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Multiple scattering of electromagnetic waves by finitely many point-like obstacles

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This paper is concerned with the direct and inverse time-harmonic electromagnetic scattering problems for a finite number of isotropic point-like obstacles in three dimensions. In the first part, we show that the representation of scattered fields obtained using the Foldy physical assumption "on the proportionality of the strength of the scattered wave on a given scatterer to the external field on it" is the same as the one derived from the model corresponding to the scattering by Dirac-like refraction indices. Using the regularization approach known in quantum mechanics, we rigorously deduce the solution operator (Green's tensor) of the last model in appropriate weighted spaces. Intermediate levels of the scattering between the Born and Foldy models are also described. In the second part, we apply the MUSIC algorithm to the inverse problem of detecting both the position of point-like scatterers and the scattering coefficients attached to them from the far-field measurements of finitely many incident plane waves, with an emphasis on discussing the effect of multiple scattering.

Keywords: Electromagnetic scattering; point-like scatterers; multiple scattering; MUSIC algorithm.

AMS Subject Classification: 35J08, 35Q61, 45Q05

1. Introduction

We consider the scattering of a time-harmonic electromagnetic plane wave from an inhomogeneous isotropic medium in $\mathbb{R}^3$ with electric permittivity $\epsilon = \epsilon(x) > 0$, magnetic permeability $\mu = \mu_0 > 0$ and electric conductivity $\sigma = \sigma(x)$. It is supposed that the inhomogeneous medium occupies a bounded domain such that $\epsilon(x) = \epsilon_0 > 0$ and $\sigma(x) = 0$ for x outside of some sufficiently large ball. Assume that the time-harmonic incident plane waves (with the time-form $\exp(-i\omega t)$) take the form

$$E^i(x;\theta,p) = p\exp(i\kappa x\cdot\theta), \quad H^i(x;\theta,p) = (\theta\times p)\exp(i\kappa x\cdot\theta), \quad \theta\perp p,$$

where $\kappa := \omega\sqrt{\epsilon_0\mu_0} > 0$ is the wavenumber corresponding to the background medium and $\theta, p \in \mathbb{S}^2 := \{x : |x| = 1\}$ stand for the propagation and polarization directions, respectively. Then, the total electric and magnetic fields E, H satisfy the reduced time-harmonic Maxwell equations

$$\operatorname{curl} E - i\kappa H = 0, \quad \operatorname{curl} H + i\kappa n(x) E = 0, \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where the refractive index $n = n(x)$ is given by

$$n(x) := \frac{1}{\epsilon_0}\left(\epsilon(x) + i\frac{\sigma(x)}{\omega}\right).$$

The scattered fields $E^s := E - E^i, H^s := H - H^i$ are required to satisfy the Silver–Müller radiation condition

$$\lim_{|x|\to\infty}(H^s\times x - |x|E^s) = 0,$$

uniformly in all directions $\hat{x} := \frac{x}{|x|} \in \mathbb{S}^2$, leading to the electric and magnetic far-field patterns $E^\infty(\hat{x}), H^\infty(\hat{x})$ given by the asymptotic behavior

$$\begin{aligned} E^s(x) &= \frac{e^{i\kappa|x|}}{|x|}\left\{E^\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \\ H^s(x) &= \frac{e^{i\kappa|x|}}{|x|}\left\{H^\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \end{aligned} \tag{1.2}$$

as $|x| \to \infty$. It is well known that E^∞ and H^∞ are both analytic functions defined on $\mathbb{S}^2$, satisfying $E^\infty \perp H^\infty$, and that they are both tangent to $\mathbb{S}^2$.

In this paper we assume that the inhomogeneous medium consists of a finite number of components and that the wavelength of incidence is much larger than the diameter of each component. The inhomogeneous medium in this situation can be regarded as the collection of a finite number of point-like obstacles centered at $y_j, j = 1, 2, \ldots, M$. These point-like obstacles are treated as isotropic,[a] so we can

[a]In other words, we allow only local interactions between them.

write the refractive index in the form

$$(n(x) - 1) = \sum_{j=1}^{M} a'_j \, \delta(x - y_j), \tag{1.3}$$

where the constant $a'_j \in \mathbb{C}$, $j = 1, \dots, M$, is the polarizability describing the scattering coefficient attached to the scatterer located at y_j, see (16) in Ref. 13. Eliminating the magnetic field from (1.1) and making use of (1.3), we find that $E = E^i + E^s$ satisfies the problem

$$\begin{cases} \operatorname{curl}\operatorname{curl} E - \kappa^2 E - \sum_{j=1}^{M} a_j \delta(x - y_j) E = 0, \quad x \text{ in } \mathbb{R}^3, \\ \operatorname{curl}\ E^s(x) - i\kappa\, |x|\, E^s(x) = o(1), \quad |x| \to \infty \text{ uniformly in all directions} \\ \qquad\qquad \hat{x} := x/|x| \quad \text{in } \mathbb{S}^2, \end{cases} \tag{1.4}$$

which models the electromagnetic scattering by M point-like obstacles, where $a_j := \kappa^2 a'_j$, $j = 1, \dots, M$. For convenience we set

$$\begin{aligned} H_\kappa E &:= \operatorname{curl}\operatorname{curl} E - \kappa^2 E - \sum_{j=1}^{M} a_j \delta(x - y_j) E, \\ y_j &= (y_{j_1}, y_{j_2}, y_{j_3})^\top \in \mathbb{R}^3 \end{aligned} \tag{1.5}$$

and $Y := \{y_1, y_2, \dots, y_M\}$.

We refer the reader to Ref. 21 for a comprehensive study of the multiple scattering in general and the scattering by point-like scatterers in particular, where practical motivations of the corresponding models and historical facts are discussed. Another close reference to our work is Ref. 1 concerning the scattering by point-like potentials in quantum mechanics, where applications to many different areas and historical references are provided. In contrast to significant progress made for the Helmholtz equation, see Refs. 1 and 21, as far as we know, only relatively little mathematical analysis for the Maxwell equations has been carried out and most of the literature come from physical and engineering community.[10,13,18,19,25,27,29] We refer to Refs. 7 and 6 for a rigorous asymptotic analysis of the multiple electromagnetic scattering by a finite number small obstacles, see also Ref. 24 for related results, using integral equation methods and to Refs. 26 and 14 using the Krein resolvent formula. A physical overview of the applications related to the model (1.4) can be found in Ref. 13, see also Ref. 5 where a different but related model is considered with a discussion on the regularity of the solutions.

The contributions of this paper are twofold. First, we establish a rigorous solvability theory for the scattering problem (1.4). To obtain the scattered waves corresponding to plane wave incidences, it is enough to provide the Green's function of the system (1.4), see Sec. 2.2.4. This Green's function, given in Theorem 2.1, is the kernel of the solution operator of $H_\kappa(u^f) = f$ in appropriate spaces. A general idea to derive such Green's function in the whole space is to Fourier transform H_κ and then Fourier transform back the inverse of the resulting operator. By virtue of the Dirac-like potentials occurring (1.5), we get a finite rank perturbation of a multiplication operator in the Fourier variables, which cannot be inverted in a straightforward way. An idea to overcome this difficulty is to first "regularize" the obtained operator in the Fourier variables, then use the Weinstein–Aronszajn's determinant formula to invert this operator and finally Fourier transform back the inverted operator. This is known as the *regularization approach* in the framework of the solvable models in quantum mechanics, see, for instance, Ref. 1. However, applying such an approach to the Maxwell system is not trivial, compared to the acoustic model or the elastic system, mainly due to a higher (and non-integrable) singularity of the Green's tensor to the Maxwell equations. To handle this singularity, we adopt an idea of Ref. 13 where the problem (1.2)–(1.4) is studied in the case of one point-scatterer and a formal computation of the scattering matrix is shown. This idea consists of decomposing the Green's tensor into its longitudinal and transversal parts, see (2.15)–(2.16) and then regularize them in the Fourier variables, see (2.26). After that, in the Fourier variables, we restrict ourselves to the tangential fields (which corresponds to taking divergence free sources f) and invert the obtained operator. As a consequence we derive an explicit form of the Green's tensor of the problem (1.4) from which the representation of far fields corresponding to plane incident waves follows, see (2.45). This representation turns out to be nothing but the scattered field obtained using the Foldy physical assumption "on the proportionality of the strength of the scattered wave on a given scatterer y_j to the external field on it", see Sec. 2.1, after adjusting accordingly the scattering strengths, i.e. taking the scattering coefficient g_j in the Foldy model as the coefficients $(a_j^{-1} - b_j)^{-1}$ with b_j appearing in (2.25), cf. (2.11) with (2.45) and see Remark 2.1. In addition, we retrieve the results in Ref. 13 as a special case, see Remark 2.2. The representation of the far field takes into account the multiple scattering between the point scatterers. Based on this model, we then describe the intermediate scattering models and the Born model as well. The analogue results for the Lamé system are shown in Ref. 17. It is worth mentioning here the work[8] where the point-like model (2.45), for the near fields, is obtained by approximating scattering from spherical well-separated inclusions. In addition, we should emphasize that if the inclusions are not spherical then the approximating model describes the scattering by anisotropic scatterers, since the corresponding polarization tensors are anisotropic. For such scatterers, our model, which describes isotropic scatterers due to the type of contrast in (1.3), is not appropriate.

Second, we study the inverse scattering problem consisting of recovering the point-scatterers as well as the attached scattering coefficients from the far fields corresponding to finitely many incident plane waves. For this, we use the three different models given by Born, Foldy and intermediate levels and discuss the effect of the multiple scattering on the resolution of the reconstructions in terms of the used wavelength, the distance between the scatterers and the scattering coefficients.

This study is a continuation of the one provided in Ref. 11 for the acoustic and elastic cases.

The paper is organized as follows. In Sec. 2.1, we briefly review the model of Foldy for electromagnetic scattering by point-like obstacles. In Sec. 2.2, we apply the regularization method to derive the Green's tensor to the electromagnetic scattering by Dirac-like refraction indices. Our main results concerning the forward model are summarized in Theorem 2.1 at the end Sec. 2.2.3. Section 3 is devoted to the studies of intermediate models and the Born approximation. The inverse problems related to these models are investigated in Sec. 4. Some technical identities used in our analysis will be proved in the Appendix.

We finish this section by introducing some notations that will be used throughout the paper. Denote by $(\cdot)^\top$ the transpose of a vector or a matrix, and by $(\cdot)^*$ the transpose and conjugate of a matrix. The symbols $e_j, j = 1, 2, 3$, denote the Cartesian unit vectors in $\mathbb{R}^3$. For $a \in \mathbb{C}$, let $|a|$ denote its modulus, and for $\mathbf{a} \in \mathbb{R}^3$, let $|\mathbf{a}|$ denote its Euclidean norm. The notation $\mathbf{a} \cdot \mathbf{b}$ stands for the inner product $\sum_{j=1}^3 a_j b_j$ of $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3) \in \mathbb{C}^3$. Let X' be the dual space of a Hilbert space X; then the set of bounded linear operators from X to Y will be denoted by $\mathcal{L}(X, Y)$.

2. The Forward Problem

We introduce the dyadic Green's function Π_κ for the Maxwell equations in the homogeneous isotropic background. It is well known that Π_κ takes the form (see, e.g. Chap. 12 in Ref. 22 and Theorem 5.2.1 in Ref. 23)

$$\Pi_\kappa(x, y) = \Phi_\kappa(x, y)\mathbf{I} + \frac{1}{\kappa^2}\nabla_y\nabla_y\Phi_\kappa(x, y) \in \mathbb{C}^{3\times 3}, \quad x \neq y, \tag{2.1}$$

satisfying

$$\operatorname{curl}_y \operatorname{curl}_y \Pi_\kappa(x, y) - \kappa^2\Pi_\kappa(x, y) = \delta(x - y)\mathbf{I}, \quad x \neq y, \tag{2.2}$$

where the notation $\mathbf{I}$ stands for the 3×3 identity matrix, $\Phi_\kappa(x, y) := (4\pi)^{-1}\exp(i\kappa|x-y|)/|x-y|$ is the fundamental function to the Helmholtz equation, i.e. $(\Delta + \kappa^2)\Phi_\kappa(x, y) = -\delta(x - y)$ in $\mathbb{R}^3$, and $\nabla_y\nabla_y\Phi_\kappa(x, y)$ is the Hessian matrix for Φ_κ defined by

$$(\nabla_y\nabla_y\Phi_\kappa(x, y))_{j,l} = \frac{\partial^2\Phi}{\partial y_j \partial y_l}, \quad 1 \le j, \ l \le 3.$$

Note that $\operatorname{curl}\Pi_\kappa$ is understood as the application of curl to each column of Π_κ. A simple calculation shows that each column of Π_κ satisfies the Silver–Müller radiation condition, leading to the far-field matrix $\Pi_\kappa^\infty(\hat{x}; y)$ of $\Pi_\kappa(x, y)$ as $|x| \to \infty$ given by

$$\Pi_\kappa^\infty(\hat{x}; y) = \frac{e^{-i\kappa\hat{x}\cdot y}}{4\pi}(\mathbf{I} - \hat{x} \otimes \hat{x}), \tag{2.3}$$

where $\hat{x} \otimes \hat{x} := \hat{x}\hat{x}^\top \in \mathbb{R}^{3\times 3}$.

2.1. *Foldy's model for the electromagnetic scattering by point-like obstacles*

Following the Foldy method (see Ref. 15 or p. 298 in Ref. 21 for the acoustic case), we represent the total field as

$$E(x) = E^i(x) + \sum_{j=1}^{M} \Pi_\kappa(x, y_j) A_j, \tag{2.4}$$

where A_j are unknown constants. The field

$$E_j(x) = E(x) - \Pi_\kappa(x, y_j) A_j = E^i(x) + \sum_{\substack{l=1 \\ l \neq j}}^{M} \Pi_\kappa(x, y_l) A_l, \tag{2.5}$$

is regarded as the external field incident on the jth scatterer in the presence of all the other scatterers. The physical assumption in Foldy method is that the strength of the scattered wave from the scatterer y_j is proportional to the external field on it. In our case this is given by the assumption that

$$A_j = g_j E_j(y_j), \tag{2.6}$$

where g_j is called the scattering coefficient of the scatterer y_j. Evaluating (2.5) at y_j, we obtain

$$E_j(y_j) = E^i(y_j) + \sum_{\substack{l=1 \\ l \neq j}}^{M} g_l \Pi_\kappa(y_j, y_l) E_l(y_l) \tag{2.7}$$

and then (2.4) becomes

$$E(x) = E^i(x) + \sum_{j=1}^{M} g_j \Pi_\kappa(x, y_j) E_j(y_j). \tag{2.8}$$

Following Ref. 15, we call the system (2.7) the fundamental system of multiple scattering.

The equations in (2.7) can be written as the algebraic linear system

$$[\tilde{\Gamma}]_{3M \times 3M} [\Lambda]_{3M \times 1} = [\mathbf{E}^I]_{3M \times 1}, \tag{2.9}$$

with $\Lambda := (E_1(y_1)^\top, E_2(y_2)^\top, \ldots, E_M(y_M)^\top)^\top \in \mathbb{C}^{3M \times 1}$, $\mathbf{E}^I := (E^i(y_1)^\top, \ldots, E^i(y_M)^\top)^\top \in \mathbb{C}^{3M \times 1}$ and

$$\tilde{\Gamma} := \tilde{\Gamma}(\kappa)$$
$$= \begin{pmatrix} \mathbf{I} & -g_2 \Pi_\kappa(y_1, y_2) & -g_3 \Pi_\kappa(y_1, y_3) & \cdots & -g_M \Pi_\kappa(y_1, y_M) \\ -g_1 \Pi_\kappa(y_2, y_1) & \mathbf{I} & -g_3 \Pi_\kappa(y_2, y_3) & \cdots & -g_M \Pi_\kappa(y_2, y_M) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -g_1 \Pi_\kappa(y_M, y_1) & -g_2 \Pi_\kappa(y_M, y_2) & -g_3 \Pi_\kappa(y_M, y_3) & \cdots & \mathbf{I} \end{pmatrix}.$$

Assuming $\det(\tilde{\Gamma}) \neq 0$ and denoting the 3×3 blocks of $\tilde{\Gamma}^{-1} \in \mathbb{C}^{3M\times 3M}$ by $[\tilde{\Gamma}^{-1}]_{lj}$ for $l, j = 1, 2, \ldots, M$, we deduce from (2.4) that the scattered field takes the form

$$E^s(x) := E(x) - E^i(x) = \sum_{l,j=1}^{M} g_j \Pi_\kappa(x, y_j)[\tilde{\Gamma}^{-1}]_{jl} E^i(y_l), \tag{2.10}$$

with the far-field pattern, using (2.3)

$$E^\infty(\hat{x}) = \frac{1}{4\pi} \sum_{l,j=1}^{M} g_j e^{-i\kappa\hat{x}\cdot y_j} e^{i\kappa\,\theta\cdot y_l} (\mathbf{I} - \hat{x} \otimes \hat{x})[\tilde{\Gamma}^{-1}]_{jl} p, \quad \hat{x} \in \mathbb{S}^2. \tag{2.11}$$

The Foldy method provides us explicit formulas (2.10) and (2.11) of the scattered near and far fields in terms of the point-like obstacles y_j and the scattering coefficients g_j, under the hypothesis of the invertibility of $\tilde{\Gamma}$ and the assumption (2.6). In the rest of this section, we will rigorously derive (2.10) and (2.11) from the model (1.4), establishing a relation between the Foldy method and the regularized approach, see Remark 2.1. Our argument, in this sense, provides a theoretical justification of Foldy's fundamental system (2.7)–(2.8) for idealized point-like obstacles. The invertibility of $\tilde{\Gamma}$ will be discussed in Sec. 4.2.

2.2. *The electromagnetic scattering by Dirac-like refraction indices*

The purpose of this section is to derive the Green's tensor to the solution operator of H_κ (see (1.5)) using the regularization approach. As mentioned in Sec. 1, we first Fourier transform the operator H_κ in Sec. 2.2.1, and then apply the Weinstein–Aronszajn's determinant formula to invert the "regularized" operator by choosing appropriate coupling coefficients in the Fourier variables, see Sec. 2.2.2. The inverted operator will be Fourier transformed back to the origin space variables in Sec. 2.2.3. Finally, in Sec. 2.2.4 we deduce the scattered near and far fields for the scattering problem (1.4) from the Green's tensor.

