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Direct and inverse problems for electromagnetic scattering by a doubly periodic structure with a partially coated dielectric

Guanghui Hu, Fenglong Qu and Bo Zhang[*†‡]

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Consider the problem of scattering of electromagnetic waves by a doubly periodic Lipschitz structure. The medium above the structure is assumed to be homogenous and lossless with a positive dielectric coefficient. Below the structure there is a perfect conductor with a partially coated dielectric boundary. We first establish the well-posedness of the direct problem in a proper function space and then obtain a uniqueness result for the inverse problem by extending Isakov's method. Copyright © 2009 John Wiley & Sons, Ltd.

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1. Introduction

The scattering theory in periodic structures has many applications in micro-optics, radar imaging and non-destructive testing. We refer to [1] for historical remarks and details of these applications. In this paper, we will consider the direct and inverse problems for electromagnetic scattering by a doubly periodic structure with a partially coated dielectric.

Physically, the propagation of time-harmonic electromagnetic waves (with the time variation of the form $e^{-i\omega t}, \omega>0$) in a homogeneous isotropic medium in $\mathbb{R}^3$ is modeled by the time-harmonic Maxwell equations:

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikE = 0 \tag{1}$$

Here, we assume that the medium is lossless, that is, k is a positive wave number given by $k=\sqrt{\varepsilon\mu}\omega$ in terms of the frequency ω, the electric permittivity ε and the magnetic permeability μ, which are assumed to be positive constants everywhere. Let the scattering profile be described by the doubly periodic surface

$$\Gamma = \{x_3 = f(x_1,x_2)\,|\, f(x_1+2n_1\pi, x_2+2n_2\pi) = f(x_1,x_2)\ \forall n=(n_1,n_2)\in\mathbb{Z}^2\}$$

of period $\Lambda=(2\pi,2\pi)$. Consider the plane wave

$$E^i = p e^{ikx\cdot d}, \quad H^i = q e^{ikx\cdot d}$$

incident on Γ from the top region $\Omega := \{x\in\mathbb{R}^3\,|\,x_3>f(x_1,x_2)\}$, where $d=(\alpha_1,\alpha_2,-\beta)=(\cos\theta_1\cos\theta_2, \cos\theta_1\sin\theta_2, -\sin\theta_1)$ is the incident wave vector whose direction is specified by θ_1 and θ_2 with $0<\theta_1\leqslant\pi, 0<\theta_2\leqslant 2\pi$ and the vectors p and q are the polarization directions satisfying that $p=\sqrt{\mu/\varepsilon}(q\times d)$ and $q\perp d$.

LSEC and Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People's Republic of China

Correspondence to: Bo Zhang, LSEC and Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People's Republic of China.

†*E-mail: b.zhang@amt.ac.cn*

‡*Professor.*

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In this paper, we assume that the boundary $\Gamma=\partial\Omega$ has a Lipschitz dissection $\Gamma=\Gamma_D\cup\Sigma\cup\Gamma_I$, where Γ_D and Γ_I are disjoint, relatively open subsets of Γ, having Σ as their common boundary. Suppose a perfect conductor is below Γ with a partially coated dielectric on Γ_I. The problem of scattering of time-harmonic electromagnetic waves in this model leads to the following problem:

$$\operatorname{curl}\operatorname{curl} E-k^2E=0 \quad \text{in } \Omega \tag{2}$$

$$\nu\times E=0 \quad \text{on } \Gamma_D \tag{3}$$

$$\nu\times\operatorname{curl} E-\mathrm{i}\lambda(\nu\times E)\times\nu=0 \quad \text{on } \Gamma_I \tag{4}$$

$$E=E^i+E^s \quad \text{in } \Omega \tag{5}$$

where ν is the unit normal pointing into Ω. Throughout this paper, we assume that λ is a positive constant and $\Gamma_I\neq\emptyset$.

Let $\alpha=(\alpha_1,\alpha_2,0)$, $x'=(x_1,x_2,0)\in\mathbb{R}^3$, $n=(n_1,n_2)\in\mathbb{Z}^2$. We require the electric field $E(x)$ to be α-quasi-periodic in the sense that $E(x_1,x_2,x_3)e^{-\mathrm{i}\alpha\cdot x'}$ are 2π periodic with respect to x_1 and x_2, respectively. We also need a radiation condition in the x_3 direction such that $E(x)$ can be composed of the incident wave E^i plus bounded outgoing plane waves E^s in the form of

$$E^s(x)=\sum_{n\in\mathbb{Z}^2}E_n e^{\mathrm{i}(\alpha_n\cdot x'+\beta_n x_3)}, \quad x_3>\max_{x_1,x_2}f(x_1,x_2) \tag{6}$$

where $\alpha_n=(\alpha_1+n_1,\alpha_2+n_2,0)\in\mathbb{R}^3$, $E_n=(E_n^{(1)},E_n^{(2)},E_n^{(3)})\in\mathbb{C}^3$ are constant vectors and

$$\beta_n=\begin{cases}(k^2-|\alpha_n|^2)^{1/2} & \text{if } |\alpha_n|<k\\ \mathrm{i}(|\alpha_n|^2-k^2)^{1/2} & \text{if } |\alpha_n|>k\end{cases}$$

with $\mathrm{i}^2=-1$. Furthermore, we assume that $\beta_n\neq0$ for all $n\in\mathbb{Z}^2$. The series expansion in (6) will be considered as the Rayleigh series of the scattered field and the condition is called the Rayleigh expansion radiation condition. From the fact that $\operatorname{div}E^s(x)=0$ it is clear that

$$\alpha_n\cdot E_n+\beta_nE_n^{(3)}=0$$

The coefficients E_n in (6) are also called the Rayleigh sequence.

The direct problem is to compute the scattered field E^s in Ω given the incident wave E^i and the diffraction grating profile Γ with the corresponding boundary conditions. Since only a finite number of terms in (6) are upward propagating plane waves and the rest are evanescent modes that decay exponentially with distance from the grating, we use the near-field data rather than the far-field data to reconstruct the surface. Thus, our inverse problem is to determine the profile Γ and the impedance coefficient λ from the knowledge of the incident wave E^i and the total tangential electric field $\nu\times E$ on a plane $\Gamma_b=\{x\in\mathbb{R}^3\,|\,x_3=b\}$ above the surface.

Scattering of electromagnetic waves by a smooth doubly periodic structure has been studied by many authors using both integral and variational methods. See, e.g. [2–7] for the results on existence, uniqueness, and numerical approximations of solutions to the direct problems. The inverse problem in a smooth doubly periodic structure has been considered in [3, 8] for the case when $\Gamma_I=\emptyset$. With a lossy medium (i.e. $\operatorname{Im}(k)>0$) above the conductor, Ammari [3] proved a global uniqueness result for the inverse problem with one incident plane wave. For the case of lossless medium (i.e. $\operatorname{Im}(k)=0$) above the conductor, a local uniqueness result was obtained by Bao and Zhou in [8] for the inverse problem with one incident plane wave by establishing a lower bound of the first eigenvalue of the curl curl operator with the boundary condition (3) in a bounded, smooth convex domain in $\mathbb{R}^3$. The stability of the inverse problem was also studied in [8]. For inverse scattering problems by bounded obstacles, the reader is referred to [9, 10].

