MULTISCALE MODEL. SIMUL.
Vol. 12, No. 3, pp. 996–1027

DIRECT AND INVERSE ACOUSTIC SCATTERING BY A COLLECTION OF EXTENDED AND POINT-LIKE SCATTERERS*

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Abstract. This paper concerns the acoustic scattering by an extended obstacle surrounded by point-like obstacles. The extended obstacle is supposed to be rigid, while the point-like obstacles are modeled by point perturbations of the exterior Laplacian. In the first part, we consider the forward problem. Following two equivalent approaches (the Foldy formal method and the Krein resolvent method), we show that the scattered field is a sum of two contributions: one is due to the diffusion by the extended obstacle, and the other arises from the linear combination of the interactions between the point-like obstacles and the interaction between the point-like obstacles with the extended one. In the second part, we deal with the inverse problem. It consists in reconstructing both the extended and point-like scatterers from the corresponding far-field pattern. To solve this problem, we describe and justify the factorization method of Kirsch. Using this method, we provide several numerical results and discuss the multiple scattering effect concerning both the interactions between the point-like obstacles and between these obstacles and the extended one.

Key words. Foldy method, Krein resolvent method, point interactions, multiple scattering, self-adjoint extension, inverse acoustic scattering, factorization method, multiscale

AMS subject classifications. 78A45, 78A46, 47B25, 81Q10, 35R30, 35J05, 35L05

DOI. 10.1137/130932107

1. Introduction.

Let D be a bounded and C^2-smooth simply connected set of $\mathbb{R}^3$. We consider the scattering of a time-harmonic acoustic plane wave from the obstacle D and an inhomogeneous isotropic medium with an index of refraction $n = n(x) > 0$ in $\mathbb{R}^3\backslash\overline{D}$. It is supposed that the inhomogeneous medium occupies a bounded domain such that $n(x) = 1$ for x outside of some sufficiently large ball containing D. The time-harmonic incident plane waves take the form

$$u^{in}(x;d) = \exp(i\kappa\, x\cdot d)\,,$$

where κ is the wave number corresponding to the background medium and $d \in \mathbb{S}^2 := \{x : |x| = 1\}$ denotes the propagation direction, while their evolution in time is defined by the modulation factor: $\exp(-i\omega t)$. Then the total acoustic fields u satisfy the reduced time-harmonic acoustic equation

$$\Delta u + \kappa^2\, n(x)u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}\,, \tag{1.1}$$

with the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial D\,. \tag{1.2}$$

*Received by the editors August 6, 2013; accepted for publication (in revised form) April 24, 2014; published electronically July 8, 2014.
http://www.siam.org/journals/mms/12-3/93210.html

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The corresponding scattered fields $u^{sc} := u - u^{in}$ are required to satisfy the Sommerfeld radiation condition

$$\lim_{|x|\to\infty} |x|(\nabla u^{sc} \cdot \hat{x} - i\kappa u^{sc}) = 0\,, \tag{1.3}$$

uniformly in all directions $\hat{x} := \frac{x}{|x|} \in \mathbb{S}^2$, leading to the far-field patterns $u^\infty(\hat{x})$ given by the asymptotic behavior

$$u^{sc}(x,d) = \frac{e^{i\kappa\,|x|}}{4\pi\,|x|}\left\{u^\infty\,(\hat{x},d) + O\left(\frac{1}{|x|}\right)\right\} \tag{1.4}$$

as $|x| \to \infty$. The function $u^\infty(\cdot,\cdot) : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{C}^2$ is called the far-field pattern of u^{sc}, and it is well known that it is an analytic function.

In this work, we assume that the inhomogeneous medium in $\mathbb{R}^3 \setminus \overline{D}$ consists of a finite number of components whose diameter is much smaller than the incidence wavelength. Then the interaction between the incident wave and the inhomogeneous part of the medium can be modeled as a scattering problem with a collection of point-like obstacles located in $Y = \{y_j\}_{j=1}^N$, where $Y \subset \mathbb{R}^3 \setminus \overline{D}$ and $\sup_j |y_j| < \infty$. As has been suggested in [11], this corresponds to introduce a formal delta-like perturbation of the refraction index

$$(n(x) - 1) = \sum_{j=1}^{N} a_j\, \kappa^2\, \delta(x - y_j) \tag{1.5}$$

as a potential term in the definition of the on-shell T-matrix. The resulting model describes a system of *N-point scatterers* (see also the definition in [12]) surrounding an extended obstacle.

There is a large literature dealing with the forward problem for the case where the extended obstacle is absent. We mention, for instance, the book [23] describing the corresponding Foldy method. Regarding the inverse problem, which consists in locating the point-like scatterers and reconstructing the scattering coefficients from the far-field pattern, we mention the works [10, 13, 14, 21, 22].

The contributions of the present work are twofold. In the first part, we study the forward problem following two equivalent approaches. As a first approach (section 3.1), we use the Foldy's formal method, introduced in [12], according to which the scattered field is the sum of two contributions: one is due to the diffusion by the extended obstacle, and the other one is a linear combination of the interactions between the point-like obstacles and the interaction between the point-like obstacles with the extended one through some *scattering coefficients* g_j, $j = 1,\ldots,N$. We obtain a closed form of the solution; see (3.10) in section 3.1. To use this formulation, we need to solve the scattering problem by the extended obstacle only, i.e., in the absence of the point-like obstacles, and then invert the algebraic system (3.7). To obtain this closed form we use the Green's function of the scattering by the extended obstacle. A slightly different strategy, still following the Foldy formal method, has been developed in [15, 17]. The authors use only the fundamental solution of the free space and write the scattered field as a sum of point sources, modeling the scattering by the point-like obstacles, and the scattered field due to the extended obstacle represented by layer potentials. The price to pay is that the coefficients and the densities appearing in this representation are solutions of a coupled system of integral equations instead of an algebraic system, as we do here. They propose and justify an iterative method to solve this system.