2.2.1. *The model in the Fourier variables*

Define the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3)^3$ by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) := (2\pi)^{-3/2} \lim_{R\to\infty} \int_{|x|\leq R} f(x) e^{-ix\cdot\xi} dx, \quad \xi = (\xi_1, \xi_2, \xi_3)^\top \in \mathbb{R}^3.$$

Its inverse transform is given by

$$(\mathcal{F}^{-1}g)(x) := (2\pi)^{-3/2} \lim_{R\to\infty} \int_{|\xi|\leq R} g(\xi) e^{ix\cdot\xi} d\xi.$$

For $u = (u_1, u_2, u_3)^\top \in L^2(\mathbb{R}^3)^3$, a simple calculation shows

$$\mathcal{F}(\operatorname{curl}\operatorname{curl} u) = (|\xi|^2\mathbf{I} - \xi \otimes \xi)\hat{u} = |\xi|^2(\mathbf{I} - \hat{\xi} \otimes \hat{\xi})\hat{u}, \quad \hat{\xi} = \xi/|\xi|,$$

where $\hat{u} := \mathcal{F}u = (\hat{u}_1, \hat{u}_2, \hat{u}_3)^\top$. Define

$$\mathcal{M}_\kappa(\xi) := |\xi|^2(\mathbf{I} - \hat{\xi} \otimes \hat{\xi}) - \kappa^2 \mathbf{I} = (|\xi|^2 - \kappa^2)\mathbf{I} - \xi \otimes \xi \in \mathbb{R}^{3\times 3}.$$

It is easy to check that the inverse matrix of $\mathcal{M}_\kappa$ takes the form

$$\mathcal{M}_\kappa^{-1}(\xi) = \frac{1}{|\xi|^2 - \kappa^2}\left(\mathbf{I} - \frac{1}{\kappa^2}\xi \otimes \xi\right), \tag{2.12}$$

and that (cf. (2.2))

$$(2\pi)^{-3/2}\mathcal{F}^{-1}[\mathcal{M}_\kappa^{-1}(\xi)] = \left(\mathbf{I} + \frac{1}{\kappa^2}\nabla_x \nabla_x\right)\frac{e^{i\kappa|x|}}{4\pi|x|} = \Pi_\kappa(x, 0), \quad x \neq 0. \tag{2.13}$$

Following Ref. 13, we decompose $\mathcal{M}_\kappa^{-1}(\xi)$ into the sum

$$\begin{aligned} \mathcal{M}_\kappa^{-1}(\xi) \;&=\; T_\kappa(\xi) + L_\kappa(\xi), \quad T_\kappa := \frac{1}{|\xi|^2 - \kappa^2}(\mathbf{I} - \hat{\xi} \otimes \hat{\xi}), \\ L_\kappa &:= -\frac{1}{\kappa^2}\hat{\xi} \otimes \hat{\xi}, \end{aligned} \tag{2.14}$$

where T_κ, and L_κ denote the transverse and longitudinal parts with respect to ξ, respectively. Accordingly, the dyadic Green's function $\Pi_\kappa(x, 0)$ admits the decomposition

$$\Pi_\kappa(x, 0) = \Pi_\kappa^T(x) + \Pi_\kappa^L(x), \quad x \neq 0,$$

with (see Part II in Ref. 13 or the Appendix)

$$\Pi_\kappa^L(x) := (2\pi)^{-3/2}\mathcal{F}^{-1}[L_\kappa(\xi)] = -\frac{\mathbf{I} - 3\hat{x} \otimes \hat{x}}{4\pi\kappa^2|x|^3}, \tag{2.15}$$

$$\begin{aligned} \Pi_\kappa^T(x) &:= (2\pi)^{-3/2}\mathcal{F}^{-1}[T_\kappa(\xi)] \\ &= \frac{\mathbf{I} - 3\hat{x} \otimes \hat{x}}{4\pi\kappa^2|x|^3} + \frac{e^{i\kappa|x|}}{4\pi|x|}[P(i\kappa|x|)\mathbf{I} + Q(i\kappa|x|)\hat{x} \otimes \hat{x}], \end{aligned} \tag{2.16}$$

where $P(t) = 1 - 1/t + 1/t^2$, and $Q(t) = -1 + 3/t - 3/t^2$.

To Fourier transform the operator H_κ, we set $\widetilde{H}_\kappa(f) := \mathcal{F}H_\kappa\mathcal{F}^{-1}(f)$ for $f = (f_1, f_2, f_3)^\top \in L^2(\mathbb{R}^3)^3$. Obviously, there holds

$$[\mathcal{F}(\mathrm{curl}\,\mathrm{curl}\, - \kappa^2)\mathcal{F}^{-1}]\hat{f} = \mathcal{M}_\kappa(\xi)\hat{f}$$

and formally

$$(\mathcal{F}\delta(x - y_j)\mathcal{F}^{-1}\hat{f})(\xi) = (\mathcal{F}\delta(x - y_j)f)(\xi) = \sum_{m=1}^{3}\langle \hat{f}, \varphi_{y_j}^m\rangle \varphi_{y_j}^m(\xi),$$

where $\varphi_{y_j}^m(\xi) := \phi_{y_j}(\xi)e_m \in \mathbb{C}^{3\times 1}$, $m = 1, 2, 3$ with $\phi_{y_j}(\xi) := (2\pi)^{-3/2}e^{-iy_j\cdot\xi}$. Here we used the inner product $\langle \hat{f}, \hat{g}\rangle := \int_{\mathbb{R}^3}\hat{f}(\xi)\cdot\overline{\hat{g}(\xi)}d\xi$ for $\hat{f}, \hat{g} \in L^2(\mathbb{R}^3)^3$. Therefore, by the definitions of H_κ and $\widetilde{H}_\kappa$, in the Fourier variables we obtain

$$\widetilde{H}_\kappa(\hat{f})(\xi) = \mathcal{M}_\kappa(\xi)\hat{f} - \sum_{j=1}^{M}\sum_{m=1}^{3}\langle a_j\hat{f}, \varphi_{y_j}^m\rangle\varphi_{y_j}^m(\xi), \tag{2.17}$$

which is a finite rank perturbation of the multiplication operator $\mathcal{M}_\kappa(\xi)$.

2.2.2. *The regularization of the model in the Fourier variables*

Since it is not easy to prove the existence of $\widetilde{H}_\kappa^{-1}$ in a straightforward way, in particular the functions $\varphi_{y_j}^m, j = 1, \ldots, M$, are not square integrable, we will regularize the operator $\widetilde{H}_\kappa$. To make the computations rigorous, we introduce the cut-off function

$$\chi_\epsilon(\xi) = \begin{cases} 1 & \text{if } |\xi| \le 1/\epsilon, \\ 0 & \text{if } |\xi| > 1/\epsilon, \end{cases} \quad \text{for some } 0 < \epsilon < 1$$

and define the regularized operator (cf. (2.17))

$$\begin{aligned} \widetilde{H}_{\kappa_\alpha}^\epsilon \hat{f} &:= \mathcal{M}_{\kappa_\alpha}(\xi)\hat{f} - \sum_{j=1}^{M}\sum_{m=1}^{3}\langle a_j(\epsilon)\hat{f}, \varphi_{y_j}^{\epsilon,m}\rangle\varphi_{y_j}^{\epsilon,m}(\xi), \\ \varphi_{y_j}^{\epsilon,m}(\xi) &:= \chi_\epsilon(\xi)\varphi_{y_j}^m(\xi), \end{aligned} \tag{2.18}$$

where $\kappa_\alpha := \kappa + i\alpha$ with $\alpha > 0$. The essence of the regularization approach in quantum mechanics (see Ref. 1) is to choose the coupling constants $a_j(\epsilon)$ in a suitable way such that $(\widetilde{H}_{\kappa_\alpha}^\epsilon)^{-1}$ has a reasonable limit as ϵ tends to zero in appropriate spaces. Let us first recall the Weinstein–Aronszajn determinant formula from Lemma B.5 of Ref. 1, which is our main tool for analyzing the inverse of $\widetilde{H}_{\kappa_\alpha}^\epsilon$.

Lemma 2.1. *Let $\mathcal{H}$ be a* (*complex*) *separable Hilbert space with a scalar product $\langle\cdot,\cdot\rangle$. Let A be a closed operator in $\mathcal{H}$ and Φ_j, $\Psi_j \in \mathcal{H}$, $j = 1, \ldots, M'$. Then*

$$\begin{aligned} &\left(A + \sum_{j=1}^{M'}\langle\cdot, \Phi_j\rangle\Psi_j - z\right)^{-1} \\ &\quad = (A-z)^{-1} - \sum_{j'=1}^{M'}[\Gamma(z)]_{j,j'}^{-1}\langle\,\cdot,\, [(A-z)^{-1}]^*\Phi_{j'}\rangle(A-z)^{-1}\Psi_j, \end{aligned} \tag{2.19}$$

for z in the resolvent of A such that $\det[\Gamma(z)] \neq 0$, *with the entries of* $\Gamma(z)$ *given by*

$$[\Gamma(z)]_{j,j'} := \delta_{j,j'} + \langle (A-z)^{-1}\Psi_{j'}, \Phi_j\rangle. \tag{2.20}$$

Note that in Lemma 2.1, the notation $[\Gamma(z)]_{j,j'}^{-1}$ denotes the (j, j')th entry of the matrix $[\Gamma(z)]^{-1}$, and $[\ ]^*$ stands for the adjoint operator of $[\ ]$. To apply Lemma 2.1, we take

$$\mathcal{H} := L^2(\mathbb{R}^3)^3, \quad A(f) := |\xi|^2(\mathbf{I} - \hat{\xi}\otimes\hat{\xi})f(\xi), \quad M' := 3M, \quad z = \kappa_\alpha^2$$

and $\Phi_j := \Phi_j^\epsilon$, $\Psi_j = -\tilde{a}_j \Phi_j^\epsilon$ for $j = 1, \dots, 3M$, with $\tilde{a}_j$ and Φ_j^ϵ defined as follows:

$$\tilde{a}_j(\epsilon) = a_l(\epsilon) \quad \text{if } j \in \{3l-1, 3l-2, 3l\},$$

$$\Phi_j^\epsilon := \begin{cases} \varphi_{y_l}^{\epsilon,1} & \text{if } j = 3l-2, \\ \varphi_{y_l}^{\epsilon,2} & \text{if } j = 3l-1, \\ \varphi_{y_l}^{\epsilon,3} & \text{if } j = 3l, \end{cases} \tag{2.21}$$

for some $l \in \{1, 2, \dots, M\}$. The multiplication operator A is closed with a dense domain

$$\mathcal{D}(A) := \{f(\xi) \in L^2(\mathbb{R}^3)^3 : |\xi|^2(\mathbf{I} - \hat{\xi} \otimes \hat{\xi}) f(\xi) \in L^2(\mathbb{R}^3)^3\},$$

in $L^2(\mathbb{R}^3)^3$ hence $\widetilde{H}_{\kappa_\alpha}^\epsilon$, with $\epsilon > 0, \alpha > 0$, is also closed with the same domain. For a complex-valued number $\kappa + i\alpha$, one can observe that $\det(\mathcal{M}_{\kappa_\alpha}(\xi)) \neq 0$ for all $\xi \in \mathbb{R}^3$, so that $(\mathcal{M}_{\kappa_\alpha})^{-1}(\xi)$ always exists. Further, it holds that

$$[(\mathcal{M}_{\kappa_\alpha})^{-1}]^* = [(\mathcal{M}_{\overline{\kappa}_\alpha})]^{-1},$$

where $\overline{\kappa}_\alpha := \kappa - i\alpha$ denotes the conjugate of κ_α. Simple calculations show that

$$(A - z)^{-1}\Psi_j = -\tilde{a}_j (\mathcal{M}_{\kappa_\alpha})^{-1} \Phi_j^\epsilon,$$

$$\delta_{j,j'} + \langle (A - z)^{-1}\Psi_{j'}, \Phi_j \rangle = \tilde{a}_j [\tilde{a}_j^{-1} \delta_{j,j'} - \langle (\mathcal{M}_{\kappa_\alpha})^{-1} \Phi_{j'}^\epsilon, \Phi_j^\epsilon \rangle].$$

Therefore, by Lemma 2.1, we arrive at an explicit expression of the inverse of $\widetilde{H}_{\kappa_\alpha}^\epsilon$ given by

$$(\widetilde{H}_{\kappa_\alpha}^\epsilon)^{-1} \hat{f} = (\mathcal{M}_{\kappa_\alpha})^{-1} \hat{f} + \sum_{j,j'=1}^{3M} [\Gamma_\epsilon(\kappa_\alpha)]_{j,j'}^{-1} \langle \hat{f}, \chi_\epsilon F_{-\overline{\kappa}_\alpha}^{(j')} \rangle \chi_\epsilon F_{\kappa_\alpha}^{(j)}, \quad \alpha > 0, \tag{2.22}$$

with

$$\Gamma_\epsilon(\kappa_\alpha) := [\tilde{a}_j^{-1} \delta_{j,j'} - \langle (\mathcal{M}_{\kappa_\alpha})^{-1} \Phi_{j'}^\epsilon, \Phi_j^\epsilon \rangle]_{j,j'=1}^{3M}, \quad \chi_\epsilon F_{\kappa_\alpha}^{(j)} := \mathcal{M}_{\kappa_\alpha}^{-1} \Phi_j^\epsilon, \tag{2.23}$$

provided that $\det[\Gamma_\epsilon(\kappa_\alpha)] \neq 0$.

In order to get $\widetilde{H}_{\kappa_\alpha}^{-1}$ for the complex wavenumber κ_α, we need to remove the cut-off function in (2.22) by evaluating the limits of $\Gamma_\epsilon(\kappa_\alpha)$ and $\langle \hat{f}, \chi_\epsilon F_{-\overline{\kappa}_\alpha}^{(j')} \rangle \chi_\epsilon F_{\kappa_\alpha}^{(j)}$ as $\epsilon \to 0$. This will be done in the subsequent lemmas.

Lemma 2.2. *The coupling coefficients* $a_l(\epsilon)$ *appearing in* (2.18) *can be chosen in such a way that the limit* $\Gamma_{B,Y}(\kappa_\alpha) := \lim_{\epsilon \to 0} \Gamma_\epsilon(\kappa_\alpha)$ *exists and takes the form*

$$\Gamma_{B,Y}(\kappa_\alpha) = \begin{pmatrix} (a_1^{-1} - b_1)\boldsymbol{I} & -\Pi_{\kappa_\alpha}(y_1, y_2) & \cdots & -\Pi_{\kappa_\alpha}(y_1, y_M) \\ -\Pi_{\kappa_\alpha}(y_2, y_1) & (a_2^{-1} - b_2)\boldsymbol{I} & \cdots & -\Pi_{\kappa_\alpha}(y_2, y_M) \\ \vdots & \vdots & \ddots & \vdots \\ -\Pi_{\kappa_\alpha}(y_M, y_1) & -\Pi_{\kappa_\alpha}(y_M, y_2) & \cdots & (a_M^{-1} - b_M)\boldsymbol{I} \end{pmatrix}, \tag{2.24}$$

where a_js are the coefficients involved in (1.4) *and* $B := (b_1, b_2, \ldots, b_M)$ *with*

$$b_j := b_j(\beta_j, \gamma_j, \kappa_\alpha) = \frac{\beta_j + i\kappa_\alpha}{6\pi} \frac{\beta_j^2}{\beta_j^2 + \kappa_\alpha^2} - \frac{(\gamma_j/\sqrt{2})^3}{6\pi\kappa_\alpha^2}, \quad \beta_j, \gamma_j \in \mathbb{R}. \tag{2.25}$$

If in addition we assume that $a_j \in \mathbb{R}$, *then* $[\Gamma_{B,Y}(\kappa_\alpha)]^* = \Gamma_{B,Y}(-\overline{\kappa}_\alpha)$.

Proof. The proof will be carried out in the following three cases of $j, j' \in \{1, \ldots, 3M\}$.

Case 1: $|j' - j| = 1$, and $j, j' \in \{3l-2, 3l-1, 3l\}$ for some $l \in \{1, \ldots, M\}$.

Assume first that $j = 3l - 2$, $j' = 3l - 1$ for some $l = 1, \ldots, M$. Then, we have $\Phi_j^\epsilon = \chi_\epsilon \phi_{y_l} e_1$, $\Phi_{j'}^\epsilon = \chi_\epsilon \phi_{y_l} e_2$. Since $\varphi_{y_j}(\xi)\,\overline{\varphi_{y_j}(\xi)} = (2\pi)^{-3}$, there holds

$$\begin{aligned}\langle (\mathcal{M}_{\kappa_\alpha})^{-1} \Phi_j^\epsilon, \Phi_{j'}^\epsilon \rangle &= (2\pi)^{-3} \langle (\mathcal{M}_{\kappa_\alpha})^{-1} \chi_\epsilon e_1, \chi_\epsilon e_2 \rangle \\ &= (2\pi)^{-3} \int_{|\xi|<1/\epsilon} [(\mathcal{M}_{\kappa_\alpha})^{-1} e_1] \cdot e_2 d\xi = 0,\end{aligned}$$

where the last identity follows from the fact that $[(\mathcal{M}_{\kappa_\alpha})(\xi)^{-1} e_1] \cdot e_2$ is odd in ξ_j, see (2.12). By symmetry, we have also $\langle (\mathcal{M}_{\kappa_\alpha})^{-1} \Phi_{j'}^\epsilon, \Phi_j^\epsilon \rangle = 0$. The other cases of $j \neq j'$ and $j, j' \in \{3l-2, 3l-1, 3l\}$ can be proved analogously. This means that the off-diagonal terms of each diagonal-by 3×3-block of $\Gamma_{B,Y}$ vanish.

Case 2: $j = j' \in \{3l-2, 3l-1, 3l\}$ for some $l \in \{1, 2, \ldots, M\}$.