The result in [3] was based on the space $H(\operatorname{curl},\Omega_{\mathrm{loc}})$ with the boundary value in the trace space $H_{\mathrm{div}}^{-1/2}(\Gamma)$ in the case when $\Gamma_I=\emptyset$ and Γ is smooth. If Γ is Lipschitz, the trace space on Γ of $H(\operatorname{curl},\Omega_{\mathrm{loc}})$ was defined in [11] (see also the references there). The validity of the Hodge decomposition and integration by parts formula were also proved in [11] for non-periodic case. In this paper, we use the quasi-periodic vector space $X(\Omega_b,\Gamma_I)$ and its tangential trace space $Y(\Gamma_D)$ on Γ_D introduced in [12, 13] to establish the well-posedness of the direct problem. These spaces will play a crucial role in the study of not only the direct problem but also the inverse problem. Our main results in this paper extend the results of [3, 12] to the case of a doubly periodic Lipschitz boundary with a partially coated dielectric. We first propose a variational formulation in a truncated domain by introducing a Dirichlet-to-Neumann map on an artificial boundary Γ_b and then use the Hodge decomposition to prove the existence of a unique solution to the direct scattering problem with the help of the Fredholm alternative. We are more interested in the inverse problem. In this paper, we use electric dipoles as incident waves to detect the unknown doubly periodic structure and establish a uniqueness theorem by employing Isakov's method (see [14]). This result seems unsatisfactory in the practical sense since it requires the information on the scattered field associated with all the electric dipoles lying on Γ_b.

The rest of the paper is organized as follows. In Section 2, we introduce some suitable quasi-periodic function spaces needed in the study of the direct problem and the Dirichlet-to-Neumann map on an artificial boundary Γ_b transforming the problem (2)–(6) into a boundary value problem in a truncated domain Ω_b. In Section 3, we establish the well-posedness of the direct problem, employing the variational method together with the Hodge decomposition and the Fredholm alternative. Section 4 is devoted to the uniqueness of the inverse problem.

2. Function spaces and the dirichlet-to-neumann map

In this section we introduce some function spaces needed to solve the scattering problem (2)–(5). We will also define the Dirichlet-to-Neumann map on an artificial boundary to truncate the unbounded domain of the scattering problem. Define

$$\Gamma = \{x_3 = f(x_1, x_2) \,|\, 0 < x_1, x_2 < 2\pi\}$$
$$\Gamma_b = \{x_3 = b \,|\, 0 < x_1, x_2 < 2\pi\}$$
$$\Omega = \{x \in \mathbb{R}^3 \,|\, x_3 > f(x_1, x_2), 0 < x_1, x_2 < 2\pi\}$$
$$\Omega_b = \{x \in \Omega \,|\, x_3 < b\}$$

We introduce the following scalar quasi-periodic Sobolev space:

$$H^1(\Omega_b) = \left\{ u(x) = \sum_{n\in\mathbb{Z}^2} u_n \exp[\mathrm{i}(\alpha_n \cdot x' + \beta_n x_3)] \,|\, u \in L^2(\Omega_b), \nabla u \in (L^2(\Omega_b))^3, u_n \in \mathbb{C} \right\}$$

Denote by $H^{1/2}(\Gamma_b)$ the trace space of $H^1(\Omega_b)$ on Γ_b with the norm

$$\|f\|^2_{H^{1/2}(\Gamma_b)} = \sum_{n\in\mathbb{Z}^2} |f_n|^2 (1 + |\alpha_n|^2)^{1/2}, \quad f \in H^{1/2}(\Gamma_b)$$

where $f_n = (f, \exp(\mathrm{i}\alpha_n \cdot x'))_{L^2(\Gamma_b)}$ and write $H^{-1/2}(\Gamma_b) = (H^{1/2}(\Gamma_b))'$, the dual space to $H^{1/2}(\Gamma_b)$.

We now introduce some vector spaces. Let

$$H(\mathrm{curl}, \Omega_b) = \left\{ E(x) = \sum_{n\in\mathbb{Z}^2} E_n \exp[\mathrm{i}(\alpha_n \cdot x' + \beta_n x_3)] \,|\, E_n \in \mathbb{C}^3, \quad E \in (L^2(\Omega_b))^3, \mathrm{curl}\, E \in (L^2(\Omega_b))^3 \right\}$$

with the norm

$$\|E\|^2_{H(\mathrm{curl},\Omega_b)} = \|E\|^2_{L^2(\Omega_b)} + \|\mathrm{curl}\, E\|^2_{L^2(\Omega_b)}$$

and let

$$H_0(\mathrm{curl}, \Omega_b) = \{E \in H(\mathrm{curl}, \Omega_b),\ \nu \times E = 0 \text{ on } \Gamma_b\}$$

Define

$$X := X(\Omega_b, \Gamma_l) = \{E \in H(\mathrm{curl}, \Omega_b),\ \nu \times E|_{G_l} \in L^2_t(\Gamma_l)\}$$

with the norm

$$\|E\|^2_X = \|E\|^2_{H(\mathrm{curl},\Omega_b)} + \|\nu \times E\|^2_{L^2_t(\Gamma_l)}$$

where $L^2_t(\Gamma) = \{E \in (L^2(\Gamma))^3,\ \nu \cdot E = 0 \,\text{on}\, \Gamma\}$. For $s \in \mathbb{R}$ define

$$H^s_t(\Gamma_b) = \left\{ E(x') = \sum_{n\in\mathbb{Z}^2} E_n \exp(\mathrm{i}\alpha_n \cdot x') \,|\, E_n \in \mathbb{C}^3, e_3 \cdot E = 0, \right.$$
$$\left. \sum_{n\in\mathbb{Z}^2} (1 + |\alpha_n|^2)^s |E_n|^2 < +\infty \right\}$$

$$H^s_t(\mathrm{div}, \Gamma_b) = \left\{ E(x') = \sum_{n\in\mathbb{Z}^2} E_n \exp(\mathrm{i}\alpha_n \cdot x') \,|\, E_n \in \mathbb{C}^3, e_3 \cdot E = 0, \right.$$
$$\left. \|E\|^2_{H^s(\mathrm{div},\Gamma_b)} = \sum_{n\in\mathbb{Z}^2} (1 + |\alpha_n|^2)^s (|E_n|^2 + |E_n \cdot \alpha_n|^2) < +\infty \right\}$$