As a second approach, we model the point scatterers as (frequency-dependent) point interactions. In section 3.2, we follow this line and look at the acoustic propagator as a point perturbation of the one modeling the scattering without the point-like obstacles. The perturbed operator is obtained as a self-adjoint extension of the symmetric operator Q acting as $-\Delta$ on the domain

$$D(Q) = \left\{ u \in H^2 \cap H_0^1 \left(\mathbb{R}^3 \backslash D\right) \mid u(y_i) = 0\,,\ y_j \in Y \right\}. \tag{1.6}$$

It is well known that these extensions are defined through boundary conditions occurring in the points y_j (e.g., in [1]). Using the boundary triples technique (e.g., in [9]), they can be parametrized by linear operators on $\mathbb{C}^N$. In particular, the "local" point interactions are those defined by using a parameter matrix $\alpha \in \mathbb{C}^{N,N}$ of the form $\alpha = \operatorname{diag}(\alpha_j)$, $\alpha_j \in \mathbb{R}$. The explicit character of the model allows one to obtain a "Krein-like" formula for the corresponding scattered field due to incident plane waves. In section 3.2 we show that this is nothing else but the one obtained using the Foldy method where the scattering coefficients g_j are explicitly related to the parameters α_j and the boundary values of the singular sources $\Phi_D^{sc}(y_j, y_j, \kappa)$ (i.e., the scattered fields by the extended obstacle D corresponding to point sources as incident fields) according to (see Remark 3.6)

$$g_j := \left(\Phi_D^{sc}(y_j, y_j, \kappa) + \frac{i\kappa}{4\pi} - \alpha_j \right)^{-1}, \qquad j = 1, \ldots, N\,. \tag{1.7}$$

The main interest in using the theory of operator extensions to build our model of point scatterers stands upon the fact that this approach naturally leads to an appropriate scattering problem. More precisely, we show that the scattered field solves the Helmholtz equation in $\mathbb{R}^3 \backslash \{D \cup Y\}$ with a Dirichlet condition on ∂D and impedance-like boundary conditions on Y, depending on α_j (see problem (3.43)).

In the second part of our work, we investigate the inverse scattering problem of reconstructing both the extended and point-like obstacles from the corresponding far-field pattern with infinitely many incident directions. Based on the novel boundary value problem (3.43) for the forward scattering, our aim is to describe and justify the factorization method. This method was first put forward by Kirsch [18] to reconstruct extended obstacles from the spectral data of far-field operators. It requires neither computation of direct solutions nor initial guesses and provides a sufficient and necessary condition for characterizing the shape of the extended obstacle and positions of the point-like scatterers (see Theorem 4.10). We refer to the monograph [19] and references therein for a detailed discussion of the various versions of the factorization method for acoustic scattering from extended impenetrable and penetrable scatterers. Using this method, we provide several numerical results and discuss the multiple scattering effects reflecting the interactions between the point-like obstacles and also between these obstacles with the extended one. It is worth noting that our arguments for the direct and inverse problems can be both extended to the case where the extended scatterer is a sound-hard obstacle, an impedance-type obstacle, or a penetrable medium; see Remark 4.12(iii).

For references related to the inverse issue we mention the work [16], where a MUSIC algorithm is used. This algorithm is designed and widely used mainly to detect point locations of small or point-like scatterers. However, there is no guarantee that it provides correct information on the shape of extended scatterers even though some explanation is provided in [30], for instance.

The paper is organized as follows. In section 2, we briefly recall some of the main ideas in the modeling of point-like scatterers, namely, the Foldy and the regularization methods, and suggest a possible link with the theory of the singular perturbations of the Dirichlet Laplacian in $\mathbb{R}^3$: this leads to an alternative approach to the modeling of point scatterers. In section 3, we study our forward problem following these two approaches. We first provide the far-field representation using the Foldy method in section 3.1, and then, in section 3.2, we construct a frequency-dependent point interaction model leading to our scattering problem. In section 4, we study the inverse problem and justify the factorization method for our problem, while in section 5 we test this method numerically.

2. Acoustic scattering by point scatterers. We next introduce the acoustic scattering model in the simplest case of point-like obstacles in $\mathbb{R}^3$. Different approaches to this problem have been developed; see [12] and [11]. Suitable assumptions on the scattering interaction involving point scatterers, and formal computations lead, in these works, to similar nonperturbative representations of the scattering waves (see (2.6) and (2.7) below). After briefly recalling the ideas underlying these methods, we show that both can be rephrased in terms of a frequency-dependent point interaction model.