In this case, we introduce the regularization parameters $\beta_l, \gamma_l \in \mathbb{R}$ and define

$$\begin{aligned} U &= U(\epsilon, \kappa_\alpha, \beta_l, \gamma_l) \\ &:= \frac{1}{(2\pi)^3} \int_{|\xi|<1/\epsilon} \left[T_{\kappa_\alpha}(\xi) \frac{|\xi|^2}{\beta_l^2 + |\xi|^2} + L_{\kappa_\alpha}(\xi) \frac{|\xi|^4}{\gamma_l^4 + |\xi|^4} \right] d\xi \in \mathbb{C}^{3\times 3}. \end{aligned} \tag{2.26}$$

Employing spherical coordinate systems, we can readily deduce from the definitions of T_{κ_α} and L_{κ_α} that the diagonal terms of $U(\epsilon, \kappa_\alpha, \beta_l, \gamma_l)$ coincide with each other, which we denote by $U_l(\epsilon)$. Moreover, there holds the asymptotic behavior $U_l(\epsilon) \sim \mathcal{O}(\epsilon^{-3})$ as $\epsilon \to 0^+$. Now we define the coupling coefficient $a_l(\epsilon)$ appearing in (2.18) as

$$a_l(\epsilon) := (a_l^{-1} + U_l(\epsilon))^{-1}, \quad l = 1, 2, \ldots, M, \tag{2.27}$$

which converges to a_l as $\epsilon \to 0^+$. In view of (2.14), we see using the inverse Fourier transformation that

$$\begin{aligned} &\lim_{\epsilon\to 0} \left[U(\epsilon, \kappa_\alpha, \beta_l, \gamma_l) - \frac{1}{(2\pi)^3} \int_{|\xi|<1/\epsilon} (\mathcal{M}_{\kappa_\alpha})^{-1}(\xi) d\xi \right] \\ &\quad = \lim_{\epsilon\to 0} \left[U(\epsilon, \kappa_\alpha, \beta_l, \gamma_l) - \frac{1}{(2\pi)^3} \int_{|\xi|<1/\epsilon} (T_{\kappa_\alpha}(\xi) + L_{\kappa_\alpha}(\xi)) d\xi \right] \\ &\quad = -\lim_{\epsilon\to 0} \frac{1}{(2\pi)^3} \int_{|\xi|<1/\epsilon} \left[T_{\kappa_\alpha}(\xi) \frac{\beta_l^2}{\beta_l^2 + |\xi|^2} + L_{\kappa_\alpha}(\xi) \frac{\gamma_l^4}{\gamma_l^4 + |\xi|^4} \right] d\xi \\ &\quad = -b_l \mathbf{I}, \end{aligned} \tag{2.28}$$

where the constant $b_l = b_l(\kappa_\alpha, \beta_l, \gamma_l)$ is given in (2.25), see Ref. 13 or the Appendix for the details. Therefore, by (2.21) we get

$$\lim_{\epsilon\to 0}[\tilde{a}_j^{-1}(\epsilon) - \langle(\mathcal{M}_{\kappa_\alpha})^{-1}\Phi_j^\epsilon, \Phi_j^\epsilon\rangle] = a_l^{-1} - b_l, \quad j = 3l-2, 3l-1, 3l. \tag{2.29}$$

To sum up Cases 1 and 2, we conclude that the lth diagonal-by 3×3-block of the matrix $\Gamma_{B,Y} := \lim_{\epsilon\to 0}\Gamma_\epsilon(\omega)$ takes the form $(a_l^{-1} - b_l)\mathbf{I}$.

Case 3: $j \in \{3l-2, 3l-1, 3l\}$, $j' \in \{3l'-2, 3l'-1, 3l'\}$ for some $l, l' \in \{1, \ldots, M\}$ such that $|l-l'| \geq 1$, i.e. the element $[\Gamma_{B,Y}]_{j,j'}$ lies in the off-diagonal-by 3×3-block of $\Gamma_{B,Y}$.

Without loss of generality we assume $j = 3l-2, j' = 3l'-2$. Define the 3×3 matrix $\Upsilon_l := (\Phi_j^\epsilon, \Phi_{j+1}^\epsilon, \Phi_{j+2}^\epsilon) = \chi_\epsilon\phi_{y_l}\mathbf{I}$. A short computation shows

$$\begin{aligned}\langle(\mathcal{M}_{\kappa_\alpha})^{-1}\Upsilon_l, \Upsilon_{l'}\rangle &= \int_{|\xi|<1/\epsilon}(\mathcal{M}_{\kappa_\alpha})^{-1}\phi_{y_l}\,\overline{\phi_{y_{l'}}}d\xi \\ &= \frac{1}{(2\pi)^3}\int_{|\xi|<1/\epsilon}(\mathcal{M}_{\kappa_\alpha})^{-1}\exp[i(y_{l'} - y_l)\cdot\xi]d\xi,\end{aligned}$$

which, by the inverse Fourier transformation, converges to $\Pi_{\kappa_\alpha}(y_{l'}, y_l)$ as $\epsilon \to 0$.

Combining Cases 1–3 finally yields the matrix (2.24). □

To be consistent with the definitions of Φ_j^ϵ and $\chi_\epsilon F_\omega^{(j)}$, we introduce the functions

$$\Phi_j(\xi) := \begin{cases}(2\pi)^{-3/2}e^{-i\xi\cdot y_l}e_1 & \text{if } j = 3l-2, \\ (2\pi)^{-3/2}e^{-i\xi\cdot y_l}e_2 & \text{if } j = 3l-1, \\ (2\pi)^{-3/2}e^{-i\xi\cdot y_l}e_3 & \text{if } j = 3l,\end{cases} \quad F_{\kappa_\alpha}^{(j)} := (\mathcal{M}_{\kappa_\alpha})^{-1}\Phi_j(\xi), \tag{2.30}$$

for $l = 1, \ldots, M$. Before carrying out the convergence analysis of $(\tilde{H}_\alpha^\epsilon)^{-1}$, we introduce the tangential subspace

$$X := \{\hat{f} \in (L^2(\mathbb{R}^3))^3 : \xi\cdot\hat{f}(\xi) = 0,\ \text{a.e. in } \mathbb{R}^3\}$$

and define the operator

$$\mathbf{g}_{\kappa_\alpha}\hat{f} := (\mathcal{M}_{\kappa_\alpha}^{-1})\hat{f} + \sum_{j,j'=1}^{3M}[\Gamma_{B,Y}(\kappa_\alpha)]_{jj'}^{-1}\langle\hat{f}, F_{-\bar{\kappa}_\alpha}^{(j')}\rangle F_{\kappa_\alpha}^{(j)}, \quad \hat{f} \in X. \tag{2.31}$$

Lemma 2.3. *Let the coefficients $a_j(\epsilon)$ and b_j be given as in Lemma* 2.2 *such that* $\det[\Gamma_{B,Y}(\kappa_\alpha)] \neq 0$. *Then we have the convergence $(\widetilde{H}_\alpha^\epsilon)^{-1} \to \boldsymbol{g}_{\kappa_\alpha}$ as $\epsilon \to 0$ in the space $\mathcal{L}(X, X')$.*

Proof. It is seen from (2.12) and (2.23) that

$$\langle\hat{f}, \chi_\epsilon F_{\kappa_\alpha}^{(j)}\rangle = \left\langle\hat{f}, \frac{\Phi_j^\epsilon}{|\xi|^2 - \kappa_\alpha^2}\right\rangle, \tag{2.32}$$

since $\langle \hat{f}, (\xi \otimes \xi)\, \Phi_j^\epsilon \rangle = 0$ for $\hat{f} \in X$. Define the bilinear form $L_\alpha^\epsilon : X \times X \to \mathbb{C}$ by

$$L_\alpha^\epsilon(\hat{f}, \hat{g}) := \left\langle \frac{\hat{f}}{|\xi|^2 - \kappa_\alpha^2}, \hat{g} \right\rangle + \sum_{j,j'=1}^{3M} [\Gamma_\epsilon(\kappa_\alpha)]_{jj'}^{-1} \left\langle \hat{f}, \frac{\Phi_{j'}^\epsilon}{|\xi|^2 - \overline{\kappa}_\alpha^2} \right\rangle \left\langle \frac{\Phi_j^\epsilon}{|\xi|^2 - \kappa_\alpha^2}, \hat{g} \right\rangle < \infty, \tag{2.33}$$

for any $\hat{f}, \hat{g} \in X$. Then, using (2.22) and (2.32), we deduce

$$\langle (\widetilde{H}_\alpha^\epsilon)^{-1} \hat{f}, \hat{g} \rangle = L_\alpha^\epsilon(\hat{f}, \hat{g}), \quad \hat{f}, \hat{g} \in X.$$

Hence $(\widetilde{H}_\alpha^\epsilon)^{-1} \in \mathcal{L}(X, X')$. Analogously, we obtain $\mathbf{g}_{\kappa_\alpha} \in \mathcal{L}(X, X')$ because of the quadratic form

$$\begin{aligned} L_\alpha(\hat{f}, \hat{g}) &:= \langle \mathbf{g}_{\kappa_\alpha} \hat{f}, \hat{g} \rangle_{X,X'} = \langle \mathcal{M}_{\kappa_\alpha}^{-1} \hat{f}, \hat{g} \rangle + \sum_{j,j'=1}^{3M} [\Gamma_{B,Y}(\kappa_\alpha)]_{jj'}^{-1} \langle \hat{f}, F_{-\overline{\kappa}_\alpha}^{(j')} \rangle \langle F_{\kappa_\alpha}^{(j)}, \hat{g} \rangle \\ &= \left\langle \frac{1}{|\xi|^2 - \kappa_\alpha^2} \hat{f}, \hat{g} \right\rangle + \sum_{j,j'=1}^{3M} [\Gamma_{B,Y}(\kappa_\alpha)]_{jj'}^{-1} \left\langle \hat{f}, \frac{\Phi_{j'}}{|\xi|^2 - \overline{\kappa}_\alpha^2} \right\rangle \left\langle \frac{\Phi_j}{|\xi|^2 - \kappa_\alpha^2}, \hat{g} \right\rangle \\ &< \infty, \end{aligned} \tag{2.34}$$

for any $\hat{f}, \hat{g} \in X$ and $\alpha > 0$. The convergence $(\widetilde{H}_\alpha^\epsilon)^{-1} \to \mathbf{g}_{\kappa_\alpha}$ as $\epsilon \to 0$ follows from (2.33), (2.34), Lemma 2.2 and the fact that

$$\int_{\mathbb{R}^3} \left| \frac{\Phi_j - \Phi_j^\epsilon}{|\xi|^2 - \kappa_\alpha^2} \right|^2 d\xi = \int_{|\xi| > 1/\epsilon} \left| \frac{\Phi_j}{|\xi|^2 - \kappa_\alpha^2} \right|^2 d\xi \to 0 \quad \text{as } \epsilon \to 0. \qquad \square$$

2.2.3. *Back to the original space variables*

Having established the convergence of the inverse of the "regularized" operator $\widetilde{H}_\alpha^\epsilon$, we are now in a position to Fourier transform back the limiting operator $\mathbf{g}_{\kappa_\alpha}$ to the original space variables, and then analyze the convergence of the resulting operator as $\alpha \to 0$ using limiting absorption principle.

Recalling the formula

$$\sum_{j,j'=1}^{3} m_{j,j'} a_{j'} b_j = (a_1, a_2, a_3) \mathbf{M} (b_1, b_2, b_3)^\top, \quad \mathbf{M} = (m_{j,j'})_{j,j'=1}^3 \in \mathbb{C}^{3\times 3},$$

we can rewrite the second term on the right-hand side of (2.31) as

$$\begin{aligned} &\sum_{j=3l-2}^{3l} \sum_{j'=3l'-2}^{3l'} [\Gamma_{B,Y}(\kappa_\alpha)]_{j,j'}^{-1} \langle \hat{f}, F_{-\bar{\kappa}_\alpha}^{(j')} \rangle F_{\kappa_\alpha}^{(j)} \\ &\qquad = \Theta_{l,\kappa_\alpha}(\xi) [\Gamma_{B,Y}^{-1}(\kappa_\alpha)]_{l,l'} \int_{\mathbb{R}^3} [\Theta_{l',-\bar{\kappa}_\alpha}(t)]^* \hat{f}(t) dt, \end{aligned} \tag{2.35}$$

where $[\Gamma^{-1}_{B,Y}(\kappa_\alpha)]_{l,l'}$, $l, l' = 1, \ldots, M$, denote the 3×3 blocks of the matrix $[\Gamma_{B,Y}(\kappa_\alpha)]^{-1}$, and

$$\begin{aligned}\Theta_{l,\kappa_\alpha}(\xi) &:= (F^{(3l-2)}_{\kappa_\alpha}, F^{(3l-1)}_{\kappa_\alpha}, F^{(3l)}_{\kappa_\alpha}) = \mathcal{M}^{-1}_{\kappa_\alpha} \exp(-i\xi \cdot y_l)(2\pi)^{-3/2} \\ &= \mathcal{F}(\Pi_{\kappa_\alpha}(\cdot - y_l))(\xi). \end{aligned} \tag{2.36}$$

If $f \in (L^2(\mathbb{R}^3))^3$ such that div $f = 0$, then $\hat{f} \in X$ and

$$\begin{aligned}\int_{\mathbb{R}^3} [\Theta_{l',-\bar{\kappa}_\alpha}(\xi)]^* \hat{f}(\xi) d\xi &= (2\pi)^{-3/2} \int_{\mathbb{R}^3} \mathcal{M}^{-1}_{\kappa_\alpha}(\xi) \hat{f}(\xi) \exp(i\xi \cdot y_{l'}) d\xi \\ &= (2\pi)^{-3/2} \int_{\mathbb{R}^3} \frac{\hat{f}(\xi)}{|\xi|^2 - \kappa_\alpha^2} \exp(i\xi \cdot y_{l'}) d\xi \\ &= (2\pi)^{-3/2} \left[\mathcal{F}^{-1}\left(\frac{1}{|\cdot|^2 - \kappa_\alpha^2}\right) \star f\right](y_{l'}) \\ &= \Phi_{\kappa_\alpha} \star f(y_{l'}). \end{aligned} \tag{2.37}$$

Here and thereafter $\star$ denotes convolution, and recall that Φ_κ is the fundamental solution to the Helmholtz equation, i.e. $(\Delta + \kappa^2)\Phi_\kappa = -\delta$ in $\mathbb{R}^3$. In addition, using the Fourier transform,

$$\mathcal{M}^{-1}_{\kappa_\alpha}(\xi) \hat{f}(\xi) = \hat{f}(\xi)/(|\xi|^2 - \kappa_\alpha^2) = \mathcal{F}(\Phi_{\kappa_\alpha} \star f)(\xi). \tag{2.38}$$

Hence, taking the inverse Fourier transform in (2.31) and using (2.35), (2.36), (2.37) and (2.38), we arrive at

$$\mathcal{G}_{\kappa_\alpha} f := \mathcal{F}^{-1}(\mathbf{g}_{\kappa_\alpha} \hat{f}) = \Phi_{\kappa_\alpha} \star f(\cdot) + \sum_{l,l'=1}^{M} \Pi_{\kappa_\alpha}(\cdot - y_l)[\Gamma^{-1}_{B,Y}(\kappa_\alpha)]_{l,l'} \Phi_{\kappa_\alpha} \star f(y_{l'}). \tag{2.39}$$

In what follows we will analyze the convergence of $\mathcal{G}_{\kappa_\alpha}$ as $\alpha \to 0^+$ in appropriate spaces. For $\sigma > 1$, introduce the Agmon-type weighted spaces

$$\begin{aligned} L^2_\sigma &:= \{f : \|(1 + |x|^2)^{\sigma/2} f\|_{(L^2(\mathbb{R}^3))^3} < \infty\}, \\ X_\sigma &= L^2_\sigma(\text{div}) := \{f : \|(1 + |x|^2)^{\sigma/2} f\|_{(L^2(\mathbb{R}^3))^3} < \infty, \text{div } f = 0\}, \\ Z_\sigma &= \mathcal{H}^2_{-\sigma}(\mathbb{R}^3) := \{f : \|(1 + |x|^2)^{-\sigma/2} \partial^\beta f\|_{(L^2(\mathbb{R}^3))^3} < \infty, |\beta| \le 2\}, \end{aligned}$$

with $\partial^\beta f := \partial^{\beta_1}_{x_1} \partial^{\beta_2}_{x_2} \partial^{\beta_3}_{x_3} f$, where $|\beta| := \beta_1 + \beta_2 + \beta_3$, $\beta_j = 0, 1, 2$. We denote the dual space of X_σ by X'_σ. Define the expression

$$\mathcal{G}_\kappa f := \Phi_\kappa \star f(\cdot) + \sum_{l,l'=1}^{M} \Pi_\kappa(\cdot - y_l)[\Gamma^{-1}_{B,Y}(\kappa)]_{l,l'} \Phi_\kappa \star f(y_{l'}), \quad f \in X_\sigma. \tag{2.40}$$

Our aim is to prove that the limiting operator of $\mathcal{G}_{\kappa_\alpha}$, as $\alpha \to 0^+$, is in $\mathcal{L}(X_\sigma, X'_\sigma)$. We claim that $\mathcal{G}_\kappa f \in X'_\sigma$. To see this we first observe that $\Phi_\kappa \star f \in \mathcal{H}^2_{-\sigma}(\mathbb{R}^3) \subset X'_\sigma$

for $f \in X_\sigma$. Then it suffices to prove that the second term on the right-hand side of (2.40) also belongs to X'_σ, i.e.

$$T_\kappa(f,g) := \int_{\mathbb{R}^3} \left[\sum_{l,l'=1}^{M} \Pi_\kappa(x-y_l)[\Gamma^{-1}_{B,Y}(\kappa)]_{l,l'}\Phi_\kappa \star f(y_{l'})\right] g(x)dx < \infty \quad \forall\, g \in X_\sigma. \tag{2.41}$$

By the imbedding of $H^2_{-\sigma}$ into the continuous space in $\mathbb{R}^3$, we know $\Phi_\kappa \star f \in C(\mathbb{R}^3)$. Then the terms $\Phi_\kappa \star f(y_{l'})$ are well defined. Below we interpret (2.41) as a linear functional over X_σ with respect to g. Let $g \in X_\sigma \cap C_0^\infty(\mathbb{R}^3 \backslash Y)$, recalling that $Y = \bigcup_{l=1}^M \{y_l\}$. By integration by parts and the fact that div $g = 0$, we obtain

$$T_\kappa(f,g) = \sum_{l,l'=1}^{M} [\Gamma^{-1}_{B,Y}(\kappa)]_{l,l'}\Phi_\kappa \star f(y_{l'}) \cdot \Phi_\kappa \star g(y_l). \tag{2.42}$$

in analogy with (2.37). Hence, $\mathcal{G}_\kappa f$ defines a linear and continuous functional on $X_\sigma \cap C_0^\infty(\mathbb{R}^3\backslash Y)$. Since $X_\sigma \cap C_0^\infty(\mathbb{R}^3\backslash Y)$ is dense in X_σ, see Lemma 2.4 below, it has a unique extension to X_σ as a continuous linear functional. Hence $\mathcal{G}_\kappa \in \mathcal{L}(X_\sigma, X'_\sigma)$.