$$H^s_t(\mathrm{curl}, \Gamma_b) = \left\{ E(x') = \sum_{n\in\mathbb{Z}^2} E_n \exp(\mathrm{i}\alpha_n \cdot x') \,|\, E_n \in \mathbb{C}^3, e_3 \cdot E = 0, \right.$$
$$\left. \|E\|^2_{H^s(\mathrm{curl},\Gamma_b)} = \sum_{n\in\mathbb{Z}^2} (1 + |\alpha_n|^2)^s (|E_n|^2 + |E_n \times \alpha_n|^2) < +\infty \right\}$$

and write $L_t^2(\Gamma_b)=H_t^0(\Gamma_b)$. Recall that

$$H_t^{-1/2}(\mathrm{div},\Gamma_b)=\{e_3\times E|_{\Gamma_b},\ E\in H(\mathrm{curl},\Omega_b)\}$$

and that the trace mapping from $H(\mathrm{curl},\Omega_b)$ to $H_t^{-1/2}(\mathrm{div},\Gamma_b)$ is continuous and surjective. The trace space of $X(\Omega_b,\Gamma_I)$ on the complementary part Γ_D is

$$Y(\Gamma_D)=\{f\in(H^{-1/2}(\Gamma_D))^3\,|\,\exists E\in H_0(\mathrm{curl},\Omega_b)\text{ such that } \nu\times E|_{\Gamma_I}\in L_t^2(\Gamma_I),\ \nu\times E|_{\Gamma_D}=f\}$$

which is a Banach space with the norm

$$\|f\|^2_{Y(\Gamma_D)}=\inf\{\|E\|^2_{H(\mathrm{curl},\Omega_b)}+\|\nu\times E\|^2_{L_t^2(\Gamma_I)}\,|$$

$$E\in H_0(\mathrm{curl},\Omega_b),\ \nu\times E|_{\Gamma_I}\in L_t^2(\Gamma_I),\ \nu\times E|_{\Gamma_D}=f\}$$

An equivalent norm to $\|\cdot\|_{Y_{\Gamma_D}}$ is given by (see [12, 15])

$$|||f|||_1=\sup_{\Phi\in X(\Omega_b,\Gamma_I)}\frac{|\langle f,\Phi\rangle_1|}{\|\Phi\|_{X(\Omega_b,\Gamma_I)}}$$

where, for $E\in H_0(\mathrm{curl},\Omega_b)$ satisfying that $\nu\times E|_{\Gamma_I}\in L_t^2(\Gamma_I)$ and $\nu\times E|_{\Gamma_D}=f$, we have

$$\langle f,\Phi\rangle_1=\int_{\Omega_b}\mathrm{curl}\,E\cdot\Phi-E\cdot\mathrm{curl}\,\Phi\,\mathrm{d}x-\int_{\Gamma_I}\nu\times E\cdot\Phi\,\mathrm{d}s(x),\quad \Phi\in X(\Omega_b,\Gamma_I) \tag{7}$$

In particular, $Y(\Gamma_D)$ is a Hilbert space and (7) can be considered as a duality between $Y(\Gamma_D)$ and its dual space $Y(\Gamma_D)'$. From (7) it can been seen that $\varphi\in Y(\Gamma_D)'$ can be extended as a function $\tilde{\varphi}\in H^{-1/2}_{\mathrm{curl}}(\Gamma)$ defined on the whole boundary Γ such that $\tilde{\varphi}|_{\Gamma_I}\in L_t^2(\Gamma_I)$.

For $\tilde{E}(x')=\sum_{n\in\mathbb{Z}^2}\tilde{E}_n\exp(\mathrm{i}\alpha_n\cdot x')\in H_t^{-1/2}(\mathrm{div},\Gamma_b)$, define the Dirichlet-to-Neumann map $\mathscr{R}:H_t^{-1/2}(\mathrm{div},\Gamma_b)\to H_t^{-1/2}(\mathrm{curl},\Gamma_b)$ by

$$(\mathscr{R}\tilde{E})(x')=(e_3\times\mathrm{curl}\,E)\times e_3\quad\text{on }\Gamma_b \tag{8}$$

where $E(x)$ is a quasi-periodic solution of the problem

$$\mathrm{curl}\,\mathrm{curl}\,E-k^2E=0,\quad x_3>b$$

$$\nu\times E=\tilde{E}(x')\quad\text{on }\Gamma_b$$

$$E(x)=\sum_{n\in\mathbb{Z}^2}E_n\exp(\mathrm{i}(\alpha_n\cdot x'+\beta_n x_3)),\quad x_3>b$$

The Dirichlet-to-Neumann map $\mathscr{R}$ is well-defined and can be used to replace the radiation condition (6) on an artificial boundary Γ_b. Let $f=-\nu\times E^i|_{\Gamma_D}\in Y(\Gamma_D)$ and let $h=-\nu\times\mathrm{curl}\,E^i|_{\Gamma_I}+\mathrm{i}\lambda(\nu\times E^i)\times\nu|_{\Gamma_I}\in L_t^2(\Gamma_I)$. Then the scattering problem (2)–(6) can be transformed into the following boundary value problem in a truncated domain Ω_b:

$$\mathrm{curl}\,\mathrm{curl}\,E-k^2E=0\quad\text{in }\Omega_b \tag{9}$$

$$\nu\times E=f\quad\text{on }\Gamma_D \tag{10}$$

$$\nu\times\mathrm{curl}\,E-\mathrm{i}\lambda E_T=h\quad\text{on }\Gamma_I \tag{11}$$

$$(\mathrm{curl}\,E)_T=\mathscr{R}(e_3\times E)\quad\text{on }\Gamma_b \tag{12}$$

where, for any vector function U, $U_T=\nu\times(\nu\times U)$ denotes its tangential component on a surface. The Dirichlet-to-Neumann map $\mathscr{R}$ has the following properties:

(1) $\mathscr{R}:H_t^{-1/2}(\mathrm{div},\Gamma_b)\to H_t^{-1/2}(\mathrm{curl},\Gamma_b)$ is continuous and has the following explicit representation (see also [2]):

$$(\mathscr{R}\tilde{E})(x')=-\sum_{n\in\mathbb{Z}^2}\frac{1}{\mathrm{i}\beta_n}[k^2\tilde{E}_n-(\alpha_n\cdot\tilde{E}_n)\alpha_n]\exp(\mathrm{i}\alpha_n\cdot x') \tag{13}$$

where $\tilde{E}(x')=\sum_{n\in\mathbb{Z}^2}\tilde{E}_n\exp(\mathrm{i}\alpha_n\cdot x')$ and throughout this paper we assume that $\beta_n\neq0$ for all $n\in\mathbb{Z}^2$.

