a connection between the electric far field patterns for translational, perfectly conducting balls, which generalizes the result of [8] for disks in $\mathbb{R}^2$. For simplicity, we write $E_\infty(\hat{x}; B(x,R)) = E_\infty(\hat{x}; d, p, B(x,R))$ and $B_R = B(0,R)$. Clearly, we have the following translation relation:

$$B(y_0, R) = \{ y \in \mathbb{R}^3 \mid y = x + y_0,\ x \in B_R \}.$$

For any $y = x + y_0 \in \partial B(y_0, R)$ with $x \in \partial B_R$, we have

$$\begin{aligned}
\nu(y) \times E^s(y, B(y_0,R)) &= -\nu(y) \times E^i(y) \\
&= -\frac{y - y_0}{\|y - y_0\|} \times p e^{iky\cdot d} \\
&= -\hat{x} \times p e^{ikx\cdot d} e^{iky_0\cdot d} \\
&= -\hat{x} \times E^i(x) e^{iky_0\cdot d} \\
&= \hat{x} \times E^s(x, B_R) e^{iky_0\cdot d} \\
&= \nu(y) \times \{ e^{iky_0\cdot d} E^s(y - y_0, B_R) \}.
\end{aligned}$$

By the uniqueness of the exterior problem of the Maxwell equations for perfectly conducting balls it follows that

$$E^s(y, B(y_0,R)) = e^{iky_0\cdot d} E^s(y - y_0, B_R), \quad \forall y \in \mathbb{R}^3 \backslash B(y_0,R). \tag{4.1}$$

This implies that for any $y = x + y_0 \in \partial B(y_0, R)$ with $x \in \partial B_R$,

$$\nu(y) \times \operatorname{curl} E^s(y, B(y_0,R)) = \frac{y - y_0}{\|y - y_0\|} \times \operatorname{curl} E^s(y - y_0, B_R) e^{iky_0\cdot d} = \nu(x) \times \operatorname{curl} E^s(x, B_R) e^{iky_0\cdot d}. \tag{4.2}$$

From (4.1) and (4.2) it is seen that the electric far field pattern associated with $B(y_0, R)$ can be characterized as follows (see [3]):

$$\begin{aligned}
E_\infty(\hat{x}, B(y_0,R)) &= \frac{ik}{4\pi} \hat{x} \times \int\limits_{\partial B(y_0,R)} \left[\nu(y) \times E^s(y, B(y_0,R)) + \nu(y) \times \frac{1}{ik} \operatorname{curl} E^s(y, B(y_0,R)) \times \hat{x} \right] e^{-ik\hat{x}\cdot y}\, ds(y) \\
&= \frac{ik}{4\pi} \hat{x} \times \int\limits_{\partial B_R} \left[\nu(x) \times E^s(x, B_R) + \nu(x) \times \frac{1}{ik} \operatorname{curl} E^s(x, B_R) \times \hat{x} \right] e^{-ik\hat{x}\cdot x}\, ds(x) e^{iky_0\cdot(d-\hat{x})} \\
&= e^{iky_0\cdot(d-\hat{x})} E_\infty(\hat{x}, B_R).
\end{aligned}$$

From this and the assumption we have

$$e^{ikx_1\cdot(d-d_j)} E_\infty(d_j, B_{R_1}) = e^{ikx_2\cdot(d-d_j)} E_\infty(d_j, B_{R_2}), \quad j = 1,2,3,4. \tag{4.3}$$

For $j = 1$ we have, on noting that $d_1 = d$, that

$$E_\infty(d, B_{R_1}) = E_\infty(d, B_{R_2}).$$

This together with Theorem 3.1 implies that $R_1 = R_2$. Let $R_1 = R_2 = R$. Then (4.3) gives

$$E_\infty(d_j, B_R) e^{ik(x_1 - x_2)\cdot(d - d_j)} = E_\infty(d_j, B_R), \quad j = 2,3,4.$$

Thus, for $j = 2,3,4,$ we have

$$(x_1 - x_2)\cdot(d - \hat{x}_j) = 0, \quad j = 2,3,4.$$

Since $d_2 - d$, $d_3 - d$, $d_4 - d$ are three linearly independent vectors in $\mathbb{R}^3$, we obtain that $x_1 = x_2$. The proof is thus complete. □

Remark 4.2. (i) From Theorems 3.1 and 4.1, it is seen that three more far field data are added to locate the center of the ball. It seems that these data are the least in the three-dimensional case since, in this case, there are totally four unknowns (the radius and three components of the center of the ball) to be determined.

(ii) Our method can be applied in the case of inverse acoustic scattering by a disk (in 2D) or a ball (in 3D) to obtain that the radius R and the center of a sound-soft disk with $R < 0.8935769/k$ or a sound-hard disk with $R < 1/k$ can be uniquely determined by three far field data and that the radius R and the center of a sound-soft ball with $R < \pi/(2k)$ or a sound-hard ball with $R < \sqrt{2}/k$ can be uniquely determined by four far field data (see [12] for the corresponding results when the center is fixed at the origin).

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