2.1. The formal methods. The notion of point scatterer arises from the work of Foldy [12], where the steady state scattering problem for a scalar wave of a single frequency κ is considered in $\mathbb{R}^3$. Assuming the interaction to act only on the spherical symmetric part of the incident waves (*isotropy prescription*), the scattered field near a point scatterer y_j behaves as

$$A_j \Phi^\kappa(\cdot, y_j),$$

where $\Phi^\kappa(x, y)$ is the fundamental solution of the three-dimensional (3D) Helmholtz equation

$$\Phi^\kappa(x,y) = \frac{e^{i\kappa|x-y|}}{4\pi\,|x-y|}\,. \tag{2.1}$$

The total field is represented by summation according to

$$u = u_0 + \sum_{j=1}^{N} A_j \Phi^\kappa(\cdot, y)\;, \tag{2.2}$$

where the incident field, u_0, is a solution of the problem $\left(\Delta + \kappa^2\right) u_0(\cdot, \kappa) = 0$ in $\mathbb{R}^3$. The external field u_j acting on a scatterer y_j is then

$$u_j = u - A_j \Phi^\kappa(\cdot, y_j) = u_0 + \sum_{j' \neq j} A_j \Phi^\kappa(\cdot, y)\;. \tag{2.3}$$

Assuming the coefficient A_j to be proportional to the external field $u_j(y_j)$ at the scatterer y_j, i.e.,

$$A_j = g_j(\kappa)\, u_j(y_j), \tag{2.4}$$

we derive the following linear algebraic system:

$$u_j\left(y_j\right) = u_0 + \sum_{j'\neq j} g_{j'}\left(\kappa\right) u_{j'}\left(y_{j'}\right) \Phi^{\kappa}(y_j, y_{j'}). \tag{2.5}$$

Following [12], (2.2)-(2.4)-(2.5) are referred to as the fundamental equations of the multiple scattering, while $g_j\left(\kappa\right)$ is referred to as "scattering coefficient" of the jth scatterer. These coefficients are assigned physical data; they can depend both on the frequency κ and on some other scattering parameter (see [12] for this point) and fix the properties of the scattering interactions. Next, assume u_0 to describe an incident plane wave of wave number κ propagating in the direction d and scattering on a single point scatterer located in y. The previous equations read as

$$u = u_0 + g\left(\kappa\right) u_0\left(y\right) \Phi^{\kappa}(\cdot, y)\,, \qquad u_0(x) = e^{i\kappa\, d\cdot x}. \tag{2.6}$$

A physical interpretation of the Foldy coefficients, for the interaction between classical waves and matter, has been given in [11]. In the acoustic case, the authors define the interaction through a pointwise perturbation of the refraction index n. Assuming $n(x) = 1 + a\delta\left(x - y\right)$, for a single point scatterer located in y, a formal computation yields the following representation of the outgoing scattering waves:

$$\psi_a\left(x, \kappa, d\right) = e^{i\kappa\, d\cdot x} + \tau\left(\kappa, a, \Lambda\right) e^{i\kappa\, d\cdot y} \Phi^{\kappa}\left(x, y\right)\,, \tag{2.7}$$

where τ depends on the wave number κ, the coupling coefficient a, and a cutoff momentum Λ (used in the formal regularization of the unperturbed Green's function) according to

$$\tau\left(\kappa, a, \Lambda\right) = \frac{\kappa^2}{1/a - \kappa^2\Lambda/4\pi - i\kappa^3/4\pi}\,. \tag{2.8}$$

(We refer the reader to the equations (22) and (37) in [11]. See also [5], where this approach is rigorously justified in the case of the Maxwell system as well). Comparing (2.7) with (2.6) leads to the equivalence of the two models once the identity $g\left(\kappa\right) = \tau\left(\kappa, a, \Lambda\right)$ is assumed for fixed values of the parameters κ, a, and Λ.

2.2. A point interaction model. In operator theory, equations similar to (2.13) are known to represent the scattering wave functions associated to pointwise singular perturbations of the 3D Laplacian (we refer the reader to [1] for an exhaustive presentation of this topic). Consider as an example the case of the self-adjoint one-point interactions centered in $y \in \mathbb{R}^3$; these are modeled by a one-parameter family of operators H_α, $\alpha \in \mathbb{R}$, defined as self-adjoint extensions of the symmetric minimal Laplacian H_0,

$$D(H_0) = \left\{ u \in H^2\left(\mathbb{R}^3\right) \mid u(y) = 0 \right\}\,, \qquad H_0 u = -\Delta u\,. \tag{2.9}$$

The defect spaces $\ker\left(H_0^* - z\right)$, of the adjoint operator H_0^*, are related to the Green's kernels: $G^z(x) = e^{i\sqrt{z}|x-y|}/4\pi\left|x - y\right|$ (here $z \in \mathbb{C}\backslash\mathbb{R}_+$ and the square root is fixed with $\operatorname{Im}\sqrt{z} > 0$). In particular, a direct computation shows that $\ker\left(H_0^* - z\right) = l.c.\left\{G^z\right\}$ (we refer the reader to [1, Chapter I.1.1] for this point), which, according to the properties of closed symmetric operators (see [27, Chapter X]), results in

$$D\left(H_0^*\right) = D(H_0) \oplus \ker\left(H_0^* - i\right) \oplus \ker\left(H_0^* + i\right)\,. \tag{2.10}$$

Then, by (2.10), any $u \in D(H_0^*)$ exhibits the asymptotics

$$u \sim \frac{c_1}{4\pi |x-y|} + c_2 + o(1)\,, \quad c_{j=1,2} \in \mathbb{C}\,, \tag{2.11}$$

near y (see also [2, 3, 20]).