(2) Let $P=\{n=(n_1,n_2)\in\mathbb{Z}^2\,|\,\beta_n \text{ is a real number}\}$. Then

$$\mathrm{Re}\,\langle\mathscr{R}\tilde{E},\tilde{E}\rangle=4\pi^2\sum_{n\in\mathbb{Z}^2\backslash P}\frac{1}{|\beta_n|}[k^2|\tilde{E}_n|^2-|\alpha_n\cdot\tilde{E}_n|^2] \tag{14}$$

$$-\mathrm{Re}\,\langle\mathscr{R}\tilde{E},\tilde{E}\rangle\geqslant C_1\|\mathrm{div}\,\tilde{E}\|^2_{H_t^{-1/2}(\Gamma_b)}-C_2\|\tilde{E}\|^2_{H_t^{-1/2}(\Gamma_b)} \tag{15}$$

where C_1 and C_2 are positive constants and $\langle\cdot,\cdot\rangle$ denotes the inner product of $L_t^2(\Gamma_b)$.

(3)

$$\operatorname{Im}\langle\mathscr{R}\widetilde{E},\widetilde{E}\rangle=4\pi^2\sum_{n\in P}\frac{1}{\beta_n}[k^2|\widetilde{E}_n|^2-|\alpha_n\cdot\widetilde{E}_n|^2]\geqslant 0 \tag{16}$$

The representation (13) of $\mathscr{R}$ can be computed directly from its definition (8), and the properties (14)–(16) can be easily obtained using this representation. Furthermore, there exists a $C>0$ such that for every $\eta>0$ and $E\in H(\operatorname{curl},\Omega_b)$, we have (see [2])

$$\|v\times E\|_{H_t^{-1/2}(\Gamma_b)}\leqslant C[\eta\|\operatorname{curl}E\|_{L^2(\Omega_b)}+(1+1/\eta)\|E\|_{L^2(\Omega_b)}] \tag{17}$$

It is well-known (see [7]) that the free-space quasi-periodic Green function of $\mathbb{R}^3$ is given by

$$G(x,y)=\frac{1}{8\pi^2}\sum_{n\in\mathbb{Z}^2}\frac{1}{\mathrm{i}\beta_n}\exp(\mathrm{i}\alpha_n\cdot(x'-y')+\mathrm{i}\beta_n|x_3-y_3|) \tag{18}$$

3. The direct problem

In this section, we will prove the following result on the well-posedness of the direct problem (9)–(12).

Theorem 3.1
If $f\in Y(\Gamma_D), h\in L_t^2(\Gamma_I)$ and $\Gamma_I\neq\emptyset$, then there exists a unique solution $E\in X(\Omega_b,\Gamma_I)$ to the problem (9)–(12). Furthermore, we have

$$\|E\|_X^2\leqslant C[\|f\|^2_{Y(\Gamma_D)}+\|h\|^2_{L_t^2(\Gamma_I)}] \tag{19}$$

where C is a positive constant depending only on b.

Proof
We first prove the uniqueness of the solution. To this end, let $f=0$, $h=0$. Multiplying both sides of (9) by $\overline{E}$ and using integration by parts (in the distribution sense), we have

$$\int_{\Omega_b}|\operatorname{curl}E|^2-k^2|E|^2\,\mathrm{d}x-\mathrm{i}\lambda\int_{\Gamma_I}|E_T|^2\,\mathrm{d}s-\int_{\Gamma_b}\mathscr{R}(e_3\times E)\cdot(e_3\times\overline{E})\,\mathrm{d}s=0 \tag{20}$$

We now take the imaginary part of the above equation and use (16) to find that

$$\lambda\int_{\Gamma_I}|E_T|^2\,\mathrm{d}s=0$$

which implies that $E_T=v\times(v\times E)=0$ on Γ_I. This together with the boundary condition (4) gives $v\times\operatorname{curl}E=0$ on Γ_I. From the proof of the representation formula in Proposition 3.3 of [7] and integration by parts in Lipschitz domains, we have the following representation:

$$\begin{aligned}E(x)&=\operatorname{curl}_x\int_\Gamma v(y)\times E(y)G(x,y)\,\mathrm{d}s(y)+\int_\Gamma v(y)\times\operatorname{curl}E(y)G(x,y)\,\mathrm{d}s(y)\\&\quad-\nabla_x\int_\Gamma v(y)\cdot E(y)G(x,y)\,\mathrm{d}s(y)\\&=\int_{\Gamma_D}v(y)\times\operatorname{curl}E(y)G(x,y)\,\mathrm{d}s(y)-\nabla_x\int_{\Gamma_D}v(y)\cdot E(y)G(x\cdot y)\,\mathrm{d}s(y)\end{aligned}$$

for any $x\in\Omega$, where use has been made of the fact that $v\times E=0$ on Γ and $v\times\operatorname{curl}E=0$ on Γ_I. From this representation it follows that E is regular across Γ_I. This together with the unique continuation principle (see [9, 16]) implies that $E\equiv 0$ in Ω.

We now prove the existence of solutions. To this end, we introduce the following subspace of $X(\Omega_b,\Gamma_I)$:

$$\widetilde{X}=\{E\in H(\operatorname{curl},\Omega_b)\,|\,v\times E|_{\Gamma_D}=0,\ \ v\times E|_{\Gamma_I}\in L_t^2(\Gamma_I)\}\subset X(\Omega_b,\Gamma_I)$$

Then the problem (9)–(12) is equivalent to the variational formulation: find $E\in X(\Omega_b,\Gamma_I)$ such that $v\times E|_{\Gamma_D}=f$ and

$$a(E,\varphi)=\int_{\Gamma_I}h\cdot\overline{\varphi}_T\,\mathrm{d}s\quad\forall\varphi\in\widetilde{X} \tag{21}$$

where $a(\cdot,\cdot):X\times X\to\mathbb{C}$ is a bilinear form defined by

$$a(w,\varphi)=\int_{\Omega_b}[\operatorname{curl}w\cdot\operatorname{curl}\overline{\varphi}-k^2w\cdot\overline{\varphi}]\,\mathrm{d}x-\mathrm{i}\lambda\int_{\Gamma_I}w_T\cdot\overline{\varphi}_T\,\mathrm{d}s-\int_{\Gamma_b}\mathscr{R}(e_3\times w)\cdot(e_3\times\overline{\varphi})\,\mathrm{d}s$$

for any $w,\varphi\in X$. Since $f\in Y(\Gamma_D)$, then by the definition of $Y(\Gamma_D)$ there exists a $U\in H_0(\mathrm{curl},\Omega_b)$ such that $\nu\times U|_{\Gamma_D}=f, \nu\times U|_{\Gamma_l}\in L_t^2(\Gamma_l)$ and $\nu\times U|_{\Gamma_b}=0$. Let $w=E-U$. Then $w\in\widetilde{X}$ and the problem (21) is equivalent to the problem: find $w\in\widetilde{X}$ such that

$$a(w,\varphi)=\langle h,\varphi_T\rangle_{\Gamma_l}-a(U,\varphi):=B(\varphi)\quad\forall\varphi\in\widetilde{X} \tag{22}$$

where $\langle\cdot,\cdot\rangle_{\Gamma_l}$ denotes the $L_t^2(\Gamma_l)$ scalar product. The proof is broken down into the following steps.