Lemma 2.4. *The set $X_\sigma \cap C_0^\infty(\mathbb{R}^3\backslash Y)$ is dense in X_σ.*

Proof. It is enough to consider $Y = \{y_0\}$ and $M = 1$. Let $\Omega_R(y_0)$ and $\Omega_\epsilon(y_0)$ be the balls centered at y_0 with radius R and ϵ, respectively. For $f \in X_\sigma$ and $R > \epsilon$, set $f_{R,\epsilon}$ as the restriction of f to $\Omega_R\backslash\bar{\Omega}_\epsilon$, i.e. $f_{R,\epsilon} := f|_{\Omega_R\backslash\bar{\Omega}_\epsilon}$. Then $f_{R,\epsilon} \in L^2(\Omega_R\backslash\bar{\Omega}_\epsilon)^3$ and div $f_{R,\epsilon} = 0$ in $\Omega_R\backslash\bar{\Omega}_\epsilon$. It is known, see, for instance, Theorem 1.1 in Ref. 9, that there exists a sequence $\{f_{R,\epsilon,n}\}_{n\in\mathbb{N}} \subset C_0^\infty(\Omega_R\backslash\bar{\Omega}_\epsilon)$ such that

$$\text{div } f_{R,\epsilon,n} = 0, \quad \|f_{R,\epsilon,n} - f_{R,\epsilon}\|_{L^2(\Omega_R\backslash\bar{\Omega}_\epsilon)} \to 0, \;\; n \to \infty.$$

Each element $f_{R,\epsilon,n}$ can be extended to $\mathbb{R}^3$ by zero in $\mathbb{R}^3\backslash(\Omega_R\backslash\bar{\Omega}_\epsilon)$, so that div $f_{R,\epsilon,n} = 0$ in $\mathbb{R}^3$. We still denote by $f_{R,\epsilon,n}$ this extension. It is clear that $\{f_{R,\epsilon,n}\}_{n\in\mathbb{N}} \subset X_\sigma \cap C_0^\infty(\mathbb{R}^3\backslash\{y_0\})$. Now,

$$\|f - f_{R,\epsilon,n}\|^2_{L^2_\sigma(\mathbb{R}^3)} = \|f_{R,\epsilon} - f_{R,\epsilon,n}\|^2_{L^2_\sigma(\Omega_R\backslash\bar{\Omega}_\epsilon)} + \|f\|^2_{L^2_\sigma(\mathbb{R}^3\backslash\Omega_R)} + \|f\|^2_{L^2_\sigma(\Omega_\epsilon)}.$$

where the spaces $L^2_\sigma(\mathbb{R}^3\backslash\Omega_R)$, $L^2_\sigma(\mathbb{R}^3\backslash\Omega_R)$ and $L^2_\sigma(\mathbb{R}^3\backslash\Omega_R)$ are defined similarly as L_σ replacing $\mathbb{R}^3$ by one of the related domains. For any $\eta > 0$, we can choose ϵ sufficiently small and R, n sufficiently large such that

$$\|f\|^2_{L^2_\sigma(\Omega_\epsilon)} < \eta^2/3, \quad \|f\|^2_{L^2_\sigma(\mathbb{R}^3\backslash\Omega_R)} < \eta^2/3, \quad \|f_{R,\epsilon} - f_{R,\epsilon,n}\|^2_{L^2_\sigma(\Omega_R\backslash\bar{\Omega}_\epsilon)} < \eta^2/3.$$

Summing up, we deduce that $\|f - f_{R,\epsilon,n}\|_{L^2_\sigma(\mathbb{R}^3)} < \eta$. The proof is thus complete. □

The last step to derive the solution operator of our original problem is to take the limit Im $\kappa_\alpha \to 0$. This is the object of the next lemma.

Lemma 2.5. *The operator* $\mathcal{G}_{\kappa_\alpha}$ *converges to* $\mathcal{G}_\kappa$ *in the operator norm of* $\mathcal{L}(X_\sigma, X'_\sigma)$ *as* $\operatorname{Im} \kappa_\alpha \to 0$, *i.e.* $\alpha \to 0$.

Proof. Combining (2.39), (2.40) and (2.16), we have

$$\begin{aligned}\|\mathcal{G}_{\kappa_\alpha} - \mathcal{G}_\kappa\|_{\mathcal{L}(X_\sigma, X'_\sigma)} &= \sup_{\|f\|_{X_\sigma}=1} \|\mathcal{G}_{\kappa_\alpha}(f) - \mathcal{G}_\kappa(f)\|_{X'_\sigma} \\ &= \sup_{\|f\|_{X_\sigma}=1} \sup_{\|g\|_{X_\sigma}=1} |\langle \mathcal{G}_{\kappa_\alpha}(f) - \mathcal{G}_\kappa(f), g\rangle_{X'_\sigma, X_\sigma}| \\ &\leq \sup_{\|f\|_{X_\sigma}=1} \sup_{\|g\|_{X_\sigma}=1} \{|\langle (\Phi_{\kappa_\alpha} - \Phi_\kappa) \star f, g\rangle_{X'_\sigma, X_\sigma}| \\ &\quad + |T_{\kappa_\alpha}(f,g) - T_\kappa(f,g)|\}.\end{aligned}$$

Inserting (2.42) into (2.41), we find

$$\begin{aligned}T_{\kappa_\alpha}(f,g) - T_\kappa(f,g) = \sum_{l.l'=1}^{M} &[[\Gamma_{B,Y}^{-1}(\kappa)]_{l,l'}(\Phi_{\kappa_\alpha} \star f - \Phi_\kappa \star f)(y_{l'}) \cdot (\Phi_{\kappa_\alpha} \star g)(y_l) \\ &+ [\Gamma_{B,Y}^{-1}(\kappa)]_{l,l'}(\Phi_\kappa \star f)(y_{l'}) \cdot (\Phi_{\kappa_\alpha} \star g - \Phi_\kappa \star g)(y_l) \\ &+ [\Gamma_{B,Y}^{-1}(\kappa_\alpha) - \Gamma_{B,Y}^{-1}(\kappa)]_{l,l'}\Phi_{\kappa_\alpha} \star f(y_{l'}) \cdot \Phi_{\kappa_\alpha} \star g(y_l)].\end{aligned}$$

Remark that $\Phi_{\kappa_\alpha} \star f = R(\kappa_\alpha)f$, where $R(\kappa_\alpha) = (\Delta - \kappa_\alpha^2)^{-1} : L^2_\sigma \to Z_\sigma$ is the resolvent of the Laplace operator. By Lemma 2.2 and the continuous injection of Z_σ into $C(\mathbb{R}^3)$, we see

$$\begin{aligned}|T_{\kappa_\alpha}(f,g) - T_\kappa(f,g)| \leq C[&\|(\Phi_{\kappa_\alpha} - \Phi_\kappa) \star f\|_{Z_\sigma}\|\Phi_{\kappa_\alpha} \star g\|_{Z_\sigma} \\ &+ \|\Phi_\kappa \star f\|_{Z_\sigma}\|(\Phi_{\kappa_\alpha} - \Phi_\kappa) \star g\|_{Z_\sigma}] \\ &+ o(1)\|\Phi_\kappa \star f\|_{Z_\sigma}\|\Phi_\kappa \star g\|_{Z_\sigma},\end{aligned}$$

where C is a constant and $o(1) \to 0$ as $\alpha \to 0^+$. Therefore,

$$\begin{aligned}\|\mathcal{G}_{\kappa_\alpha} - \mathcal{G}_\kappa\|_{\mathcal{L}(X_\sigma, X'_\sigma)} \leq \|R(\kappa_\alpha) - R(\kappa)\|_{\mathcal{L}(X_\sigma, Z_\sigma)}&[1 + C\|R(\kappa_\alpha)\|_{\mathcal{L}(L^2_\sigma, Z_\sigma)} \\ &+ C\|R(\kappa)\|_{\mathcal{L}(L^2_\sigma, Z_\sigma)}] + o(1)\|R(\kappa)\|^2_{\mathcal{L}(L^2_\sigma, Z_\sigma)}.\end{aligned}$$

Since $\|R(\kappa_\alpha) - R(\kappa)\|_{\mathcal{L}(L^2_\sigma, Z_\sigma)} \to 0$ as $\alpha \to 0$ (see Ref. 28), we finish the proof of the convergence $\mathcal{G}_{\kappa_\alpha} \to \mathcal{G}_\kappa$. □

From (2.40), we can formulate the Green's tensor corresponding to the operator $\mathcal{G}_\kappa : X_\sigma \to X'_\sigma$ as

$$\Xi^\kappa_{B,Y}(x,y) := \Phi_\kappa(x,y)\mathbf{I} + \sum_{l,l'=1}^{M} \Pi_\kappa(x, y_l)[\Gamma_{B,Y}^{-1}(\kappa)]_{l,l'}\Phi_\kappa(y_{l'}, y).$$

We summarize our findings in the following theorem.

Theorem 2.1. *Suppose that the coupling constants* $a_l(\epsilon)$ *are given by* (2.27), *with* $\beta_l, \gamma_l \in \mathbb{R}$ *for* $l = 1, 2, \ldots, M$. *Let* $\Gamma_{B,Y}$ *and* $F^{(j)}_{\kappa_\alpha}$ *be defined by* (2.24) *and* (2.30), *respectively. Then we have the following*:

(i) *The operator* $(\widetilde{H}^{\epsilon}_{\alpha})^{-1}$ *converges in the norm of* $\mathcal{L}(X, X')$ *to the operator* $\boldsymbol{g}_{\kappa_\alpha}$ *as* $\epsilon \to 0$, *where* $\boldsymbol{g}_{\kappa_\alpha} \in \mathcal{L}(X, X')$ *is given by* (2.31). *That is, for* $\alpha > 0$ *such that* $\det[\Gamma_{B,Y}(\kappa_\alpha)] \neq 0$,

$$\boldsymbol{g}_{\kappa_\alpha} = (\mathcal{M}_{\kappa_\alpha})^{-1} + \sum_{j,j'=1}^{3M} [\Gamma_{B,Y}(\kappa_\alpha)]^{-1}_{j,j'} \langle \cdot, F^{(j')}_{-\overline{\kappa}_\alpha} \rangle F^{(j)}_{\kappa_\alpha},$$

where $[\Gamma_{B,Y}(\kappa_\alpha)]^{-1}_{j,j'}$ *denotes the* (j, j')*th entry of the matrix* $[\Gamma_{B,Y}(\kappa_\alpha)]^{-1}$.

(ii) *The operator* $\mathcal{G}_{\kappa_\alpha} := \mathcal{F}^{-1}\boldsymbol{g}_{\kappa_\alpha}$ *converges to* $\mathcal{G}_\kappa$ *in the operator norm of* $\mathcal{L}(X_\sigma, X'_\sigma)$ *as* $\operatorname{Im}\,\kappa_\alpha \to 0$, *where*

$$\mathcal{G}_\kappa f = \Phi_\kappa \star f(\cdot) + \sum_{l,l'=1}^{M} \Pi_\kappa(\cdot - y_l)[\Gamma^{-1}_{B,Y}(\kappa)]_{l,l'} \Phi_\kappa \star f(y_{l'}), \quad f \in X_\sigma.$$

(iii) *We take* $\mathcal{G}_\kappa$ *to be the solution operator, from* X_σ *to* X'_σ, *to the problem* $\operatorname{curl}\operatorname{curl}\mathcal{G}_\kappa f - \kappa^2\mathcal{G}_\kappa f - \sum_{j=1}^{M} a_j\delta(x - y_j)\mathcal{G}_\kappa f = f$. *The Green's tensor corresponding to* $\mathcal{G}_\kappa$ *is given by*

$$\Xi^{\kappa}_{B,Y}(x, y) := \Phi_\kappa(x, y)\boldsymbol{I} + \sum_{l,l'=1}^{M} \Pi_\kappa(x, y_l)[\Gamma^{-1}_{B,Y}(\kappa)]_{l,l'} \Phi_\kappa(y_{l'}, y). \quad (2.43)$$

Here $[\Gamma^{-1}_{B,Y}(\kappa)]_{l,l'}$ *denote the* 3×3 *blocks of the matrix* $[\Gamma_{B,Y}(\kappa)]^{-1}$.

2.2.4. *The scattering theory by Dirac-like refraction indices*

In classical scattering theory, (2.43) describes the electromagnetic total field by the Dirac-like refraction indices supported by the collection of points $Y = \{y_1, y_2, \ldots, y_M\}$ corresponding to the incident point source located at y. We are also interested in the case of plane wave incidence. By making use of (2.3) in (2.43), we obtain

$$U(x, \theta) = e^{i\kappa x\cdot\theta}\mathbf{I} + \sum_{l,l'=1}^{M} \Pi_\kappa(x, y_l)[\Gamma^{-1}_{B,Y}(\kappa)]_{l,l'} e^{i\kappa y_{l'}\cdot\theta}\mathbf{I},$$

with $\theta := -\hat{y}$ and $U(x, \theta) := 4\pi \lim_{|y|\to\infty} e^{-i\kappa|y|}|y|\Xi^{\kappa}_{B,Y}(x, y)$. In particular, multiplying the previous identity by the polarization direction $p \in \mathbb{S}^2 (p \perp \theta)$ gives the total field

$$E(x, \theta, p) = E^i(x, \theta, p) + \sum_{l,l'=1}^{M} \Pi_\kappa(x, y_l)[\Gamma^{-1}_{B,Y}(\kappa)]_{l,l'} E^i(y_{l'}, \theta, p), \quad (2.44)$$

corresponding to a plane wave incidence $E^i(x, \theta, p) = p\exp(i\kappa x\cdot\theta)$ with $E(x, \theta, p) := U(x, \theta) \cdot p$. The far field corresponding to the scattered field is then given by

$$E^\infty(\hat{x}; \theta, p) = \frac{1}{4\pi} \sum_{l,l'=1}^{M} \exp(-i\kappa\hat{x} \cdot y_l)(\mathbf{I} - \hat{x} \otimes \hat{x})[\Gamma^{-1}_{B,Y}(\kappa)]_{l,l'} E^i(y_{l'}, \theta, p). \quad (2.45)$$

Remark 2.1. The representations (2.44) and (2.45) in the Foldy method are nothing but the ones in (2.10) and (2.11) respectively, if we take as the scattering coefficients g_l, $l = 1, \dots, M$, the following parameters $(a_l^{-1} - b_l)^{-1}$, $l = 1, \dots, M$, where a_l, $l = 1, \dots, M$, are the coefficients modeling the Dirac-like refraction indices in (1.4) and b_l, $l = 1, \dots, M$, are the parameters appearing in the regularization method, see (2.25).

Remark 2.2. (1) We remark that the Green's tensor (2.43) can be replaced by

$$\tilde{\Xi}^{\kappa}_{B,Y}(x,y) := \Pi_\kappa(x,y) + \sum_{l,l'=1}^{M} \Pi_\kappa(x,y_l)[\Gamma^{-1}_{B,Y}(\kappa)]_{l,l'}\Pi_\kappa(y_{l'},y), \tag{2.46}$$

whenever the solution operator $\mathcal{G}_\kappa$ acts on the space $\mathcal{H}^2_\sigma(\mathrm{div})$ defined as

$$\mathcal{H}^2_\sigma(\mathrm{div}) := \{f : \|(1+|x|^2)^{\frac{\sigma}{2}}\partial^\alpha f\|_{(L^2(\mathbb{R}^3))^3} < \infty, |\alpha| \le 2,\ \mathrm{div}\ f = 0\}.$$

To make this observation more rigorous, we define $\Pi_\kappa : \mathcal{H}^2_\sigma(\mathrm{div}) \to \mathcal{H}^{-2}(\mathbb{R}^3)$ as follows:

$$\Pi_\kappa(f) := \langle \Pi_\kappa(\cdot,y), f(y)\rangle_{\mathcal{H}^{-2}(\mathbb{R}^3),\mathcal{H}^2_\sigma(\mathrm{div})(\mathbb{R}^3)},$$

which makes sense since $\Pi_\kappa(x,\cdot) \in \mathcal{H}^{-2}(\mathbb{R}^3)$ (but not in $L^2(\mathbb{R}^3)$). Note that $\Pi_\kappa(x,y) \sim \mathcal{O}(1/|x-y|^3)$ as $|x-y| \to 0$ and $\mathcal{H}^2_\sigma(\mathrm{div}) \subset \mathcal{H}^2(\mathbb{R}^3)$ for $\sigma > 1$. Now, if $f \in C_0^\infty(\mathrm{div})(\mathbb{R}^3) := \{f \in (C_0^\infty(\mathbb{R}^3))^3,\ \mathrm{div}\, f = 0\}$, then

$$\begin{aligned}
\Pi_\kappa(f) &:= \langle \Pi_\kappa(\cdot,y), f(y)\rangle_{\mathcal{H}^{-2},\mathcal{H}^2_\sigma(\mathrm{div})} \\
&= \langle \Phi_\kappa(\cdot,y), f(y)\rangle - \frac{1}{\kappa^2}\langle \nabla\Phi_\kappa(\cdot,y), \nabla\cdot f(y)\rangle \\
&= \langle \Phi_\kappa(\cdot,y), f(y)\rangle = \Phi_\kappa \star f.
\end{aligned}$$

Arguing the same as in Lemma 2.4, we can show that $\overline{C_0^\infty(\mathrm{div})(\mathbb{R}^3)}^{\mathcal{H}^2_\sigma(\mathbb{R}^3)} = \mathcal{H}^2_\sigma(\mathrm{div})$ (see, e.g. Ref. 9). Hence, there holds $\Pi_\kappa(f) = \Phi_\kappa \star f$, $\forall\, f \in \mathcal{H}^2_\sigma(\mathrm{div})$, implying that the actions of (2.43) and (2.46) on $\mathcal{H}^2_\sigma(\mathrm{div})$ are the same.