The coefficients $c_{j=1,2}$ can be used to describe the extension H_α as a restriction of H_0^* to the set of functions satisfying the boundary conditions: $\alpha c_1 = c_2$ (for this point see [20] and the references therein). This yields the representation

$$D(H_\alpha) = \{u \in D(H_0^*) \mid (2.13) \text{ holds}\}\,, \tag{2.12}$$

(2.13)

$$\alpha \lim_{x\to y} 4\pi |x-y|\, u(x) = \lim_{x\to y} \left(u(x) - \frac{\lim_{x\to y} 4\pi |x-y|\, u(x)}{4\pi |x-y|} \right) \quad \forall\, u \in D(H_\alpha)\,.$$

According to the theory of 3D point interactions, H_α acts nontrivially only in the s-wave (i.e., spherically symmetric states in $D(H_\alpha)$), and the related s-wave scattering length ℓ_α depends on the parameter α according to $\ell_\alpha = -(4\pi\alpha)^{-1}$ (see equation (1.4.10) in [1] and the subsequent remarks). In addition, the total field corresponding to an incident plane wave propagating in the direction d with wave number κ has the form (equation (1.4.11) in [1])

$$\Psi_\alpha(x,\kappa,d) = e^{i\kappa\, d\cdot x} + (\alpha - i\kappa/4\pi)^{-1} e^{i\kappa\, d\cdot y} \Phi^\kappa(\cdot, y)\,. \tag{2.14}$$

Comparing this relation with (2.6) shows that the scattering waves for an acoustic point scatterer, defined by the scattering coefficient $g(\kappa)$, coincide, for each frequency κ, with those associated to a one-center point interaction Hamiltonian $H_{\alpha(\kappa)}$ with the frequency-dependent parameter $\alpha(\kappa)$:

$$(\alpha(\kappa) - i\kappa/4\pi)^{-1} = g(\kappa)\,. \tag{2.15}$$

Moreover, from the comparison between (2.7)–(2.8) and (2.14) we get

$$\alpha := \frac{1}{\kappa^2 a} - \frac{\Lambda}{4\pi}\,. \tag{2.16}$$

For fixed Λ and κ, this provides a link between a pointwise perturbation of the refraction index and the scattering length ℓ_α of the corresponding point interaction model. In particular, from (2.16) we have $\ell_\alpha^{-1} = \Lambda - 4\pi\left(\kappa^2 a\right)^{-1}$.

Let us finish this section by showing how the scattering by a point scatterer can be described in terms of a boundary value problem. The total field u, corresponding to the incident wave $u_0 = e^{i\kappa\, d\cdot x}$, is given in (2.6) or (2.14). Taking into account (2.15), the scattered field $u^{sc} := \Psi_\alpha(x,\kappa,d) - e^{i\kappa\, d\cdot x}$ solves the problem

$$\begin{cases} \left(\Delta + \kappa^2\right) u^{sc}(x) = 0\,, \quad \text{in } \mathbb{R}^3 \setminus \{y\}\,, \\ \lim_{x\to y} 4\pi |x-y|\, u^{sc}(x) = (\alpha(\kappa) - i\kappa/4\pi)^{-1} e^{i\kappa\, d\cdot y}\,, \\ \lim_{|x|\to\infty} |x| \left(\nabla u^{sc}\cdot \hat{x} - i\kappa u^{sc}\right) = 0 \qquad \forall\, \hat{x} \in \mathcal{S}\,. \end{cases} \tag{2.17}$$

By straightforward computations, the condition across the scatterer y can be replaced by the following "impedance-type condition" for the total field u,

$$\Gamma_2(u) = \alpha(\kappa)\ \Gamma_1(u)\,, \tag{2.18}$$

where

$$\Gamma_1(u) := \lim_{x\to y} 4\pi|x-y|u(x) \quad \text{and} \quad \Gamma_2(u) := \lim_{x\to y}\left(u(x) - \frac{\Gamma_1(u)}{4\pi|x-y|}\right).$$

The relation (2.18) is of course nothing but (2.13). The characterization of the scattered fields as a solution of the problem of type (2.17) with "impedance" boundary condition (2.18) across the point scatterer y will be very useful in studying the inverse problem in section 4.

3. The model for acoustic scattering by extended and point-like scatterers. In what follows, we adapt the methods described in sections 2.1 and 2.2 to the case of point scatterers in the exterior domain $\mathbb{R}^3 \setminus \overline{D}$.

3.1. The Foldy method. Let u_D denote a solution of the scattering problem (1.1)–(1.3), when $n = 1$, i.e., homogeneous medium, corresponding to an incident field u^{in}; the scattered field u_D^{sc} is defined according to

$$u_D^{sc} = u_D - u^{in}\,. \tag{3.1}$$

The Green's function Φ_D^κ associated to this problem is the unique solution of the boundary value problem

$$\begin{cases} \left(\Delta + \kappa^2\right)\Phi_D^\kappa = -\delta(\cdot, y) & \text{in } \mathbb{R}^3 \setminus \overline{D}\,, \\ \Phi_D^\kappa(\cdot, y) = 0 & \text{in } \partial D\,, \end{cases} \tag{3.2}$$

such that the scattered field

$$\Phi_D^{sc}(\cdot, y, \kappa) = \Phi_D^\kappa(\cdot, y) - \Phi^\kappa(\cdot, y)\,, \tag{3.3}$$

with Φ^κ given in (2.1), satisfies the Sommerfeld radiation condition.