Step 1: To establish the Hodge decomposition:

$$\widetilde{X}=X_0\oplus\nabla S \tag{23}$$

where $S=\{p\in H^1(\Omega_b),\ \ p=0\ \text{ on }\ \Gamma\}$ and

$$\begin{aligned}X_0&=\{w_0\in\widetilde{X}\,|\,a(w_0,\nabla p)=0\ \forall p\in S\}\\&=\{w_0\in\widetilde{X}\,|\,\mathrm{div}\,w_0=0,\ \ w_0\cdot e_3=\mathscr{D}(e_3\times w_0)\ \text{ on }\ \Gamma_b\}\end{aligned}$$

Here, for $E=\sum_{n\in\mathbb{Z}^2}E_n\exp(\mathrm{i}\alpha_n\cdot x)$, the operator $\mathscr{D}:H_t^{-1/2}(\mathrm{div}\,\Gamma_b)\to H^{-1/2}(\Gamma_b)$ is defined by

$$\mathscr{D}(E)(x)=-\sum_{n\in\mathbb{Z}^2}\frac{1}{\beta_n}(e_3\times\alpha_n)\cdot E_n\exp(\mathrm{i}\alpha_n\cdot x)\quad\forall x\in\Gamma_b$$

By the Poincaré inequality and the property of $\mathscr{R}$ it follows that for any $p\in S$,

$$|\mathrm{Re}\,a(\nabla p,\nabla p)|\geqslant k^2\|\nabla p\|^2_{L^2(\Omega_b)}\geqslant C\|p\|^2_{H^1(\Omega_b)} \tag{24}$$

that is, $a(\cdot,\cdot)$ is coercive on S. On the other hand, by the properties of $\mathscr{R}$ and the trace theorem we know that for any $w\in\widetilde{X}$ there is a constant C independent of w and ξ such that

$$|a(w,\nabla\xi)|\leqslant C\|w\|_X\|\xi\|_{H^1(\Omega_b)}\quad\forall\xi\in S \tag{25}$$

that is, $a(w,\nabla\xi)$ is a bounded linear functional on S. Thus, by the Lax–Milgram Theorem we know that for any $w\in\widetilde{X}$ there exists a unique $p\in S$ such that

$$a(\nabla p,\nabla\xi)=a(w,\nabla\xi)\quad\forall\xi\in S$$

Let $w_0=w-\nabla p$. Then by the definition of X_0 we have $w_0\in X_0$. Now let $w_0\in X_0\cap\nabla S$. Then $w_0=\nabla p$ for some $p\in S$, and by the definition of X_0 it follows that

$$a(\nabla p,\nabla\xi)=a(w_0,\nabla\xi)=0\quad\forall\xi\in S$$

This together with (24) implies that $p=0$ and $w_0=0$, which means that $X_0\cap\nabla S=\emptyset$. Thus, the Hodge decomposition (23) holds.

We now prove the second characterization of X_0. If $a(w_0,\nabla p)=0$ for all $p\in S$, then

$$-k^2\int_{\Omega_b}w_0\cdot\nabla\overline{p}\,\mathrm{d}x-\int_{\Gamma_b}\mathscr{R}(e_3\times w_0)\cdot(e_3\times\nabla\overline{p})\,\mathrm{d}s=0\quad\forall p\in S$$

where we have made use of the fact that, since $p=0$ on Γ, we have $(\nabla p)_T=0$ on Γ. This together with the divergence theorem gives

$$\int_{\Omega_b}(\mathrm{div}\,w_0)\overline{p}\,\mathrm{d}x+\int_{G_b}\left\{\frac{1}{k^2}\mathrm{Div}_{\Gamma_b}[\mathscr{R}(w_0\times e_3)\times e_3]-w_0\cdot e_3\right\}\overline{p}\,\mathrm{d}s=0\quad\forall p\in S$$

where Div_{Γ_b} denotes the surface divergence operator on Γ_b, which implies that

$$\begin{aligned}\mathrm{div}\,w_0&=0\quad\text{ in }\Omega_b\\w_0\cdot e_3&=\frac{1}{k^2}\mathrm{Div}_{\Gamma_b}(\mathscr{R}(e_3\times w_0)\times e_3)\quad\text{ on }\Gamma_b\end{aligned}$$

A direct calculation gives that for $E(x)=\sum_{n\in\mathbb{Z}^2}E_n\exp(\mathrm{i}\alpha_n\cdot x)\in H_t^{-1/2}(\mathrm{div},\Gamma_b)$,

$$\mathrm{Div}_{\Gamma_b}(\mathscr{R}(E)\times e_3)=-k^2\sum_{n\in\mathbb{Z}^2}\frac{1}{\beta_n}(e_3\times\alpha_n)\cdot E_n\exp(\mathrm{i}\alpha_n\cdot x)\in H^{-1/2}(\Gamma_b)$$

Thus, $w_0\cdot e_3=\mathscr{D}(e_3\times w_0)$ on Γ_b. This completes the proof of Step 1.

Step 2: For any $\xi\in S, B(\nabla\xi)=-a(U,\nabla\xi)$ so, by (25) we have

$$|B(\nabla\xi)|\leqslant C\|U\|_X\|\xi\|_{H^1(\Omega_b)}\quad\forall\xi\in S \tag{26}$$

for some constant $C>0$, that is, $B(\nabla\xi)$ is a bounded linear functional defined on S. Thus by (24) and the Lax–Milgram Theorem there exists a unique $p\in S$ such that $a(\nabla p,\nabla\xi)=B(\nabla\xi)$ for all $\xi\in S$. Further, it follows by (26) that

$$\|p\|_{H^1(\Omega_b)}\leqslant C\|U\|_X \tag{27}$$

By (23) we may assume that $w=w_0+\nabla p$, $\varphi=\varphi_0+\nabla\xi$ with $w_0,\varphi\in X_0$ and $p,\xi\in S$. Thus, the variational form (22) becomes the problem: find $w_0\in X_0$ such that

$$a(w_0,\varphi_0)=\widetilde{B}(\varphi_0)\quad \forall\varphi_0\in X_0 \tag{28}$$

where $\widetilde{B}(\varphi_0):=B(\varphi_0)-a(\nabla p,\varphi_0)$.