(2) If we choose, as in Ref. 13, the regularization parameters β_j sufficiently large (compared to the fixed wavenumber κ), then the coefficient b_j in Lemma 2.2 takes the form

$$b_j := b_j(\beta_j,\gamma_j,\kappa) = \frac{\beta_j + i\kappa}{6\pi} - \frac{(\gamma_j/\sqrt{2})^3}{6\pi\kappa^2} + \mathcal{O}\left(\frac{\kappa}{\beta_j}\right), \quad \frac{\kappa}{\beta_j} \ll 1. \tag{2.47}$$

Additionally, suppose that there exists only one point-like scatterer located at the origin (i.e. $M = 1, y_1 = O$). Then by neglecting the term $\mathcal{O}(\kappa/\beta_j)$ in (2.47), the identity (2.46) becomes

$$\tilde{\Xi}^{\kappa}_{B,Y}(x,y) = \Pi_\kappa(x,y) + t\Pi_\kappa(x,0)\Pi_\kappa(0,y),$$

where t is given by (see (2.25) in Lemma 2.2)

$$t = (a_1^{-1} - b_1)^{-1} = \frac{1}{a_1^{-1} - (\beta + i\kappa)(6\pi)^{-1} + \gamma^3(6\pi\kappa^2)^{-1}\,2^{-3/2}}, \quad \beta, \gamma \in \mathbb{R}.$$

The number t is exactly the one characterizing the scattering *T matrix* in Ref. 13 (cf. Sec. III, A (33) and (19) in Ref. 13), remarking that in Ref. 13, $\beta/\sqrt{2}$ is taken as β, i.e. $1/\sqrt{2}$ is absorbed in β, and our coefficient a_1 corresponds to the number $\kappa^2\alpha_B$ there. The additional sign $(-)$ appearing in the formula is due to the fact that they used "negative" fundamental solutions, i.e. $\Phi_\kappa(x, y) := -\frac{e^{i\kappa|x-y|}}{4\pi|x-y|}$, instead of the positive one we used here, i.e. $\Phi_\kappa(x, y) := \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}$.

(3) The Green's function in (2.43) (or (2.46)) is modeled by the parameters a_l, β_l and γ_l. The physical meaning of these parameters is discussed in Ref. 13.

Remark 2.3. The far field corresponding to scattered field given in (2.45) satisfies the reciprocity relation, i.e. $p_2 \cdot E^\infty(\hat{x}; \theta, p_1) = p_1 \cdot E^\infty(-\theta; -\hat{x}, p_2)$, for the polarization directions $p_1, p_2 \in \mathbb{S}^2$ such that $p_1 \perp \theta$ and $p_2 \perp \hat{x}$, see, for instance, Ref. 12. It holds true since $\Gamma_{B,Y}^{-1}(\kappa)$ is symmetric and independent of $\hat{x}$ and θ which follows directly from (2.24).

3. The Born Approximation and the Intermediate Models

3.1. *Born approximation*

In the Born approximation, we only need to replace $E_j(y_j)$ by the value $E^i(y_j)$ of the incident field in (2.7). Therefore, $E(x)$ can be represented as

$$E(x) = E^i(x) + \sum_{j=1}^{M} g_j \Pi_\kappa(x, y_j) E^i(y_j), \tag{3.1}$$

and the far-field pattern of the scattered field is given by

$$E^\infty(\hat{x}) = \frac{1}{4\pi}\left(\sum_{j=1}^{M} g_j e^{i\kappa(\theta-\hat{x})\cdot y_j}\right)(\mathbf{I} - \hat{x} \otimes \hat{x})p, \quad \hat{x} \in \mathbb{S}^2. \tag{3.2}$$

The Born (weak) approximation neglects the multiple scattering between the point-like obstacles. Hence (3.1) is a good approximation only if the distance between y_j and y_l $(l \neq j)$ is relatively large with the wavelength.

3.2. *Intermediate levels of approximations*

Between the Born and Foldy models, we can define the kth $(k \in \mathbb{N})$ level of the total field $E^{(k)}$ as follows:

$$E^{(k)}(x) = E^i(x) + \sum_{j=1}^{M} g_j \Pi_\kappa(x, y_j) E_j^{(k)}(y_j), \tag{3.3}$$

where the value $E_j^{(k)}(y_j)$ can be computed recursively via

$$E_j^{(k+1)}(y_j) := E^i(y_j) + \sum_{\substack{l=1 \\ l\neq j}}^{M} g_l \Pi_\kappa(y_j, y_l)\, E_l^{(k)}(y_l), \quad j = 1, 2, \ldots, M, \tag{3.4}$$

with the zeroth level approximations $E_j^{(0)}(y_j)$ given by $E_j^{(0)}(y_j) := E^i(y_j)$, $j = 1, 2, \ldots, M$. Thus $E^{(0)}$ is just the Born approximation (3.1). When $k = +\infty$, we define $E_j^{(\infty)}(y_j)$ as

$$E_j^{(\infty)}(y_j) := E^i(y_j) + \sum_{\substack{l=1 \\ l\neq j}}^{M} g_l \Pi_\kappa(y_j, y_l) E_l^{(\infty)}(y_l), \quad j = 1, 2, \ldots, M.$$

Then, we see that, for $k = +\infty$, the total field in (3.3) coincides with the total field in (2.8), i.e. the Foldy regime. The kth level approximation $E^{(k)}$ only takes into account k time interactions between the scatterers, and thus is considered as the intermediate level.

Remark that the system (3.4) is nothing but the $(k+1)$th iteration of the Foldy algebraic system (2.7). From (3.3), we write the following form of the scattered field in kth level:

$$E^{(k,s)}(x) = \sum_{j=1}^{M} g_j \Pi_\kappa(x, y_j) E_j^{(k)}(y_j). \tag{3.5}$$

To write (3.5) and the corresponding far fields in a more useful form, we define the vector $\Lambda^{(k)} \in \mathbb{C}^{3M}$ with components $E_j^{(k)}(y_j)$ arranged elementwise as in the pattern of Λ in (2.9). Define $\tilde{\mathbf{I}} \in \mathbb{C}^{3M\times 3M}$ as an identity matrix, then the 3×3 diagonal blocks of $\tilde{\mathbf{I}}$, $\tilde{\mathbf{I}}_{jl}$ are $\mathbf{I}$, and the non-diagonal blocks are zero matrices. Set $\tilde{\mathbf{M}} := \tilde{\Gamma} - \tilde{\mathbf{I}}$,[b] then (3.4) can be written in a compact form as $\Lambda^{(k)} = \sum_{l=0}^{k}(-\tilde{\mathbf{M}})^l \mathbf{E}^I$ for $k = 0, 1, \ldots$.

Define the matrix $\tilde{C}_k \in \mathbb{C}^{3M\times 3M}$ by $\tilde{C}_k := \sum_{l=0}^{k}(-\tilde{\mathbf{M}})^l$ for $k = 0, 1, \ldots$. Denote the 3×3 blocks of $\tilde{C}_k \in \mathbb{C}^{3M\times 3M}$ by $[\tilde{C}_k]_{lj}$ for $l, j = 1, 2, \ldots, M$. With this setting we deduce from (3.4),(3.5) that the scattered field in kth intermediate level takes the form

$$E^{(k,s)}(x) = \sum_{l,j=1}^{M} g_j \Pi_\kappa(x, y_j)[\tilde{C}_k]_{jl} E^i(y_l) \tag{3.6}$$

and so the far-field pattern of the scattered field in the kth intermediate level is

$$E^{(k,\infty)}(\hat{x}) = \frac{1}{4\pi}\sum_{l,j=1}^{M} g_j e^{-i\kappa\hat{x}\cdot y_j} e^{i\kappa\theta\cdot y_l}(\mathbf{I} - \hat{x}\otimes\hat{x})[\tilde{C}_k]_{jl}\, p, \quad \hat{x}\in\mathbb{S}^2. \tag{3.7}$$

[b]In the case that the norm of $\mathbf{M}$ is less than one, the inverse of Γ can be approximated by the truncated Neumann series.

It is again worth mention to remind that (3.3)–(3.4) is already stated and used by Watson, see the formulas (3.1), (3.4) and (3.9) of Ref. 27.

4. The Inverse Problems for the Born, Foldy and Intermediate Models

From (3.2), (2.11) and (3.7), we can write the far-field pattern corresponding to scattered field in various models as

$$E^{\infty}(\hat{x};\theta,p) = \frac{1}{4\pi}\sum_{l,j=1}^{M} g_j e^{-i\kappa\hat{x}\cdot y_j} e^{i\kappa\theta\cdot y_l}(\mathbf{I} - \hat{x}\otimes\hat{x})[\tilde{\mathbf{T}}]_{jl}p, \tag{4.1}$$

with

$$\tilde{\mathbf{T}} := \begin{cases} \tilde{\mathbf{I}} & \text{Born approximation,} \\ \tilde{\Gamma}^{-1} & \text{Foldy method,} \\ \tilde{C}_k & k\text{th intermediate level.} \end{cases} \tag{4.2}$$

The above-mentioned far-field patterns are vectors. We define the following scalar far field, denoted by $\dot{E}^{\infty}(\hat{x})$, which will be useful in the statement and the justification of the MUSIC algorithm, see, for instance, Refs. 11 and 16:

$$\dot{E}^{\infty}(\hat{x};\theta,p) := \hat{x}^{\perp}\cdot E^{\infty}(\hat{x}) = \frac{1}{4\pi}\sum_{l,j=1}^{M} g_j e^{-i\kappa\hat{x}\cdot y_j} e^{i\kappa\theta\cdot y_l}\hat{x}^{\perp^{\top}}[\tilde{\mathbf{T}}]_{jl}p. \tag{4.3}$$

Here, $\hat{x}^{\perp} \in \mathbb{S}^2$ is any orthogonal direction to the observational direction $\hat{x} \in \mathbb{S}^2$. Since p is any direction in $\mathbb{S}^2$ perpendicular to θ, it has two orthogonal components called horizontal and vertical polarization directions denoted by p^h and p^v, respectively. So, $p := \theta^{\perp} = \underline{\theta^{\perp}}/|\underline{\theta^{\perp}}|$ with $\underline{\theta^{\perp}} := c_1\, p^h + c_2\, p^v$ for arbitrary constants c_1 and c_2. To give the explicit forms of p^h and p^v, we recall the Euclidean basis $\{e_1, e_2, e_3\}$ where $e_1 := (1,0,0)^{\top}, e_2 := (0,1,0)^{\top}$ and $e_3 := (0,0,1)^{\top}$, write $\theta := (\theta_x, \theta_y, \theta_z)^{\top}$ and set $r^2 := \theta_x^2 + \theta_y^2$. Let $\mathcal{R}_3$ be the rotation map transforming θ to e_3. Then in the basis $\{e_1, e_2, e_3\}, \mathcal{R}_3 = \mathcal{R}_3(\theta)$ is given by the matrix

$$\mathcal{R}_3 = \frac{1}{r^2}\begin{bmatrix} \theta_y^2 + \theta_x^2\theta_z & -\theta_x\theta_y(1-\theta_z) & -\theta_x r^2 \\ -\theta_x\theta_y(1-\theta_z) & \theta_x^2 + \theta_y^2\theta_z & -\theta_y r^2 \\ \theta_x r^2 & \theta_y r^2 & \theta_z r^2 \end{bmatrix}. \tag{4.4}$$

It satisfies $\mathcal{R}_3^{\top}\mathcal{R}_3 = I$ and $\mathcal{R}_3\theta = e_3$. Correspondingly, we write $p^h := \mathcal{R}_3^{\top} e_1$ and $p^v := \mathcal{R}_3^{\top} e_2$. These two directions represent the horizontal and the vertical directions of the polarized direction and they are given by

$$\begin{aligned} p^h &:= \theta^{\perp_h} = \frac{1}{r^2}(\theta_y^2 + \theta_x^2\theta_z, \theta_x\theta_y(\theta_z - 1), -r^2\theta_x)^{\top}, \\ p^v &= \theta^{\perp_v} = \frac{1}{r^2}(\theta_x\theta_y(\theta_z - 1), \theta_x^2 + \theta_y^2\theta_z, -r^2\theta_y)^{\top}. \end{aligned} \tag{4.5}$$

Hence, p can be written in terms of θ and then we can write $\dot{E}^{\infty}$ as a function of $\hat{x}$ and θ only.

Our concern in this section is to study the following inverse problem.

Inverse Problem: *Given the far-field pattern $\dot{E}^\infty(\hat{x},\theta)$ for several incident and observation directions θ and $\hat{x}$, find the locations $y_1, y_2, \ldots, y_M$ and the scattering coefficients $g_1, g_2, \ldots, g_M$.*

In the next sections, we deal with the mentioned models, (4.2)–(4.3), providing a detailed study of the resolution of the reconstruction depending on the distance between the scatterers, the frequency used and the scattering coefficients.

4.1. *MUSIC algorithm for the Maxwell system*

The MUSIC algorithm is a method to determine the locations $y_j, j = 1, 2, \ldots, M$, of the scatterers from the measured far-field pattern $\dot{E}^\infty(\hat{x},\theta)$ for a finite set of incidence and observation directions, i.e. $\hat{x},\theta \in \{\theta_j, j = 1, \ldots, N\} \subset \mathbb{S}^2$. We refer the reader to Refs. 2 and 20 for more information about this algorithm and to Refs. 4 and 3 for its stability and resolution analysis. We follow the way presented in Refs. 11 and 20. We assume that the number of scatterers is not larger than the number of incident and observation directions, in particular $N \geq 3M$. We define the response matrix $F \in \mathbb{C}^{N\times N}$ by $F_{st} := \dot{E}^\infty(\theta_s, \theta_t)$. Then by making use of (4.3), the response matrix F can be factorized as

$$F = H^*TH, \tag{4.6}$$

with the matrices $T \in \mathbb{C}^{3M\times 3M}$ and $H \in \mathbb{C}^{3M\times N}$ are given by

$$T := \mathbf{g}\tilde{\mathbf{T}}, \quad \mathbf{g} := \mathrm{Diag}(g_1\mathbf{I}, g_2\mathbf{I}, \ldots, g_M\mathbf{I}) \tag{4.7}$$

and

$$H := \begin{pmatrix} \theta_1^\perp e^{i\kappa\theta_1\cdot y_1} & \theta_2^\perp e^{i\kappa\theta_2\cdot y_1} & \cdots & \theta_N^\perp e^{i\kappa\theta_N\cdot y_1} \\ \theta_1^\perp e^{i\kappa\theta_1\cdot y_2} & \theta_2^\perp e^{i\kappa\theta_2\cdot y_2} & \cdots & \theta_N^\perp e^{i\kappa\theta_N\cdot y_2} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_1^\perp e^{i\kappa\theta_1\cdot y_M} & \theta_2^\perp e^{i\kappa\theta_2\cdot y_M} & \cdots & \theta_N^\perp e^{i\kappa\theta_N\cdot y_M} \end{pmatrix}.$$

In order to determine the locations y_j, we consider a grid of sampling points $z \in \mathbb{R}^3$ in a region containing the scatterers $y_1, y_2, \ldots, y_M$. For each point z, we define the vectors $\phi_z^u \in \mathbb{C}^M$ by

$$\phi_z^m := ((\theta_1^\perp \cdot e_m)e^{-i\kappa\theta_1\cdot z}, (\theta_2^\perp \cdot e_m)e^{-i\kappa\theta_2\cdot z}, \ldots, (\theta_N^\perp \cdot e_m)e^{-i\kappa\theta_N\cdot z})^\top \quad \forall\, m = 1, 2, 3. \tag{4.8}$$

MUSIC characterization of the scatterers. Recall that MUSIC is essentially based on characterizing the range of the response matrix F, forming projections onto its null space, and computing its singular value decomposition. Under the assumption that the matrix T in the factorization (4.6) of F is non-singular, the standard linear algebraic argument yields that, $\mathcal{R}(H^*)$ and $\mathcal{R}(F)$ coincide for

$N \geq 3M$, if the matrix H has maximal rank $3M$. So, let us discuss the invertibility of the matrix T. From the definition of T in (4.7), its invertibility depends only on the non-singularity of $\tilde{\mathbf{T}}$.

- In case of the Born approximation, it is clear that T is invertible as $\tilde{\mathbf{T}} = \tilde{\mathbf{I}}$ from the definition (4.2).
- In case of the Foldy method, from (4.2), we have $\tilde{\mathbf{T}} = \tilde{\Gamma}^{-1}$. So, the invertibility of T depends on the existence of $\tilde{\Gamma}^{-1}$. It can be observed that $\tilde{\Gamma}$ can be factorized as $\tilde{\Gamma} = \bar{\Gamma}\mathbf{g}$ with

$$\bar{\Gamma} := \begin{pmatrix} \frac{1}{g_1}\mathbf{I} & -\Pi_\kappa(y_1, y_2) & -\Pi_\kappa(y_1, y_3) & \cdots & -\Pi_\kappa(y_1, y_M) \\ -\Pi_\kappa(y_2, y_1) & \frac{1}{g_2}\mathbf{I} & -\Pi_\kappa(y_2, y_3) & \cdots & -\Pi_\kappa(y_2, y_M) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\Pi_\kappa(y_M, y_1) & -\Pi_\kappa(y_M, y_2) & -\Pi_\kappa(y_M, y_3) & \cdots & \frac{1}{g_M}\mathbf{I} \end{pmatrix}.$$

  Then $\tilde{\Gamma}$ is invertible when $\bar{\Gamma}$ is invertible and $T = \bar{\Gamma}^{-1}$. Let us assume it holds and postpone this issue to Sec. 4.2.
- In case of the approximation by intermediate level k, we have $\tilde{\mathbf{T}} = \tilde{C}_k = \sum_{l=0}^{k}(-\tilde{\mathbf{M}})^l$. One can observe that the norm of $\tilde{\mathbf{M}}$ less than half is a sufficient condition for the invertibility of T in every level of scattering.

Hence, under the assumption of the non-singularity of T, we can state the MUSIC related theorem for the electromagnetic wave scattering by point-like scatterers as follows.

Theorem 4.1. *Let $\{\theta_s : s \in \mathbb{N}\} \subset \mathbb{S}^2$ be a countable set of directions such that any analytic function on $\mathbb{S}^2$ that vanishes in θ_s for all $s \in \mathbb{N}$ vanishes identically. Let $\mathbf{K}$ be a compact subset of $\mathbb{R}^3$ containing $\{y_j : j = 1, \ldots, M\}$. Then there exists $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$ the following characterization holds for every $z \in \mathbf{K}$:*

$$z \in \{y_1, \ldots, y_M\} \Leftrightarrow \phi_z^m \in \mathcal{R}(H^*), \quad \text{for some } m = 1, 2, 3. \tag{4.9}$$

Furthermore; the ranges of H^* and F coincide and thus

$$z \in \{y_1, \ldots, y_M\} \Leftrightarrow \phi_z^m \in \mathcal{R}(F) \Leftrightarrow \mathcal{P}\phi_z^m = 0, \quad \text{for some } m = 1, 2, 3, \tag{4.10}$$

where $\mathcal{P} : \mathbb{C}^N \to \mathcal{R}(F)^\perp = \mathcal{N}(F^*)$ is the orthogonal projection onto the null space $\mathcal{N}(F^*)$ of F^*.