In what follows, we use the Foldy method to study the scattering problem from the sound-soft obstacle D surrounded by a system of point scatterers located in $Y = \{y_j\}_{j=1}^N$, where $Y \subset \mathbb{R}^3 \setminus \overline{D}$ and $\sup_j |y_j| < \infty$ (see also [12] or [23, p. 298] for the acoustic case without extended obstacles). Proceeding as in section 2.1, the total field is represented as

$$u(x) = u_D(x) + \sum_{j=1}^N \Phi_D^\kappa(x, y_j)\, A_j\,, \tag{3.4}$$

where A_j are unknown constants. The field

$$u_j(x) = u(x) - \Phi_D^\kappa(x, y_j)\, A_j = u_D(x) + \sum_{\substack{l=1\\ l\neq j}}^N \Phi_D^\kappa(x, y_l)\, A_l \tag{3.5}$$

is now regarded as the external field incident on the jth scatterer in the presence of all the other scatterers. The physical assumption in the Foldy method is that the strength of the scattered wave from the scatterer y_j is proportional to the external field on it. In our case this is given by the assumption that

$$A_j = g_j\, u_j(y_j)\,, \tag{3.6}$$

where g_j is the scattering coefficient of the scatterer y_j. Evaluating (3.5) at y_j, we obtain

$$u_j(y_j) = u_D(y_j) + \sum_{\substack{l=1\\ l\neq j}}^{N} g_l \Phi_D^\kappa(y_j, y_l)\, u_l(y_l)\,, \tag{3.7}$$

and then (3.4) becomes

$$u(x) = u_D(x) + \sum_{j=1}^{N} g_j \Phi_D^\kappa(x, y_j)\, u_j(y_j)\,. \tag{3.8}$$

Following the seminal paper [12], we refer to (3.7)–(3.8) as the fundamental system of multiple scattering.

In particular, (3.7) can be written as the algebraic linear system

$$[\tilde{\Gamma}]_{N\times N}\, [\Lambda]_{N\times 1} = [\mathbf{u}]_{N\times 1}\,,$$

with $\Lambda := (u_1(y_1), u_2(y_2), \ldots, u_N(y_N))^\top \in \mathbb{C}^{N\times 1}$, $\mathbf{u} := (u_D(y_1), \ldots, u_D(y_N))^\top \in \mathbb{C}^{N\times 1}$, and

$$\tilde{\Gamma} := \begin{pmatrix} \mathbf{I} & -g_2\Phi_D^\kappa(y_1,y_2) & -g_3\Phi_D^\kappa(y_1,y_3) & \cdots & -g_N\Phi_D^\kappa(y_1,y_N) \\ -g_1\Phi_D^\kappa(y_2,y_1) & \mathbf{I} & -g_3\Phi_D^\kappa(y_2,y_3) & \cdots & -g_N\Phi_D^\kappa(y_2,y_N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -g_1\Phi_D^\kappa(y_N,y_1) & -g_2\Phi_D^\kappa(y_N,y_2) & -g_3\Phi_D^\kappa(y_N,y_3) & \cdots & \mathbf{I} \end{pmatrix}. \tag{3.9}$$

Assuming $\det(\tilde{\Gamma}) \neq 0$ and denoting the elements of $\tilde{\Gamma}^{-1} \in \mathbb{C}^{N\times N}$ by $[\tilde{\Gamma}^{-1}]_{lj}$ for $l, j = 1, 2, \ldots, N$, we deduce from (3.4) that the scattered field takes the form

$$u^{sc}(x) := u(x) - u^{in}(x) = u_D^{sc}(x) + \sum_{l,j=1}^{N} g_j\, \Phi_D^\kappa(x, y_j)\, [\tilde{\Gamma}^{-1}]_{jl}\, u_D(y_l)\,, \tag{3.10}$$

with the far-field pattern

$$u^\infty(\hat{x}) = u_D^\infty(\hat{x}) + \sum_{l,j=1}^{N} g_j\, (\Phi_D^\infty(\hat{x}, y_j, \kappa) + e^{-i\kappa\hat{x}\cdot y_j})\, [\tilde{\Gamma}^{-1}]_{jl} u_D(y_l)\,, \quad \hat{x} \in \mathbb{S}^2\,,$$

where $\Phi_D^\infty(\hat{x}, y_j, \kappa) + e^{-i\kappa\hat{x}\cdot y_j}$ is the far field corresponding to the incident source $\Phi_D^\kappa(\cdot, y_j) = \Phi_D^{sc}(\cdot, y_j, \kappa) + \Phi^\kappa(\cdot, y_j)$ and $u_D^\infty(\hat{x})$ is the far field corresponding to the scattered field u_D^{sc} by the obstacle D.

3.2. A point interaction model in $\mathbb{R}^3\backslash\overline{D}$. As has been remarked in section 2, scattering by point scatterers can be described in terms of point interactions models provided that a suitable correspondence between the scattering coefficients and some operator extension parameter is established. In what follows, we develop this approach in the case of a set of point scatterers, located in $Y = \{y_j\}_{j=1}^N$, and surrounding an extended sound-soft obstacle whose support, D, is assumed to fulfill the conditions discussed in the introduction. The corresponding point interaction models, obtained as singular perturbations of the Dirichlet Laplacian in $\mathbb{R}^3\backslash D$, are defined as a self-adjoint extension of the symmetric operator Q,

$$\left\{\begin{array}{l} D\left(Q\right)=\left\{u\in H^2\cap H_0^1\left(\mathbb{R}^3\backslash D\right) \mid u(y_i)=0\,,\ y_i\in Y\right\}\,, \\ \\ Qu=-\Delta u, \end{array}\right. \tag{3.11}$$

and their physical properties are encoded by conditions occurring in the boundary points y_i. The extensions of symmetric operators and the related spectral properties are the objects of a permanent interest, both from theoretical as well as practical points of view. Focusing on the case of self-adjoint point interaction models, a large and exhaustive introduction can be found in [1]. The case of point interactions in bounded domains is discussed in [4].