Step 3: Let M be a positive constant to be determined later and let

$$a_1(w_0,\varphi_0)=\int_{\Omega_b}\left[\operatorname{curl} w_0\cdot\operatorname{curl}\overline{\varphi_0}+Mw_0\cdot\overline{\varphi_0}\right]\mathrm{d}x-i\lambda\int_{\Gamma_I} w_{0_T}\cdot\overline{\varphi}_{0_T}\,\mathrm{d}s$$
$$-\int_{\Gamma_b}\mathscr{R}(e_3\times w_0)\cdot(e_3\times\overline{\varphi}_0)\,\mathrm{d}s$$
$$a_2(w_0,\varphi_0)=-(M+k^2)\int_{\Omega_b} w_0\cdot\overline{\varphi_0}\,\mathrm{d}x$$

Then $a(w_0,\varphi_0)=a_1(w_0,\varphi_0)+a_2(w_0,\varphi_0)$ for $w_0,\ \varphi_0\in X_0$. By (15) and (17) it follows that

$$-\mathrm{Re}\,\langle\mathscr{R}(e_3\times w_0),e_3\times\overline{w}_0\rangle$$
$$\geqslant C_1\|\operatorname{div}(e_3\times w_0)\|^2_{H_t^{-1/2}(\Gamma_b)}-C_2\|e_3\times w_0\|^2_{H_t^{-1/2}(\Gamma_b)}$$
$$\geqslant C_1\|\operatorname{div}(e_3\times w_0)\|^2_{H_t^{-1/2}(\Gamma_b)}-C_3\eta^2\|\operatorname{curl} w_0\|^2_{L^2(\Omega_b)}-C_3\left(1+\frac{1}{\eta}\right)^2\|w_0\|^2_{L^2(\Omega_b)}$$

where C_1, C_2 and C_3 are three positive constants and $\eta>0$ is arbitrary. Thus, we have

$$\mathrm{Re}\,a_1(w_0,w_0)\geqslant\|\operatorname{curl} w_0\|^2_{L^2(\Omega_b)}+M\|w_0\|^2_{L^2(\Omega_b)}$$
$$-C_3\eta^2\|\operatorname{curl} w_0\|^2_{L^2(\Omega_b)}-C_3(1+1/\eta)^2\|w_0\|^2_{L^2(\Omega_b)}$$
$$=(1-C_3\eta^2)\|\operatorname{curl} w_0\|^2_{L^2(\Omega_b)}+(M-C_3(1+1/\eta)^2)\|w_0\|^2_{L^2(\Omega_b)}$$

Choose η sufficiently small and M sufficiently large so that

$$\mathrm{Re}\,a_1(w_0,w_0)\geqslant C_0\|\operatorname{curl} w_0\|^2_{L^2(\Omega_b)} \tag{29}$$

for some constant $C_0>0$. Since $\mathrm{Im}\,a_1(\cdot,\cdot)=-\lambda\|v\times w_0\|^2_{L_t^2(\Gamma_I)}$, we obtain from (29) and the definition of X that

$$|a_1(w_0,w_0)|\geqslant C\|w_0\|_X^2$$

for some constant $C>0$. Thus, by the Lax–Milgram Theorem, $a_1(\cdot,\cdot)$ defines a bijective operator on X_0. On the other hand, it is seen from Corollary 3.49 of [13] and the definition of X_0 that X_0 is compactly imbedded in $(L^2(\Omega_b))^3$, so $a_2(\cdot,\cdot)$ defines a compact operator on X_0. Consequently, $a(\cdot,\cdot)$ defines an operator that can be split into a bijective operator plus a compact operator on X_0. Then a standard argument implies that the Fredholm alternative can be used to prove the existence of solutions to the problem (28) if we can prove the uniqueness of solutions to the problem (28). In fact, if $a(w_0,\varphi_0)=0$ for all $\varphi_0\in X_0$, then by the Hodge decomposition of $\widetilde{X}$ (see (23)) we have $a(w_0,\varphi)=0$ for all $\varphi\in\widetilde{X}$, which, together with (22) and the uniqueness of the direct problem, implies that $w_0=0$. This proves the uniqueness of solutions to the problem (28). Hence, the problem (28) has a unique solution $w_0\in X_0\subset X$ satisfying that

$$\|w_0\|_X\leqslant C(\|h\|_{L_t^2(\Gamma_I)}+\|U\|_X)$$

for some generic positive constant C, where use has been made of the fact that, by (26), (27) and the boundedness of $a(U,\varphi_0)$ we have

$$|\widetilde{B}(\varphi_0)|\leqslant C(\|h\|_{L_t^2(\Gamma_I)}+\|p\|_{H^1(\Omega_b)}+\|U\|_X)\|\varphi_0\|_X$$
$$\leqslant C(\|h\|_{L_t^2(\Gamma_I)}+\|U\|_X)\|\varphi_0\|_X$$

for any $\varphi_0\in X_0$ and some generic positive constant C. Consequently, $E=U+w_0+\nabla p\in X(\Omega_b,\Gamma_I)$ is a unique solution of the problem (9)–(12) with the estimate

$$\begin{aligned}\|E\|_X&\leqslant(\|U\|_X+\|w_0\|_X+\|\nabla p\|_X)\\&\leqslant C(\|h\|_{L_t^2(\Gamma_I)}+\|U\|_X)\end{aligned}\tag{30}$$

with some generic positive constant C. From the definition of $Y(\Gamma_D)$ it follows that for every $\varepsilon>0$ there is a $U_\varepsilon\in H_0(\text{curl},\Omega_b)$ such that $\nu\times U_\varepsilon|_{\Gamma_D}=f$, $\nu\times U_\varepsilon|_{\Gamma_I}\in L_t^2(\Gamma_I)$, $\nu\times U_\varepsilon|_{\Gamma_b}=0$ and

$$\|U_\varepsilon\|_X\leqslant\|f\|_{Y(\Gamma_D)}+\varepsilon$$

Since the unique solution of the problem (9)–(12) is independent of the choice of U, the estimate (30) implies that

$$\|E\|_X\leqslant C(\|h\|_{L_t^2(\Gamma_I)}+\|f\|_{Y(\Gamma_D)}+\varepsilon)\quad\forall\varepsilon>0$$

Since ε is arbitrary, we have

$$\|E\|_X\leqslant C(\|h\|_{L_t^2(\Gamma_I)}+\|f\|_{Y(\Gamma_D)})$$

where C is a positive constant depending only on b. This completes the proof of Theorem 3.1. □

4. The inverse problem

For $P,y\in\mathbb{R}^3$ let us define the electric dipole

$$E_y^{\text{in}}:=E^{\text{in}}(x,y)=\text{curl}_x\text{curl}_x[PG(x,y)],\quad x\neq y$$

where $G(x,y)$ is the free-space quasi-periodic Green function defined in (18), and let us denote by $V_P=\{E_y^{\text{in}}\,|y\in\Gamma_b\}$ the set of all incident waves. Then we have the following uniqueness result on the inverse scattering problem.

Theorem 4.1
Let $\Gamma_I\neq\emptyset$ and let $P_i\ \ (i=1,2,3)\in\mathbb{R}^3$ be three linearly independent vectors. Assume that $\nu\times E_1^s(x;y)|_{\Gamma_b}=\nu\times E_2^s(x;y)|_{\Gamma_b}$ for all incident waves $E_y^{\text{in}}\in\bigcup_{i=1}^3V_{P_i}$. Then

$$f_1(x_1,x_2)=f_2(x_1,x_2)\quad\text{for any }(x_1,x_2)\in\mathbb{R}^2\quad\text{and}\quad\lambda_1=\lambda_2$$

Here, $E_j^s(x;y)\ \ (j=1,2)$ is the unique quasi-periodic scattered solution of the Maxwell equations in $\Omega_j:=\{x\in\mathbb{R}^3|x_3>f_j(x_1,x_2)\}$ with E_y^{in} being the incident wave.