Proof. One can prove this theorem in the lines of the proof of Theorem 4.1 in Ref. 20 concerning the Born approximation for the acoustic model and more closely Theorem 3.1 and Theorem 3.2 in Ref. 11 concerning the acoustic and elastic wave scattering, respectively, related to the Born, Foldy and the intermediate models, by proving the maximal rank property of the matrix H and using the test vector ϕ_z^m. □

Remark 4.1. We can observe in (4.3) that either horizontal polarized directions p^h or vertical polarized directions p^v are sufficient for the reconstruction. In addition, either the horizontal observation directions or the vertical observation directions are also sufficient for the reconstruction.

4.2. *Invertibility of the matrix* $\bar{\Gamma}$

To discuss the invertibility of $\bar{\Gamma}$, we distinguish two situations.

Diagonally dominant condition. As mentioned in Ref. 11, concerning acoustic and elastic cases, when the scatterers are relatively far away from each other comparing to the scattering coefficients, then the invertibility condition of $\bar{\Gamma}$ is the diagonally dominant condition and it is given by

$$\sum_{\substack{j=1\\ j\neq l}}^{M} \|\Pi_\kappa(y_j, y_l)\|_1 < \frac{1}{|g_l|} \quad \forall\, l = 1, 2, \ldots, M. \tag{4.11}$$

Here $\|\cdot\|_1$ is the 1-norm and it is defined for a matrix $\mathbf{L} = [L_{nm}] \in \mathbb{C}^{N\times M}$, as $\|\mathbf{L}\|_1 := \max_{1\leq m\leq M} \sum_{n=1}^{N} |L_{nm}|$.

Non-diagonally dominant condition. Using the form (2.1), we can write $\Pi_\kappa(x, y)$ explicitly as

$$\Pi_\kappa(x, y) = \Phi_\kappa(x, y)\mathbf{I} + \frac{1}{\kappa^2}\frac{\Phi_\kappa(x, y)}{r^2}[-\kappa^2 R \otimes R + (1 - i\kappa r)(3\hat{R} \otimes \hat{R} - \mathbf{I})], \tag{4.12}$$

where $R = x - y, r = |x - y|$ and $\hat{R} = \frac{R}{r}$. We remark that the entries of $\Pi_\kappa(y_j, y_l)$, $j, l = 1, \ldots, M$, are analytic in terms of the variables $\eta_{jlm} = (y_j - y_l)_m$, $j, l = 1, \ldots, M$ and $m = 1, 2, 3$ for $\eta_{jlm} \in \mathbb{R}\backslash\{0\}$. Remark also that the determinant of $\bar{\Gamma}$, $\det(\bar{\Gamma})$, is given by the products and sums of g_j^{-1} and the entries of $\Pi_\kappa(y_j, y_l)$ for $j, l = 1, \ldots, M$. Then $\det(\bar{\Gamma})$ is analytic in terms of the $3\frac{M(M-1)}{2}$ real variables η_{jlm} for $j, l = 1, \ldots, M$ with $j < l$, $m = 1, 2, 3$. Fix the wavenumber κ and the scattering coefficients g_j, for $j = 1, \ldots, M$, we deduce then that except for few distributions of the scatterers, $y_1, \ldots, y_M$, the matrix $\bar{\Gamma}$ is invertible. The wavenumbers κ for which $\bar{\Gamma}$ is not invertible are called resonances, see Ref. 1 for a study of these resonances concerning the acoustic case.

4.3. *Recovering the scattering coefficients* g_j

In this section we discuss how one can recover the scattering coefficients g_j attached to the scatterers y_j for $j = 1, \ldots, M$ for the given far-field pattern, i.e. response matrix F. Recall that F has the factorization $F = H^*TH$ where H and T are defined as in Sec. 4.1. As the matrix H is of maximal rank $3M$ and $N \geq 3M$ the matrix $HH^* \in \mathbb{C}^{3M\times 3M}$ is invertible. Let us denote this inverse by I_H. Once we locate the scatterers $y_1, y_2, \ldots, y_M$ by using MUSIC algorithm for the given far-field patterns, we can recover the matrix $T \in C^{3M\times 3M}$ as $T = I_H HFH^* I_H$, $I_H H$ is the pseudo-inverse of H^* from whose entries we can deduce the values of g_j, $j = 1, \ldots, M$.[c] We explain how this procedure of recovering $g_1, \ldots, g_M$ is going to work in each model.

- In the Born approximation we have $T = \mathbf{g}$, and hence the diagonal entries of T give $g_1, \ldots, g_M$.
- In the Foldy method we have $T = \bar{\Gamma}^{-1}$, and hence the reciprocals of the diagonal entries of T^{-1} produce $g_1, \ldots, g_M$.
- In the intermediate level, k, approximation we have $T = \mathbf{g}\sum_{l=0}^{k}(-\tilde{\mathbf{M}})^l$. We have already seen how one can recover the scattering coefficients for $k = 0$ (Born) and for $k = \infty$ (Foldy). In the case $k = 1$, we have $T = \mathbf{g} - \mathbf{g}\tilde{\mathbf{M}}$. As we know that $\mathbf{g}$ is a diagonal matrix and the 3×3 diagonal blocks of $\tilde{\mathbf{M}}$ are zero, the diagonal entries of T are equal to the scattering coefficients of the M scatterers. But for intermediate level approximation $k > 1$, it is difficult to recover the scattering coefficients due to the complicated structure of the matrices $(-\tilde{\mathbf{M}})^l$, for $l = 2, \ldots,$ and hence of T. For this reason, as in the acoustic and elastic cases of Ref. 11, we restrict ourselves to the special case of two point-like obstacles y_1, y_2 with the corresponding scattering coefficients g_1, g_2. In this case using the symmetry relation of the fundamental matrix $\Pi_\kappa(x, y)$, i.e. $\Pi_\kappa(x, y) = [\Pi_\kappa(y, x)]^\top$, we have the explicit form of $(-\tilde{\mathbf{M}})^l$ for $l = 0, 1, 2, \ldots,$ as follows:

$$(-\tilde{\mathbf{M}})^l = \begin{cases} \begin{bmatrix} g_1^{\frac{l}{2}} g_2^{\frac{l}{2}} \Pi_\kappa^l(y_1, y_2) & \mathbf{0} \\ \mathbf{0} & g_1^{\frac{l}{2}} g_2^{\frac{l}{2}} \Pi_\kappa^l(y_1, y_2) \end{bmatrix}, & l \in 2\mathbb{N} \cup \{0\}, \\[2ex] \begin{bmatrix} \mathbf{0} & g_1^{\frac{l-1}{2}} g_2^{\frac{l+1}{2}} \Pi_\kappa^l(y_1, y_2) \\ g_1^{\frac{l+1}{2}} g_2^{\frac{l-1}{2}} \Pi_\kappa^l(y_1, y_2) & \mathbf{0} \end{bmatrix}, & l \in 2\mathbb{N} - 1. \end{cases} \tag{4.13}$$

Here, $\mathbf{0}$ is the zero matrix of order 3. The matrix $(-\tilde{\mathbf{M}})^l$ is either diagonal or anti-diagonal by blocks of the size 3×3. This structure is not valid anymore for the case of more than two scatterers. From this structure, we obtain the explicit

[c] We can recover the scattering coefficients $a'_j, \ldots, a'_M$ associated to Dirac model from the relations $g_j = (a_j^{-1} - b_j)^{-1}$ and $a_j = \kappa^2 a'_j$ for $j = 1, \ldots, M$, see (1.4) and Remark 2.1. As three unknown parameters a_j, β_j, γ_j are appearing in the equations $g_j = (a_j^{-1} - b_j)^{-1}$, see (2.25), in order to get the scattering coefficients a'_j two wavenumbers κ_1 and κ_2 can be used as follows. As a_js are real, first we compare the imaginary parts of the equation $g_j = (a_j^{-1} - b_j)^{-1}$ to obtain β_j. Then we compare the real parts of the equation $g_j = (a_j^{-1} - b_j)^{-1}$ for the wavenumbers κ_1 and κ_2 to obtain γ_j and a_j, from which we can recover the scattering coefficients a'_j.

form of $T = \mathbf{g}\sum_{l=0}^{k}(-\tilde{\mathbf{M}})^l$ in the kth order scattering as follows:

$$T = \begin{cases} \begin{bmatrix} g_1\mathbf{I} & \mathbf{0} \\ \mathbf{0} & g_2\mathbf{I} \end{bmatrix}, & k = 0, \\ \begin{bmatrix} g_1\sum_{l=0}^{\frac{k}{2}} g_1^l g_2^l \Pi_\kappa^{2l}(y_1,y_2) & \sum_{l=1}^{\frac{k}{2}} g_1^l g_2^l \Pi_\kappa^{2l-1}(y_1,y_2) \\ \sum_{l=1}^{\frac{k}{2}} g_1^l g_2^l \Pi_\kappa^{2l-1}(y_1,y_2) & g_2\sum_{l=0}^{\frac{k}{2}} g_1^l g_2^l \Pi_\kappa^{2l}(y_1,y_2) \end{bmatrix}, & k \in 2\mathbb{N}, \\ \begin{bmatrix} g_1\sum_{l=0}^{\frac{k-1}{2}} g_1^l g_2^l \Pi_\kappa^{2l}(y_1,y_2) & \sum_{l=0}^{\frac{k-1}{2}} g_1^{l+1} g_2^{l+1} \Pi_\kappa^{2l+1}(y_1,y_2) \\ \sum_{l=1}^{\frac{k-1}{2}} g_1^{l+1} g_2^{l+1} \Pi_\kappa^{2l+1}(y_1,y_2) & g_2\sum_{l=0}^{\frac{k-1}{2}} g_1^l g_2^l \Pi_\kappa^{2l}(y_1,y_2) \end{bmatrix}, & k \in 2\mathbb{N}-1, \\ \begin{bmatrix} \frac{1}{g_1}\mathbf{I} & -\Pi_\kappa(y_1,y_2) \\ -\Pi_\kappa(y_1,y_2) & \frac{1}{g_2}\mathbf{I} \end{bmatrix}^{-1}, & k = \infty. \end{cases} \tag{4.14}$$

From the explicit form of T, we observe the following points.

- The diagonal entries of T give g_1 and g_2 in the Born approximation, i.e. $k = 0$.
- Substituting the non-diagonal entries in the diagonal entries gives the scattering coefficients in every even level scattering k, i.e. $k \in 2\mathbb{N}$. Indeed, define $\breve{g} := \sum_{l=1}^{\frac{k}{2}} g_1^l g_2^l \Pi_\kappa^{2l-1}(y_1,y_2)$ then the non-diagonal entries of T are equal to $\breve{g}$. Also the diagonal entries T_{11} and T_{22} of T are equal to $g_1(1+\Pi_\kappa(y_1,y_2)\breve{g})$ and $g_2(1+\Pi_\kappa(y_1,y_2)\breve{g})$ respectively. Now, with the knowledge of the scatterers y_1 and y_2 from the MUSIC algorithm and by substituting the value of $\breve{g}$ in the diagonal entries, we can evaluate g_1, g_2.
- Substituting the diagonal entries in the non-diagonal entries gives the scattering coefficients in every odd level scattering k, i.e. $k \in 2\mathbb{N}-1$. Indeed, define $\breve{b}_1 := g_1\sum_{l=0}^{\frac{k-1}{2}} g_1^l g_2^l \Pi_\kappa^{2l}(y_1,y_2)$ and $\breve{b}_2 := g_2\sum_{l=0}^{\frac{k-1}{2}} g_1^l g_2^l \Pi_\kappa^{2l}(y_1,y_2)$ then the diagonal entries T_{11} and T_{22} of T are equal to $\breve{b}_1$ and $\breve{b}_2$ respectively. Also the non-diagonal entries T_{12} and T_{21} of T are the same and are equal to $g_1\breve{b}_2\Pi_\kappa(y_1,y_2) = g_2\breve{b}_1\Pi_\kappa(y_1,y_2)$. Now again with the knowledge of the scatterers y_1 and y_2 from the MUSIC algorithm and by substituting the diagonal entries in the non-diagonal entries of T, we can evaluate g_1, g_2.
- The diagonal entries of T^{-1} give g_1 and g_2 in the method of Foldy, i.e. $k = \infty$.

4.3.1. *Numerical results and discussions*

For the convenience of visualization, we show the results for the scatterers in XY-plane. For our calculations, we consider 50 incident and observational directions and the point-like scatterers attached to same scattering coefficients located at the points $y_1 = (0, 0, 0)$, $y_2 = (0, 0.5, 0)$, $y_3 = (0.5, 0, 0)$, $y_4 = (0.5, 0.5, 0)$, $y_5 = (1, 1, 0)$, $y_6 = (1, -1, 0)$, $y_7 = (-1, -1, 0)$, $y_8 = (-1, 1, 0)$, $y_9 = (1, -1.5, 0)$, $y_{10} = (1.5, 0.5, 0)$ and $y_{11} = (-1.5, 1, 0)$. Let d_{GL} stands for the degree of Gauss–Legendre polynomial. We used the $2d_{\mathrm{GL}}^2$ $(= 50)$ incident and the observational directions obtained from the Gauss–Legendre polynomial of degree d_{GL} $(= 5)$, i.e. if we denote the zeros of the Gauss–Legendre polynomial of degree by GL_k, for $k = 1, \ldots, d_{\mathrm{GL}}$ then the azimuth and the zenith angles θ and ϕ are given by

$$\phi = \cos^{-1}(\mathrm{GL}_k), \quad k = 1, \ldots, d_{\mathrm{GL}},$$
$$\theta = j * \frac{\pi}{d_{\mathrm{GL}}}, \qquad j = 0, 1, \ldots, 2d_{\mathrm{GL}} - 1.$$

Combinations of these spherical coordinates will allow us to find the incident and the observational directions given by $(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$. These directions are shown in Fig. 1. To show numerically that horizontal, p^h, or vertical, p^v, polarization direction is enough for the reconstruction, we used the directions p^h and p^v as per the definition (4.5).

Since MUSIC algorithm is an exact method, the reconstruction is very accurate in the absence of noise in measured data, for Born, Foldy and intermediate models.

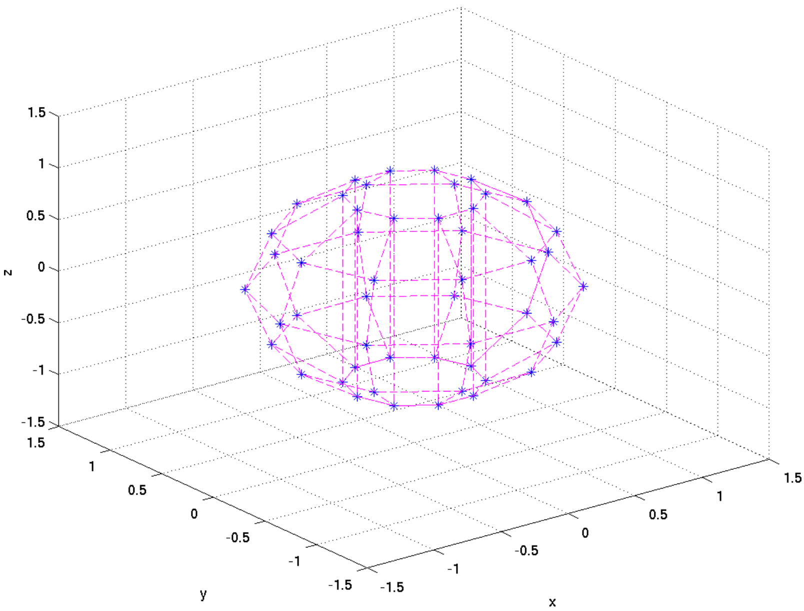

Fig. 1. The incident and observational directions.

To analyze the effect of the noise level on the resolution of the algorithm, different noise levels are used. To distinguish the differences between the Born approximation and the Foldy model, we used different scattering coefficients, noise levels and distance between the scatterers.

Figures 2 and 3 are related to the six scatterers located at the points y_1, y_2, y_5, y_6, y_7 and y_8 of each having scattering coefficient 1 for each with 1% random noise in the measured far-field pattern. Figure 2 shows the pseudo-spectrum of the mentioned six scatterers for the wavenumber $\kappa = 2\pi$ whereas Fig. 3 shows the pseudo-spectrum for the wavenumber $\kappa = \pi$, i.e. minimum distance between the scatterers is half of the wavelength and quarter of the wavelength respectively. We observe that due to the higher wavenumber, Fig. 2 has the better reconstruction comparing to Fig. 3 with respect to p^h and p^v respectively. Also, we can observe that the scatterers satisfy largely the condition (4.11) and the reconstruction looks

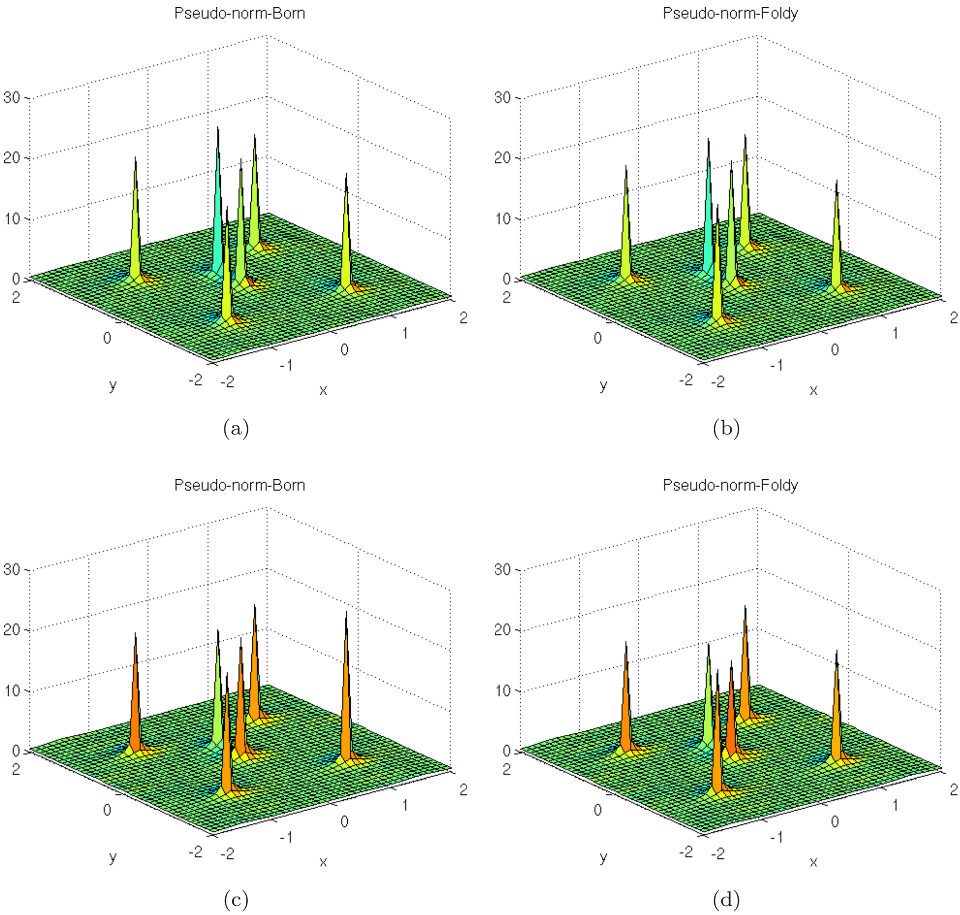

Fig. 2. Born (a), (c) and Foldy (b), (d)-based reconstructions with 1% noise, $g_j = 1$ and $\kappa = 2\pi$ for six scatterers. Left part (a), (b) — p^h, right part (c), (d) — p^v.