3.2.1. The Green's functions. Since $Q\subset Q^*$, the self-adjoint extensions of Q identify with a class of restrictions of Q^* fulfilling prescribed linear relations on a "boundary space." Following this line, we next consider the adjoint operator. Making use of the von Neumann decomposition formula (e.g., in [27, Chapter X]), this writes as

$$D\left(Q^*\right)=D\left(Q\right)\oplus\mathcal{N}_i\oplus\mathcal{N}_{-i}\,, \tag{3.12}$$

where $\mathcal{N}_z$, the defect spaces of Q, are defined by

$$\mathcal{N}_z=\ker\left(Q^*-z\right)\,. \tag{3.13}$$

Then, due to the inclusion $Q\subset Q^*$, Q^* acts as $-\Delta$ on the regular part of its domain, while from the above definition we have

$$Q^*u=\pm iu \qquad \text{as } u\in\mathcal{N}_{\pm i}\,. \tag{3.14}$$

Next we give an explicit representation of $\mathcal{N}_z$ in terms the Green's functions Φ_D^ζ introduced before. Let $\zeta\in\mathbb{C}^+$ and $x,y\in\mathbb{R}^3\backslash D$ such that $x\neq y$. To be consistent with (3.3) we write

$$\Phi_D^{sc}(x,y,\zeta):=\Phi_D^\zeta\left(x,y\right)-\Phi^\zeta\left(x,y\right)\,, \tag{3.15}$$

where

$$\Phi^\zeta\left(x,y\right)=\frac{e^{i\zeta|x-y|}}{4\pi\left|x-y\right|}\,. \tag{3.16}$$

Then the function $\Phi_D^{sc}(\cdot,y,\zeta)$ solves the boundary value problem

$$\left\{\begin{array}{ll} \left(\Delta+\zeta^2\right)\Phi_D^{sc}\left(\cdot,y,\zeta\right)=0 & \text{in } \mathbb{R}^3\backslash\overline{D}, \\ \\ \Phi_D^{sc}\left(\cdot,y,\zeta\right)|_{\partial D}=-\Phi^\zeta\left(\cdot,y\right)|_{\partial D}\,, & \\ \\ \lim_{|x|\to\infty}|x|\left(\hat{x}\cdot\nabla_x-i\zeta\right)\Phi_D^{sc}\left(\cdot,y,\zeta\right)=0\,. & \end{array}\right. \tag{3.17}$$

In what follows, Q_0 denotes the Dirichlet Laplacian in the exterior domain $\mathbb{R}^3\backslash D$, i.e.,

$$Q_0:\left\{\begin{array}{l} D\left(Q_0\right)=H^2\left(\mathbb{R}^3\backslash\overline{D}\right)\cap H_0^1\left(\mathbb{R}^3\backslash\overline{D}\right)\,, \\ \\ Q_0u=-\Delta u. \end{array}\right. \tag{3.18}$$

The Green's function $\Phi_D^\zeta\left(x,y\right)$ enjoys the following properties.

LEMMA 3.1. *Let* $x, y \in \mathbb{R}^3 \backslash \overline{D}$.

1. *For* $x \neq y$, *the map* $\zeta \to \Phi_D^\zeta(x, y)$ *is holomorphic in* $\mathbb{C}^+$ *and continuously extends to the whole real axis.*
2. *For a fixed* $\zeta \in \mathbb{C}^+$, *this gives* $\Phi_D^\zeta(\cdot, y) \in L^2(\mathbb{R}^3 \backslash D)$, *while for* $\zeta \in \overline{\mathbb{C}^+}$ *the functions* $\Phi_D^\zeta(\cdot, y)$ *are* $\mathbb{C}^\infty$*-smooth in* $\mathbb{R}^3 \backslash (\overline{D} \cup y)$ *and satisfy the Sommerfeld radiation condition*

$$\left(\frac{x}{|x|}\nabla_x - i\zeta\right)\Phi_D^\zeta(x, y) = o\left(\frac{1}{|x|}\right). \tag{3.19}$$

Proof. From the definition (3.16), the result holds in the case of $\Phi^\zeta(\cdot, y)$. We consider next $\Phi_D^{sc}(\cdot, y, \zeta)$. Recall that it is the unique solution of the scattering problem (3.17). This solution can be represented using the layer potentials approach (see, for instance, [7, 24]) as follows:

$$-\Phi_D^{sc}(\cdot, y, \zeta) = \int_{\partial D} \left\{\partial_\nu \Phi^\zeta(\cdot, y') - i\eta\Phi^\zeta(\cdot, y')\right\} \varphi(y', y)\, ds(y'), \tag{3.20}$$

where ∂_ν denotes the normal derivative oriented towards the exterior of D, while η is fixed in $\mathbb{R}^+$. The potential $\varphi(\cdot, y)$ is the unique solution of the integral equation of second kind,

$$\big(I + K - i\eta S\big)\varphi(\cdot, y) = 2\Phi^\zeta(\cdot, y) \quad \text{on } \partial D, \tag{3.21}$$

where S and K are, respectively, the single- and double-layer potential operators defined by

$$S(u)(x) := 2\int_{\partial D} \Phi^\zeta(x, y)u(y)ds(y) \quad \text{for } x \in \partial D,$$

$$K(u)(x) := 2\int_{\partial D} \partial_{\nu(y)}\Phi^\zeta(x, y)u(y)ds(y) \quad \text{for } x \in \partial D.$$