The following denseness result plays a center role in the proof of Theorem 4.1.

Lemma 4.2
Let $\Gamma_I\neq\emptyset$ and let $P_i\,(i=1,2,3)$ be three linearly independent vectors. Let $z_0=(z_{01},z_{02},z_{03})\in\Omega_a$ satisfy that $z_{03}>\|f\|_\infty$. Then for every compact set $K\subset\mathbb{R}^3\backslash\overline{\Omega}$ there exists a sequence $y_n\in\Gamma_b$ such that $E_{y_n}^{\text{in}}(x)$ converges to $E_{z_0}^{\text{in}}(x)$ uniformly in $X(K,\Gamma_I)$.

Proof
Note first that both $E_y^{\text{in}}(x)\,(y\in\Gamma_b)$ and $E_{z_0}^{\text{in}}(x)$ propagate downward and satisfy the Rayleigh expansion (6) with $-\beta_n$ in $\mathbb{R}^3\backslash\overline{\Omega}$. A similar argument as in the proof of Theorem 3.1 can be used to show the existence of a unique solution to the scattering problem in the region below the doubly periodic structure with the impedance coefficient $-\lambda$ and the Dirichlet-to-Neumann map on the artificial boundary Γ_{-b} below the structure.

Now, for $y\in\Gamma_b$ define the function $H_y^P(x)\in Y(\Gamma_D)\times L_t^2(\Gamma_I)$ by

$$H_y^P(x)=\begin{cases}\nu(x)\times E_y^{\text{in}}(x) & \text{on }\Gamma_D\\ \nu(x)\times\text{curl}_xE_y^{\text{in}}(x)+i\lambda E_y^{\text{in}}(x)_T & \text{on }\Gamma_I\end{cases}$$

To prove the lemma it is enough to show that $\overline{\text{Span}\{H_y^{P_i}|\ y\in\Gamma_b,i=1,2,3\}}$ is dense in $Y(\Gamma_D)\times L_t^2(\Gamma_I)$. To this end, for $f\times h\in B^*:=Y(\Gamma_D)'\times L_t^2(\Gamma_I)$ we are going to prove that $f=0,h=0$ under the assumption that $\langle H_y^{P_i},f\times h\rangle_{B,B^*}=0$ for any $y\in\Gamma_b,i=1,2,3$. Recalling that the dual relation between $Y(\Gamma_D)$ and $Y(\Gamma_D)'$ is defined by (7) and the duality between $L_t^2(\Gamma_I)$ and $L_t^2(\Gamma_I)$ is the L^2 scalar product, we have

$$\begin{aligned}0=&\int_{\Gamma_D}\nu(x)\times\text{curl}_x\text{curl}_x[P_i\overline{G(x,y)}]\cdot f(x)\,\text{d}s(x)\\&+\int_{\Gamma_I}\{\nu(x)\times\text{curl}_x\text{curl}_x\text{curl}_x[P_i\overline{G(x,y)}]+\text{i}\lambda(\text{curl}_x\text{curl}_x[P_i\overline{G(x,y)}])_T\}\cdot h(x)\,\text{d}s(x)\end{aligned}$$

Since $f\in Y(\Gamma_D)'$, there is an extension $\widetilde{f}\in H_{\rm curl}^{-1/2}(\Gamma)$ of f defined on Γ satisfying that $\widetilde{f}|_{\Gamma_I}\in L_t^2(\Gamma_I)$. Thus we write the above equation as

$$0=\int_{\Gamma_I}\nu(x)\times{\rm curl}_x{\rm curl}_xP_i\overline{G(x,y)}\cdot\widetilde{f}(x)\,{\rm d}s(x)-\int_{\Gamma_I}\nu(x)\times{\rm curl}_x{\rm curl}_xP_iG(x,y)\cdot\widetilde{f}(x)\,{\rm d}s(x)$$
$$+\int_{\Gamma_I}\{\nu(x)\times{\rm curl}_x{\rm curl}_x{\rm curl}_xP_iG(x,y)+{\rm i}\lambda({\rm curl}_x{\rm curl}_xP_iG(x,y))_T\}\cdot h(x)\,{\rm d}s(x)$$

Making use of the vector identity

$$\{{\rm curl}_x{\rm curl}_x[PG(x,y)]\}\cdot h(x)=\{{\rm curl}_y{\rm curl}_y[h(x)G(x,y)]\}\cdot P$$

we obtain by a direct calculation that for any $y\in\Gamma_b$ and for $i=1,2,3$

$$k^2E(y)\cdot P_i=0$$

where

$$E(y)=\frac{1}{k^2}\left\{{\rm curl}_y{\rm curl}_y\int_{\Gamma}G(x,y)\widetilde{f}(x)\times\nu(x)\,{\rm d}s(x)-{\rm curl}_y{\rm curl}_y\int_{\Gamma_I}G(x,y)\widetilde{f}(x)\times\nu(x)\,{\rm d}s(x)\right.$$
$$\left.+k^2{\rm curl}_y\int_{\Gamma_I}G(x,y)h(x)\times\nu(x)\,{\rm d}s(x)+i\lambda{\rm curl}_y{\rm curl}_y\int_{\Gamma_I}G(x,y)h(x)\,{\rm d}s(x)\right\}$$

Since $P_i\ \ (i=1,2,3)$ are three linearly independent vectors in $\mathbb{R}^3$, it follows that $E(y)\equiv0$ on Γ_b. Furthermore, we have

$$ {\rm curl}\,{\rm curl}\,E-k^2E=0\quad y\in\mathbb{R}^3\backslash\Gamma$$
$$\nu\times E=0\quad y\in\Gamma_b$$

By the uniqueness of the exterior Dirichlet problem for the upward radiating solution and the analytic continuation of the solution of the Maxwell equations, it is found that $E(y)\equiv0$ for $y_3>f(y_1,y_2)$. When $y\to\Gamma$, the following jump relations hold on Γ:

$$\nu\times E^+-\nu\times E^-=0\quad\text{on }\Gamma_D$$
$${\rm i}\lambda E_T^+-{\rm i}\lambda E_T^-=-{\rm i}\lambda h\quad\text{on }\Gamma_I$$
$$\nu\times{\rm curl}\,E^+-\nu\times{\rm curl}\,E^-={\rm i}\lambda h\quad\text{on }\Gamma_I$$

where the superscripts $+$ and $-$ indicate the limit obtained from Ω and $\mathbb{R}^3\backslash\overline{\Omega}$, respectively. It should be remarked that, since $\widetilde{f}\in H_{\rm curl}^{-1/2}(\Gamma)$, the first integral over Γ in the definition of $E(y)$ is well-defined with a $H^{-1/2}(\Gamma)$ density (see [17]) and the corresponding jump conditions are interpreted in the sense of L^2 limit. Thus, combining these jump relations and using the fact that $\nu\times E^+=\nu\times{\rm curl}\,E^+=0$ we obtain that

$$ {\rm curl}\,{\rm curl}\,E-k^2E=0\quad y_3<f(y_1,y_2)$$
$$\nu\times E^-=0\quad\text{on }\Gamma_D$$
$$\nu\times{\rm curl}\,E^-+{\rm i}\lambda E_T^-=0\quad\text{on }\Gamma_I$$