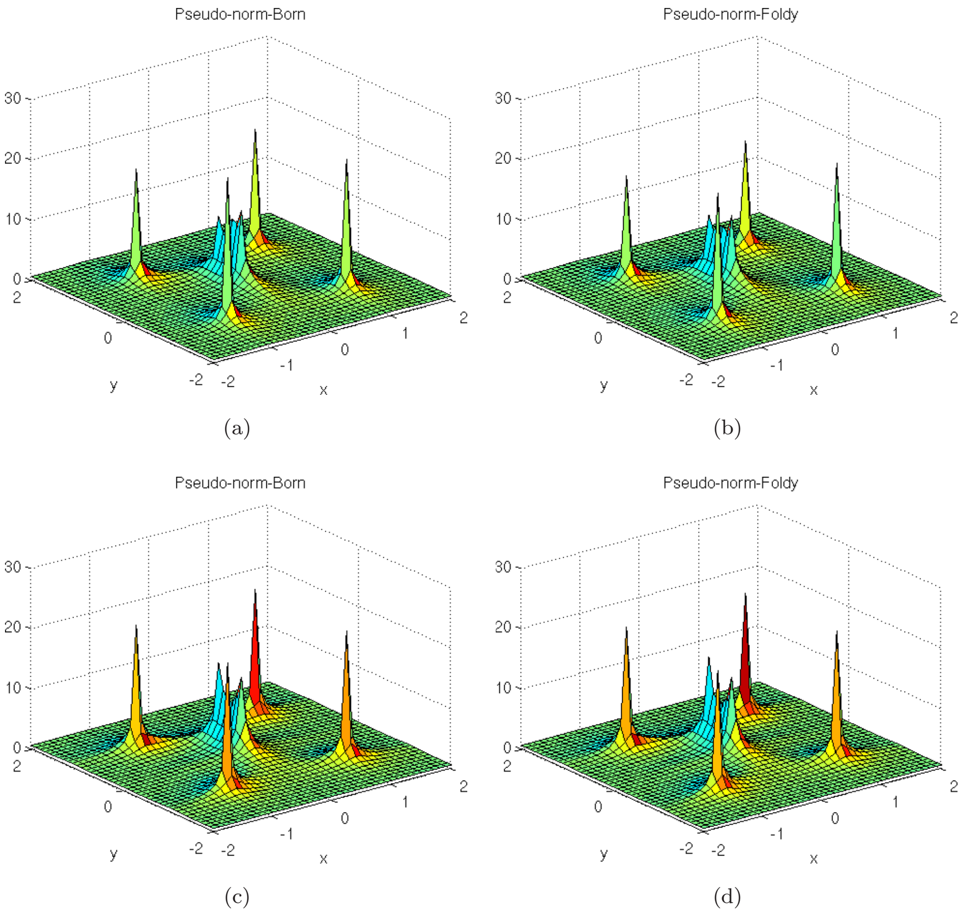

Fig. 3. Born (a), (c) and Foldy (b), (d)-based reconstructions with 1% noise, $g_j = 1$ and $\kappa = \pi$ for six scatterers. Left part (a), (b) — p^h, right part (c), (d) — p^v.

similar in both the Born approximation and the Foldy model. Hence, if the scatterers are well separated with low scattering coefficients there is not much difference in the reconstruction between the Born approximation and the Foldy model.

Now, we present an example where the scatterers do not satisfy the condition (4.11). Figure 4 shows the pseudo-spectrum of the six scatterers again located at y_1, y_2, y_5, y_6, y_7 and y_8 of each having scattering coefficient 10 for $\kappa = 2\pi$ with 6% random noise in the measured far-field patterns with respect to the Born approximation and the Foldy method. Compared to Figs. 2 and 3, we see in Fig. 4 how the reconstruction deteriorates due to the effect of multiple scattering created by the close obstacles. In this case, we can see the differences between the Born approximation and the Foldy model.

As a conclusion, we have seen that if the condition (4.11) is satisfied largely, then the effect of the multiple scattering is quite low and the reconstruction is similar in

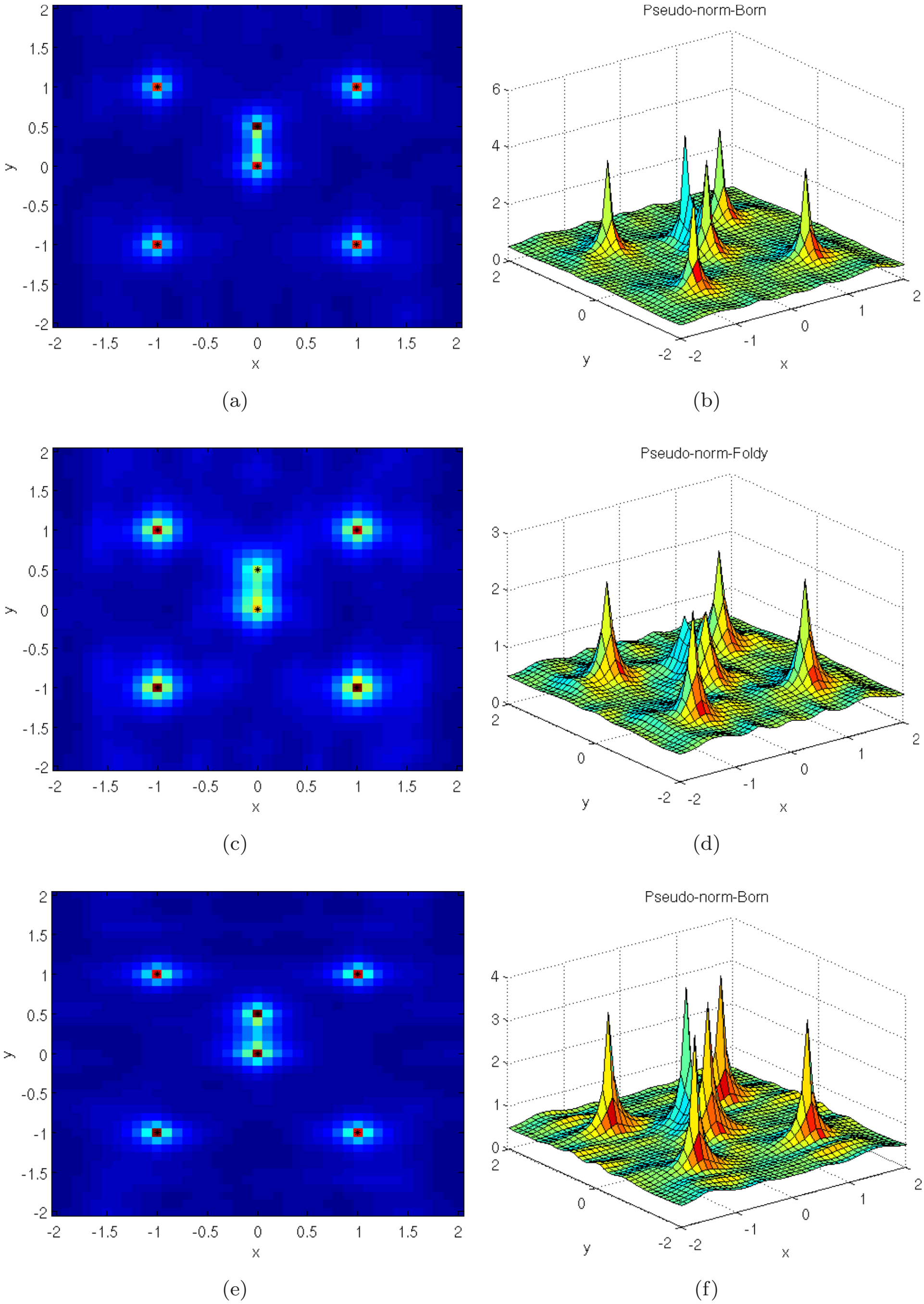

Fig. 4. Born (a), (b), (e), (f) and Foldy (c), (d), (g), (h)-based reconstructions with 6% noise, $g_j = 10$ and $\kappa = 2\pi$ for six scatterers. Upper part (a)–(d) — p^h, lower part (e)–(h) — p^v.

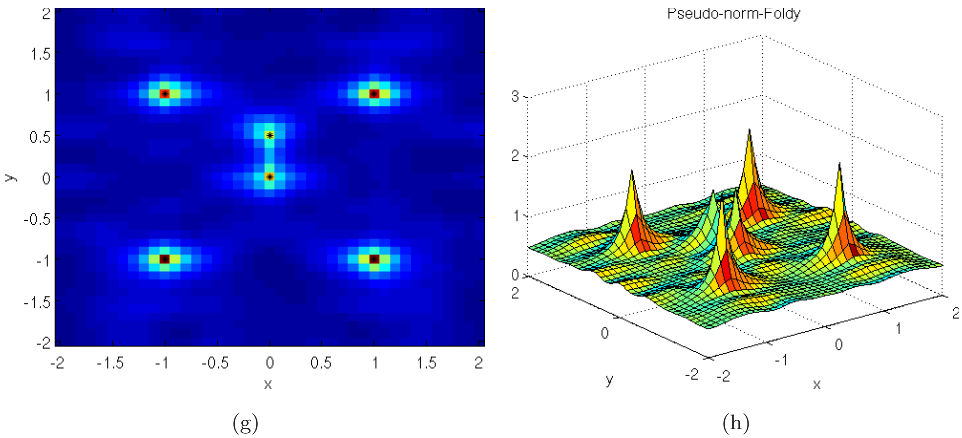

Fig. 4. (*Continued*)

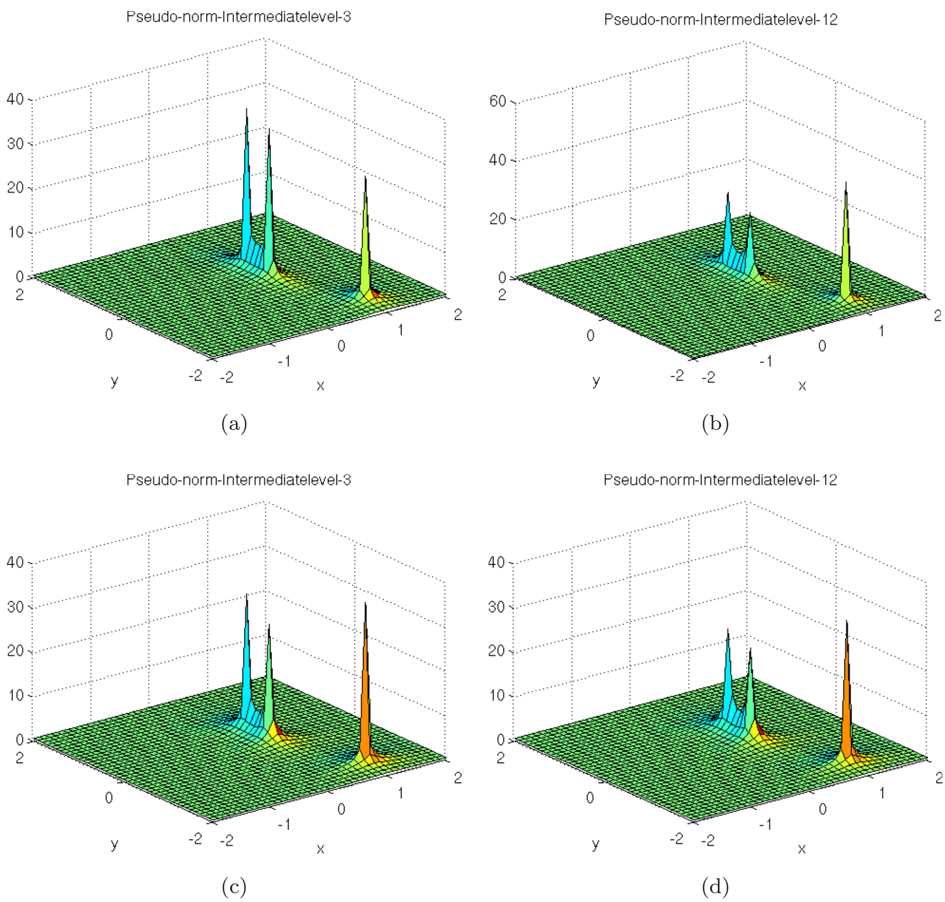

Fig. 5. Reconstruction of three scatterers with 1% noise, $g_j = 7$ and $\kappa = \pi$. 3rd level (a), (c) and 12th level (b), (d) approximations. Left part (a), (b) — p^h, right part (c), (d) — p^v.

both Born and Foldy but above the condition (4.11) the use of the Born approximation gives better reconstruction than the use of the Foldy method. However in the latter case, Born approximation is not valid as the scatterers are relatively close.

We have similar kind of difference between the intermediate level approximations as the level k increases with respect to the condition (4.11). We can observe this in Fig. 5 which shows the numerical reconstruction of the three scatterers, based on 3rd and 12th level approximations, located at y_3, y_4 and y_9 and having scattering coefficient 7 with $\kappa = \pi$ and of 1% random noise in the measured far-field pattern. Finally, let us remind that the reconstruction depends on the choice of the signal and noise subspaces of the multiscale response matrix, see, for instance, Ref. 11 for a discussion on this issue concerning the acoustic and elastic cases.

Appendix

For the reader's convenience we show the proofs of (2.15), (2.16) and (2.28).

Proofs of (2.16) and (2.15). The Green's tensor $\Pi_\kappa(x,0)$ can be written as

$$\Pi_\kappa(x,0) = \frac{e^{i\kappa|x|}}{4\pi|x|}\{P(i\kappa|x|)\mathbf{I} + Q(i\kappa|x|)\hat{x}\otimes\hat{x}\}, \tag{A.1}$$

where the functions P and Q are defined in Sec. 2.2.

In the following we will only prove (2.16), since (2.15) follows automatically from (2.16), (2.13) and (A.1). By the definitions of T_κ and the inverse Fourier transform,

$$\begin{aligned}
&(2\pi)^{-3/2}\mathcal{F}^{-1}(T_\kappa(\xi))\\
&\quad= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{1}{|\xi|^2-\kappa^2}e^{i\xi\cdot x}d\xi\mathbf{I} - \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{1}{|\xi|^2-\kappa^2}\hat{\xi}\otimes\hat{\xi}e^{i\xi\cdot x}d\xi\\
&\quad= \frac{e^{i\kappa|x|}}{4\pi|x|}\mathbf{I} - \frac{1}{(2\pi)^3}\nabla_x\nabla_x\int_{\mathbb{R}^3}\frac{1}{(\kappa^2-|\xi|^2)|\xi|^2}e^{i\xi\cdot x}d\xi\\
&\quad= \frac{e^{i\kappa|x|}}{4\pi|x|}\mathbf{I} - \frac{1}{(2\pi)^3\kappa^2}\nabla_x\nabla_x\int_{\mathbb{R}^3}\left(\frac{1}{\kappa^2-|\xi|^2}+\frac{1}{|\xi|^2}\right)e^{i\xi\cdot x}d\xi\\
&\quad= \frac{e^{i\kappa|x|}}{4\pi|x|}\mathbf{I} - \frac{1}{(2\pi)^{3/2}\kappa^2}\nabla_x\nabla_x\left(\mathcal{F}^{-1}\left(\frac{1}{|\xi|^2}\right) - \mathcal{F}^{-1}\left(\frac{1}{|\xi|^2-\kappa^2}\right)\right).
\end{aligned}$$

Employing $(2\pi)^{-3/2}\mathcal{F}^{-1}(\frac{1}{|\xi|^2-\kappa^2})(x) = \frac{e^{i\kappa|x|}}{4\pi|x|}$, we get

$$\begin{aligned}
(2\pi)^{-3/2}\mathcal{F}^{-1}(T_\kappa(\xi)) &= \frac{e^{ik|x|}}{4\pi|x|}\mathbf{I} - \frac{1}{4\pi\kappa^2}\nabla_x\nabla_x\left(\frac{1-e^{i\kappa|x|}}{|x|}\right)\\
&= (\mathbf{I} + \frac{1}{\kappa^2}\nabla_x\nabla_x)\frac{e^{i\kappa|x|}}{4\pi|x|} - \frac{1}{4\pi\kappa^2}\nabla_x\nabla_x\frac{1}{|x|}\\
&= \Pi_\kappa(x,0) - \frac{1}{4\pi\kappa^2}\nabla_x\nabla_x\frac{1}{|x|}.
\end{aligned}$$

Simple calculations show

$$\nabla_x \nabla_x \frac{1}{|x|} = -\frac{1}{|x|^3}\{\mathbf{I} - 3\hat{x} \otimes \hat{x}\}. \tag{A.2}$$

With the help of (A.1), we finally obtain

$$\begin{aligned}\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{1}{|\xi|^2-\kappa^2}(\mathbf{I} - \hat{\xi}\otimes\hat{\xi})e^{i\xi\cdot x}d\xi \\ = \frac{\mathbf{I} - 3\hat{x}\otimes\hat{x}}{4\pi\kappa^2|x|^3} + \frac{e^{i\kappa|x|}}{4\pi|x|}\{P(i\kappa|x|)\mathbf{I} + Q(i\kappa|x|)\hat{x}\otimes\hat{x}\}.\end{aligned}$$