The integral operator $I + K - i\eta S : L^2(\partial D) \to L^2(\partial D)$ (respectively, $C(\partial D) \to C(\partial D)$) is Fredholm, and since $\eta \neq 0$ it is injective; see, for instance, [7, 24]. In addition it is holomorphic in $\mathbb{C}$. From the Fredholm theory, its inverse is meromorphic in $\mathbb{C}$. This implies that φ is also meromorphic as a function from $\mathbb{C}$ to $L^2(\partial D)$. From the representation (3.20), we deduce that for every x, y in $\mathbb{R}^3 \backslash \overline{D}$, $\Phi_D^{sc}(x, y, \zeta)$ is also meromorphic in $\mathbb{C}$. Next, we show that the eventual poles are located in $\mathbb{C}^-$. We introduce the auxiliary function $s(\cdot, y, \zeta) := \Phi^\zeta(\cdot, y)v(\cdot)$, where $v \in C_0^\infty(\mathbb{R}^3)$ such that $v = 1$ in $\overline{D}$. Setting $-\rho^\zeta = \Phi_D^{sc}(\cdot, y, \zeta) + s$, problem (3.17) is rephrased as

$$\begin{cases} \left(-\Delta - \zeta^2\right)\rho^\zeta(\cdot, y) = F^\zeta(\cdot, y) & \text{in } \mathbb{R}^3 \backslash D, \\ \rho^\zeta(\cdot, y)\big|_{\partial D} = 0, \qquad F^\zeta(\cdot, y) = \left(-\Delta - \zeta^2\right)s(\cdot, y, \zeta). \end{cases} \tag{3.22}$$

Correspondingly, we denote the solution of the above problem as

$$\rho^\zeta(\cdot, y) = \left(Q_0 - \zeta^2\right)^{-1} F^\zeta(\cdot, y). \tag{3.23}$$

Recall that (by the limiting absorption principle) the operator $\left(Q_0 - \zeta^2\right)^{-1} : L^2_\sigma(\mathbb{R}^3 \backslash \overline{D}) \to H^2_{-\sigma}(\mathbb{R}^3 \backslash \overline{D})$, $\sigma > 1$, where $L^2_\sigma(\mathbb{R}^3 \backslash \overline{D})$ and $H^2_{-\sigma}(\mathbb{R}^3 \backslash \overline{D})$ are the Agmon spaces, is well defined for $\zeta \in \overline{\mathbb{C}^+}$ and $\left(Q_0 - \zeta^2\right)^{-1} : \overline{\mathbb{C}^+} \to \mathcal{L}(L^2_\sigma(\mathbb{R}^3\backslash\overline{D}), H^2_{-\sigma}(\mathbb{R}^3\backslash\overline{D}))$ is analytic in $\mathbb{C}^+$ and Hölder continuous up to $\overline{\mathbb{C}^+}$; see, for instance, [29]. Hence $\rho^\zeta(\cdot, y) : \overline{\mathbb{C}^+} \to H^2_{-\sigma}(\mathbb{R}^3\backslash\overline{D})$ is also analytic in $\mathbb{C}^+$ and Hölder continuous up to $\overline{\mathbb{C}^+}$. By the continuous injection of $H^2_{loc}(\mathbb{R}^3\backslash\overline{D})$ in the space of continuous functions $C(\mathbb{R}^3)$, we deduce that for x, y in $\mathbb{R}^3\backslash\overline{D}$ fixed, the function $\rho^\zeta(x, y)$, and hence $\Phi^{sc}_D(x, y, \zeta)$, is analytic in $\mathbb{C}^+$ and Hölder continuous up to $\overline{\mathbb{C}^+}$. The first point in the lemma follows now since $\Phi^\zeta_D(x, y) = \Phi^\zeta(x, y) + \Phi^{sc}_D(\cdot, y, \zeta)$. Let us consider the second point. Since $\Phi^\zeta_D(x, y)$ identifies with the integral kernel of $\left(Q_0 - \zeta^2\right)^{-1}$, the absence of poles in the upper complex half-plane implies that $\zeta^2 \in res(Q_0)$ (the resolvent set of Q_0) for any ζ in $\mathbb{C}^+$. Hence, for $\zeta \in \mathbb{C}^+$, $\left(Q_0 - \zeta^2\right)^{-1}$ is a bounded map: $L^2\left(\mathbb{R}^3\backslash D\right) \to D(Q_0)$ and the relation (3.23) gives $\rho^\zeta(\cdot, y) \in D(Q_0)$. Taking into account that $\Phi^\zeta(\cdot, y) \in L^2(\mathbb{R}^3)$, this leads to $\Phi^\zeta_D(\cdot, y) \in L^2\left(\mathbb{R}^3\backslash D\right)$. For the x-regularity of this function when $\zeta \in \overline{\mathbb{C}^+}$, let us assume, in addition to the previous conditions, $s(\cdot, y, \zeta)$ to be defined through a smooth cutoff function v such that supp $v \subset \mathbb{R}^3\backslash\mathcal{B}_\delta(y)$, with $\mathcal{B}_\delta(y)$ denoting the ball of center y and radius δ small enough. In this case, following the definition in (3.22), the function $F^\zeta(\cdot, y)$ is infinitely many times differentiable, and each derivative is in $L^2\left(\mathbb{R}^3\backslash D\right)$. Then, by the limiting absorption principle, we have $\rho^\zeta(\cdot, y) \in H^m_{loc}(\mathbb{R}^3\backslash\overline{D})$ for any $m \in \mathbb{N}$. Finally, the radiation condition (3.19), holding for Φ^ζ, also holds for the "regular part" $\Phi^{sc}_D(\cdot, y, \zeta)$ as a direct consequence of the representation (3.20). ∎