Noting that the unite normal ν points into Ω, the application of the uniqueness to the above problem yields that $E(y)\equiv0$ for $y_3<f(x_1,x_2)$. Thus,

$$f=[{\rm curl}\,E]_{\Gamma_D}=0,\qquad h=-[\nu\times E]_{\Gamma_I}=0$$

where $[\cdot]_{\Gamma_A}$ stands for the jump across Γ_A of a function with $A=D,I$. The proof of Lemma 4.2 is then completed. □

Proof of Theorem 4.1.
Theorem 4.1 can be proved by contradiction. Suppose $f_1\neq f_2$. Without loss of generality we may choose $z_0=(z_{01},z_{02},z_{03})\in\Gamma_1$ such that $z_{03}>f_2(z_{01},z_{02})$ and $z_0+e_3\varepsilon$ lies above the surface Γ_1 and Γ_2 for any $\varepsilon>0$, where $\Gamma_j=\{x_3=f_j(x_1,x_2)\,|\,0<x_1,x_2<2\pi\}$ with $j=1,2$. From the assumption that $\nu\times E_1^s(x;y)|_{\Gamma_b}=\nu\times E_2^s(x;y)|_{\Gamma_b}$ for all incident waves $E_y^{\rm in}\in\bigcup_{i=1}^3V_{P_i}$, it follows by the uniqueness of the exterior Dirichlet problem and the analytic continuation that $E_1^s(x;y)=E_2^s(x;y)$ in $\Omega_1\cap\Omega_2$. From Theorem 3.1 and Lemma 4.2 it is easy to see that for any $\varepsilon>0$

$$E_1^s(x;z_0+e_3\varepsilon)=E_2^s(x;z_0+e_3\varepsilon)\quad\text{in }\ \Omega_1\cap\Omega_2$$

However, since $z_0\in\Gamma_1$, we have

$$\lim_{\varepsilon\to0}\|\nu\times E_1^s(x;z_0+e_3\varepsilon)\|_{L_t^2(\Gamma_1)}=+\infty$$

which contradicts the fact that

$$\lim_{\varepsilon\to 0}\|\nu\times E_2^s(x;z_0+e_3\varepsilon)\|_{L_t^2(\Gamma_1)}<+\infty$$

Consequently, $f_1=f_2$, that is, Γ_1 coincides with Γ_2. Hence, we have $E^s_{1,y_n}=E^s_{2,y_n}:=E^s_{y_n}$ in Ω and $\nu\times E^s_{1,y_n}=\nu\times E^s_{2,y_n}$, $\nu\times\mathrm{curl}\,E^s_{1,y_n}=\nu\times\mathrm{curl}\,E^s_{2,y_n}$ on Γ. We claim that $\Gamma_{1D}\cap\Gamma_{2I}$ must be empty since, otherwise, a similar argument as above deduces that the total electric field vanishes in Ω, which is impossible.

Finally, it is seen from the boundary condition

$$\nu\times\mathrm{curl}\,(E^{\mathrm{in}}_{y_n}+E^s_{y_n})-\mathrm{i}\lambda_j(E^{\mathrm{in}}_{y_n}+E^s_{y_n})_T=0 \quad \text{on } \Gamma_I,\ j=1,2$$

that $(\lambda_1-\lambda_2)(E^{\mathrm{in}}_{y_n}+E^s_{y_n})_T=0$, which implies that $\lambda_1=\lambda_2$. The proof of Theorem 4.1 is thus completed.

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References

1. Petit R (ed.). *Electromagnetic Theory of Gratings*. Springer: Berlin, 1980.
2. Abboud T. Formulation variationnelle des équations de Maxwell dans un réseau bipéiodique de R^3. *Comptes Rendus de l'Académie Des Sciences* 1993; **317**:245–248.
3. Ammari H. Uniqueness theorems for an inverse problems in a doubly periodic structure. *Inverse Problems* 1995; **11**:823–833.
4. Bao G. Variational approximation of Maxwell's equations in biperiodic structures. *SIAM Journal on Applied Mathematics* 1997; **57**:364–381.
5. Dobson D. A variational method for electromagnetic diffraction in biperiodic structures. *Mathematical Modelling and Numerical Analysis* 1994; **28**:419–439.
6. Dobson D, Friedman A. The time-harmonic Maxwell equations in a doubly periodic structure. *Journal of Mathematical Analysis and Applications* 1992; **166**:507–528.
7. Nédélec JC, Starling F. Integral equation method in a quasi-periodic diffraction problem for the time-harmonic Maxwell equations. *SIAM Journal on Mathematical Analysis* 1991; **22**:1679–1701.
8. Bao G, Zhou Z. An inverse problem for scattering by a doubly periodic structure. *Transactions of the American Mathematical Society* 1998; **350**: 4089–4103.
9. Colton D, Kress R. *Inverse Acoustic and Electromagnetic Scattering Theory* (2nd edn). Springer: Berlin, 1998.
10. Kress R. Uniqueness in inverse obstacle scattering for electromagnetic waves. *Proceedings of the URSI General Assembly*, Maastricht, 2002. Available from: http://www.num.math.uni-goettingen.de/kress/researchlist.html.
11. Buffa A, Costabel M, Sheen D. On traces for $H(\mathbf{curl}\ \Omega)$ in Lipschitz domains. *Journal of Mathematical Analysis and Applications* 2002; **276**:847–867.
12. Cakoni F, Colton D, Monk P. The electromagnetic inverse scattering problem for partially coated Lipschitz domains. *Proceedings of the Royal Society of Edinburgh* 2004; **134A**:661–682.
13. Monk P. *Finite Element Method for Maxwell's Equations*. Oxford University Press: Oxford, 2003.
14. Kirsch A, Kress R. Uniqueness in inverse obstacle scattering. *Inverse Problems* 1993; **9**:285–299.
15. Chen Z, Du Q, Zou J. Finite element methods with matching and nonmatching meshes for Maxwell's equations with discontinuous coefficients. *SIAM Journal on Numerical Analysis* 2000; **37**:1542–1570.
16. Leis R. *Initial Boundary Value Problems in Mathematical Physics*. Wiley: Chichester, 1986.
17. Mclean W. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press: Cambridge, 2000.