The identity (2.16) is thus proven. □

Proof of (2.28). We first compute the integral $(2\pi)^{-3}\int_{\mathbb{R}^3} T_\kappa(\xi)f(\beta,\xi)d\xi$ with $f(\beta,\xi) = \frac{\beta^2}{\beta^2+|\xi|^2}$. Again using the definition of T_κ, we find

$$\begin{aligned}T_\kappa(\xi)f(\beta,\xi) &= T_\kappa(\xi)f(\beta,\kappa) + T_\kappa(\xi)(f(\beta,\xi) - f(\beta,\kappa)) \\ &= \left[T_\kappa(\xi) + T_\kappa(\xi)\frac{\kappa^2-\xi^2}{\beta^2+\xi^2}\right] f(\beta,\kappa) \\ &= \left[\frac{1}{\xi^2-\kappa^2}(\mathbf{I} - \hat{\xi}\otimes\hat{\xi}) - \frac{1}{\xi^2+\beta^2}(\mathbf{I} - \hat{\xi}\otimes\hat{\xi})\right] f(\beta,\kappa).\end{aligned}$$

Recalling that $\Pi_\kappa^T(x) = (2\pi)^{-3/2}\mathcal{F}^{-1}[\frac{1}{\xi^2-\kappa^2}(\mathbf{I} - \hat{\xi}\otimes\hat{\xi})]$, we have

$$(2\pi)^{-3/2}\mathcal{F}^{-1}[T_\kappa(\xi)f(\beta,\xi)] = [\Pi_\kappa^T(x) - \Pi_{i\beta}^T(x)]f(\beta,\kappa).$$

It is easy to see

$$\left[\frac{\mathbf{I} - 3\hat{x}\otimes\hat{x}}{4\pi\kappa^2|x|^3} - \frac{\mathbf{I} - 3\hat{x}\otimes\hat{x}}{-4\pi\beta^2|x|^3}\right]\frac{\beta^2}{\beta^2+\kappa^2} = \frac{\mathbf{I} - 3\hat{x}\otimes\hat{x}}{4\pi\kappa^2|x|^3}.$$

Using the previous identity, it follows from the expression of $\Pi_\kappa^T(x)$ that

$$\begin{aligned}(2\pi)^{-3/2}\mathcal{F}^{-1}[T_\kappa(\xi)f(\beta,\xi)] = \frac{\mathbf{I} - 3\hat{x}\otimes\hat{x}}{4\pi\kappa^2|x|^3} + \left\{\frac{e^{i\kappa|x|}}{4\pi|x|}[P(i\kappa|x|)\mathbf{I} + Q(i\kappa|x|)\hat{x}\otimes\hat{x}]\right. \\ \left. - \frac{e^{-\beta|x|}}{4\pi|x|}[P(-\beta|x|)\mathbf{I} + Q(-\beta|x|)\hat{x}\otimes\hat{x}]\right\}\frac{\beta^2}{\beta^2+\kappa^2}.\end{aligned} \tag{A.3}$$

Now, by the inverse Fourier transformation and the expressions for P and Q,

$$\begin{aligned}\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} T_\kappa(\xi)f(\beta,\xi)d\xi &= (2\pi)^{-3/2}\lim_{|x|\to 0}\mathcal{F}^{-1}[T_\kappa(\xi)\,f(\beta,\xi)](x) \\ &= \frac{\beta+i\kappa}{6\pi}\frac{\beta^2}{\beta^2+\kappa^2}\mathbf{I}.\end{aligned} \tag{A.4}$$

Indeed, recall that $P(z) = 1 - \frac{1}{z} + \frac{1}{z^2}$, $Q(z) = -1 + \frac{3}{z} - \frac{3}{z^2}$.

- Consider the term $\frac{e^{i\kappa|x|}}{4\pi|x|}Q(i\kappa|x|) - \frac{e^{-\beta|x|}}{4\pi|x|}Q(-\beta|x|) =: A_Q$, then

$$A_Q = \frac{e^{i\kappa|x|}}{4\pi|x|}\left[-1 + \frac{3}{i\kappa|x|} + \frac{3}{\kappa^2|x|^2}\right] - \frac{e^{-\beta|x|}}{4\pi|x|}\left[-1 - \frac{3}{\beta|x|} - \frac{3}{\beta^2|x|^2}\right].$$

Using Taylor series, we obtain the following after few computations:

$$\frac{e^{i\kappa|x|}}{4\pi|x|}\left[-1 + \frac{3}{i\kappa|x|} + \frac{3}{\kappa^2|x|^2}\right] = \frac{1}{8\pi|x|} + \frac{3}{4\pi\kappa^2|x|^3} + o(|x|),$$

$$\frac{e^{-\beta|x|}}{4\pi|x|}\left[-1 - \frac{3}{\beta|x|} - \frac{3}{\beta^2|x|^2}\right] = \frac{1}{8\pi|x|} - \frac{3}{4\pi\beta^2|x|^3} + o(|x|).$$

By substituting the above expressions in A_Q, we obtain

$$\begin{aligned} A_Q &= \frac{3}{4\pi|x|^3}\left[\frac{1}{\kappa^2} + \frac{1}{\beta^2}\right] + o(|x|) \\ &= \frac{3}{4\pi\kappa^2|x|^3}\frac{1}{f(\beta,\kappa)} + o(|x|);\, f(\beta,\kappa) := \frac{\beta^2}{\beta^2+\kappa^2}. \end{aligned} \tag{A.5}$$

- Consider the term $\frac{e^{i\kappa|x|}}{4\pi|x|}P(i\kappa|x|) - \frac{e^{-\beta|x|}}{4\pi|x|}P(-\beta|x|) =: A_P$, then

$$A_P = \frac{e^{i\kappa|x|}}{4\pi|x|}\left[1 - \frac{1}{i\kappa|x|} - \frac{1}{\kappa^2|x|^2}\right] - \frac{e^{-\beta|x|}}{4\pi|x|}\left[1 + \frac{1}{\beta|x|} + \frac{1}{\beta^2|x|^2}\right].$$

Again by using the Taylor series, we obtain the following after few computations,

$$\frac{e^{i\kappa|x|}}{4\pi|x|}\left[1 - \frac{1}{i\kappa|x|} - \frac{1}{\kappa^2|x|^2}\right] = \frac{i\kappa}{6\pi} + \frac{1}{8\pi|x|} - \frac{1}{4\pi\kappa^2|x|^3} + o(|x|),$$

$$\frac{e^{-\beta|x|}}{4\pi|x|}\left[1 + \frac{1}{\beta|x|} + \frac{1}{\beta^2|x|^2}\right] = -\frac{\beta}{6\pi} + \frac{1}{8\pi|x|} + \frac{1}{4\pi\beta^2|x|^3} + o(|x|).$$

By substituting the above expressions in A_P, we obtain

$$\begin{aligned} A_P &= \frac{\beta + i\kappa}{6\pi} - \frac{1}{4\pi|x|^3}\left[\frac{1}{\kappa^2} + \frac{1}{\beta^2}\right] + o(|x|) \\ &= \frac{\beta + i\kappa}{6\pi} - \frac{1}{4\pi\kappa^2|x|^3}\frac{1}{f(\beta,\kappa)} + o(|x|). \end{aligned} \tag{A.6}$$

Gathering (A.3), (A.5) and (A.6) will produce

$$\begin{aligned} (2\pi)^{-3/2}\mathcal{F}^{-1}[T_\kappa(\xi)f(\beta,\xi)] &= \frac{\mathbf{I} - 3\hat{x}\otimes\hat{x}}{4\pi\kappa^2|x|^3} \\ &\quad + \left\{\frac{\beta+i\kappa}{6\pi}\mathbf{I} + \frac{3\hat{x}\otimes\hat{x} - \mathbf{I}}{4\pi\kappa^2|x|^3}\frac{1}{f(\beta,\kappa)}\right\} f(\beta,\kappa) + o(|x|) \\ &= \frac{\beta+i\kappa}{6\pi}f(\beta,\kappa)\mathbf{I} + o(|x|). \end{aligned}$$

Hence, (A.4) holds.

It remains to check the relation

$$\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} L_\kappa(\xi)\frac{\gamma^4}{\gamma^4+\xi^4}d\xi = -\frac{\tilde{\gamma}^3}{6\pi\kappa^2}\mathbf{I}, \quad \tilde{\gamma}=\gamma/\sqrt{2}. \tag{A.7}$$

From the definition of L_k, we see

$$\begin{aligned}
&\frac{1}{(2\pi)^{3/2}}\mathcal{F}^{-1}\left[L_\kappa(\xi)\frac{\gamma^4}{\gamma^4+|\xi|^4}\right](x)\\
&\quad = \frac{-1}{(2\pi)^3\kappa^2}\int_{\mathbb{R}^3}\frac{\gamma^4}{\gamma^4+|\xi|^4}\hat{\xi}\otimes\hat{\xi}e^{i\xi\cdot x}d\xi\\
&\quad = \frac{1}{(2\pi)^3\kappa^2}\nabla_x\nabla_x\int_{\mathbb{R}^3}\left(\frac{1}{|\xi|^2}-\frac{|\xi|^2}{\gamma^4+|\xi|^4}\right)e^{i\xi\cdot x}d\xi\\
&\quad = \frac{-1}{(2\pi)^{3/2}\kappa^2}\nabla_x\nabla_x\left(\mathcal{F}^{-1}\left[\frac{|\xi|^2}{\gamma^4+|\xi|^4}\right](x)-\mathcal{F}^{-1}\left[\frac{1}{|\xi|^2}\right](x)\right).
\end{aligned} \tag{A.8}$$

In view of (A.2),

$$\frac{1}{(2\pi)^{3/2}\kappa^2}\nabla_x\nabla_x\left(\mathcal{F}^{-1}\left[\frac{1}{|\xi|^2}\right](x)\right) = \frac{1}{\kappa^2}\nabla_x\nabla_x\frac{1}{4\pi|x|} = -\frac{\mathbf{I}-3\hat{x}\otimes\hat{x}}{4\pi\kappa^2|x|^3}.$$

To evaluate the first term on the right-hand side of (A.8), we need the integral identity

$$\begin{aligned}
\int_{\mathbb{R}^3}\frac{|\xi|^2}{\gamma^4+|\xi|^4}e^{i\xi\cdot x}d\xi &= \int_0^\infty\int_0^{2\pi}\int_0^\pi\frac{|\xi|^4}{\gamma^4+|\xi|^4}\sin\theta e^{i|\xi|\,|x|\cos\theta}d\theta d\phi d|\xi|\\
&= 4\pi\int_0^\infty\frac{|\xi|^3\sin(|\xi|\,|x|)}{(\gamma^4+|\xi|^4)|x|}d|\xi| = 2\pi^2\frac{e^{-\tilde{\gamma}|x|}\cos(\tilde{\gamma}|x|)}{|x|},
\end{aligned}$$

with $\tilde{\gamma}=\gamma/\sqrt{2}$, where the last equality follows from the Fourier sine transform of the odd function $t^3/(\gamma^4+t^4)$. It then follows that

$$\begin{aligned}
\frac{-1}{(2\pi)^{3/2}\kappa^2}\nabla_x\nabla_x\mathcal{F}^{-1}\left[\frac{|\xi|^2}{\gamma^4+|\xi|^4}\right](x) &= \frac{-1}{4\pi\kappa^2}\nabla_x\nabla_x\frac{e^{-\tilde{\gamma}|x|}\cos(\tilde{\gamma}|x|)}{|x|}\\
&= \frac{1}{4\pi\kappa^2}\{g(|x|)\mathbf{I}+|x|\,g'(|x|)\hat{x}\otimes\hat{x}\},
\end{aligned}$$

where $g(t)=\{e^{-\tilde{\gamma}t}[\cos(\tilde{\gamma}t)+\tilde{\gamma}t(\cos(\tilde{\gamma}t)+\sin(\tilde{\gamma}t))]\}/t^3$. After elementary calculations, we obtain

$$\begin{aligned}
\frac{1}{(2\pi)^{3/2}}\mathcal{F}^{-1}\left[L_\kappa(\xi)\frac{\gamma^4}{\gamma^4+|\xi|^4}\right](x) &= -\frac{\mathbf{I}-3\hat{x}\otimes\hat{x}}{4\pi\kappa^2}\{1-g(|x|)\}\\
&\quad -\frac{\tilde{\gamma}^2e^{-\tilde{\gamma}|x|}\sin(\tilde{\gamma}|x|)}{2\pi\kappa^2|x|}\hat{x}\otimes\hat{x},
\end{aligned}$$

and arguing similarly to the justification of (A.4), we obtain

$$\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} L_\kappa(\xi)\frac{\gamma^4}{\gamma^4+\xi^4}d\xi = \lim_{|x|\to 0}\frac{1}{(2\pi)^{3/2}}\mathcal{F}^{-1}\left[L_\kappa(\xi)\frac{\gamma^4}{\gamma^4+|\xi|^4}\right](x)$$
$$= -\frac{\tilde{\gamma}^3}{6\pi\kappa^2}\mathbf{I}.$$

This proves (A.7). Finally, combining (A.7) and (A.4) yields (2.28). □

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References

1. S. Albeverio, F. Gesztesy, R. Høegh-Krohn and H. Holden, *Solvable Models in Quantum Mechanics*, 2nd edn. (AMS Chelsea Publishing, 2005). With an Appendix by Pavel Exner.
2. H. Ammari, *An Introduction to Mathematics of Emerging Biomedical Imaging*, Mathématiques and Applications (Berlin) [Mathematics and Applications], Vol. 62 (Springer, 2008).
3. H. Ammari, J. Garnier, V. Jugnon and H. Kang, Stability and resolution analysis for a topological derivative based imaging functional, *SIAM J. Control Optim.* **50** (2012) 48–76.
4. H. Ammari, J. Garnier, H. Kang, W.-K. Park and K. Sølna, Imaging schemes for perfectly conducting cracks, *SIAM J. Appl. Math.* **71** (2011) 68–91.
5. H. Ammari, K. Hamdache and J.-C. Nédélec, Chirality in the Maxwell equations by the dipole approximation, *SIAM J. Appl. Math.* **59** (1999) 2045–2059 (electronic).
6. H. Ammari, E. Iakovleva, D. Lesselier and G. Perrusson, MUSIC-type electromagnetic imaging of a collection of small three-dimensional inclusions, *SIAM J. Sci. Comput.* **29** (2007) 674–709 (electronic).
7. H. Ammari and A. Khelifi, Electromagnetic scattering by small dielectric inhomogeneities, *J. Math. Pures Appl.* **82** (2003) 749–842.
8. H. Ammari, M. S. Vogelius and D. Volkov, Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter. II. The full Maxwell equations, *J. Math. Pures Appl.* **80** (2001) 769–814.
9. C. Amrouche, P. G. Ciarlet and P. Ciarlet, Jr., Weak vector and scalar potentials: Applications to Poincaré's theorem and Korn's inequality in Sobolev spaces with negative exponents, *Anal. Appl.* (*Singapore*) **8** (2010) 1–17.
10. K. Belkebir, P. C. Chaumet and A. Sentenac, Influence of multiple scattering on three-dimensional imaging with optical diffraction tomography, *J. Opt. Soc. Amer. A* **23** (2006) 586–595.
11. D. P. Challa and M. Sini, Inverse scattering by point-like scatterers in the Foldy regime, *Inverse Problems* **28** (2012) 125006.

12. G. Dassios and R. Kleinman, *Low Frequency Scattering*, Oxford Mathematical Monographs (Oxford Univ. Press, 2000).
13. P. de Vries, D. V. van Coevorden and A. Lagendijk, Point scatterers for classical waves, *Rev. Mod. Phys.* **70** (1998) 447–466.
14. H. J. S. Dorren and A. Tip, Maxwell's equations for nonsmooth media; fractal-shaped and pointlike objects, *J. Math. Phys.* **32** (1991) 3060–3070.
15. L. L. Foldy, The multiple scattering of waves. I. General theory of isotropic scattering by randomly distributed scatterers, *Phys. Rev.* **67** (1945) 107–119.
16. D. Gintides, M. Sini and N. T. Thành, Detection of point-like scatterers using one type of scattered elastic waves, *J. Comput. Appl. Math.* **236** (2012) 2137–2145.
17. G. Hu and M. Sini, Elastic scattering by finitely many point-like obstacles, *J. Math. Phys.* **54** (2013) 042901.
18. A. R. Jones, Electromagnetic wave scattering by assemblies of particles in the Rayleigh approximation, *Proc. Roy. Soc. Lond. A* **366** (1979) 111–127.
19. A. R. Jones, Scattering efficiency factors for agglomerates for small spheres, *J. Phys. D: Appl. Phys.* **12** (1979) 1661.
20. A. Kirsch and N. Grinberg, *The Factorization Method for Inverse Problems*, Oxford Lecture Series in Mathematics and its Applications (Oxford Univ. Press, 2008).
21. P. A. Martin, *Multiple Scattering: Interaction of Time-Harmonic Waves with N Obstacles*, Encyclopedia of Mathematics and its Applications, Vol. 107 (Cambridge Univ. Press, 2006).
22. P. Monk, *Finite Element Methods for Maxwell's Equations* (Oxford Univ. Press, 2003).
23. J.-C. Nédélec, *Acoustic and Electromagnetic Equations: Integral Representations for Harmonic Problems*, Applied Mathematical Sciences, Vol. 144 (Springer, 2001).
24. A. G. Ramm, *Wave Scattering by Small Bodies of Arbitrary Shapes* (World Scientific, 2005).
25. R. Song, R. Chen and X. Chen, Imaging three-dimensional anisotropic scatterers in multilayered medium by multiple signal classification method with enhanced resolution, *J. Opt. Soc. Amer. A* **29** (2012) 1900–1905.
26. J. F. van Diejen and A. Tip, Scattering from generalized point interactions using self-adjoint extensions in pontryagin spaces, *J. Math. Phys.* **32** (1991) 630–641.
27. K. M. Watson, Multiple scattering of electromagnetic waves in an underdense plasma, *J. Math. Phys.* **10** (1969) 688–702.
28. R. Weder, *Spectral and Scattering Theory for Wave Propagation in Perturbed Stratified Media*, Applied Mathematical Sciences, Vol. 87 (Springer, 1991).
29. Y. Zhong and X. Chen, MUSIC imaging and electromagnetic inverse scattering of multiple-scattering small anisotropic spheres, *IEEE Trans. Antennas Propag.* **55** (2007) 3542–3549.