As in the whole space case (cf. [1]), the defect spaces $\mathcal{N}_z$ are N-dimensional and can be represented in terms of the Green's kernel of $(Q_0 - z)^{-1}$. This is shown in the next lemma.

Lemma 3.2. *Let $\mathcal{N}_z$ be defined according to* (3.11) *and* (3.13). *For any $z \in \mathbb{C}\backslash\mathbb{R}_+$, $\mathcal{N}_z$ is N-dimensional and generated by the linearly independent set $\{\Phi^\zeta_D(\cdot, y),\ y \in Y\}$, with $\zeta \in \mathbb{C}^+$ and $\zeta^2 = z$.*

Proof. Consider the problem $u \in \ker(Q^* - z)$. Since $\mathcal{C}^\infty_0\left(\mathbb{R}^3\backslash(D \cup Y)\right)$ is dense in $D(Q)$, this is equivalent to

$$
\begin{cases} \langle (Q^* - z)\, u, \varphi\rangle = 0 \qquad \forall\, \varphi \in \mathcal{C}^\infty_0\left(\mathbb{R}^3\backslash\left(\overline{D \cup Y}\right)\right), \\ u \in L^2\left(\mathbb{R}^3\backslash\overline{D}\right). \end{cases} \tag{3.24}
$$

The above equation can be written as

$$
\int_{\mathbb{R}^3\backslash\overline{D}} \bar{u}(x)\left((-\Delta - z)\,\varphi(x)\right) dx = 0 \qquad \forall\, \varphi \in \mathcal{C}^\infty_0\left(\mathbb{R}^3\backslash\left(\overline{D \cup Y}\right)\right), \tag{3.25}
$$

with $u \in L^2\left(\mathbb{R}^3\backslash\overline{D}\right)$. This implies

$$
(-\Delta - z)\, u(x) = \mu \in H^{-2}\left(\mathbb{R}^3\backslash D\right), \text{ where } \mu(\varphi) = 0 \quad \forall\, \varphi \in \mathcal{C}^\infty_0\left(\mathbb{R}^3\backslash\left(\overline{D \cup Y}\right)\right). \tag{3.26}
$$

Thus, the only possible solutions of (3.24) are those $u \in L^2\left(\mathbb{R}^3\backslash\overline{D}\right)$ such that $(-\Delta - z)\, u$ coincides with a bounded measure supported in Y. Let $\mu \in H^{-2}\left(\mathbb{R}^3\backslash D\right)$ be such a measure and $f \in H^2\left(\mathbb{R}^3\backslash D\right)$; since μ acts linearly on $H^2\left(\mathbb{R}^3\backslash D\right)$ and its action on f depends only on the boundary values of the function in Y, we have

$$
\mu(f) = \sum c_j f(y_j).
$$

This allows one to identify μ with a linear superposition of delta measures concentrated in Y and, according to the definition (3.15)–(3.17), we obtain $\dim \mathcal{N}_z = \#Y = N$, with

$$\mathcal{N}_z = \left\{ \Phi_D^{\zeta}(\cdot, y) \ , \ y \in Y \right\}$$

for $\zeta^2 = z \in \mathbb{C}$. □

Remark 3.3. Take $\zeta, \zeta' \in \mathbb{C}\backslash\mathbb{R}$ and $\alpha \in \mathbb{R}_+$. Since $(-\Delta+\alpha)(\Phi_D^{\zeta}(\cdot, y_j) - \Phi_D^{\zeta'}(\cdot, y_j)) = (\zeta^2+\alpha)\Phi_D^{\zeta}(\cdot, y_j) - ((\zeta')^2+\alpha)\Phi_D^{\zeta'}(\cdot, y_j)$ with a homogeneous boundary condition on ∂D, and since both $\Phi_D^{\zeta'}(\cdot, y_j)$ and $\Phi_D^{\zeta'}(\cdot, y_j)$ are in $L^2(\mathbb{R}^3\backslash\overline{D})$, we deduce from the elliptic regularity that

$$\Phi_D^{\zeta}(\cdot, y_j) - \Phi_D^{\zeta'}(\cdot, y_j) \in H^2(\mathbb{R}^3\backslash\overline{D}) \cap H_0^1\left(\mathbb{R}^3\backslash\overline{D}\right) . \tag{3.27}$$

Then the decomposition (3.12) can be rephrased as $D(Q^*) = H^2(\mathbb{R}^3\backslash\overline{D}) \cap H_0^1\left(\mathbb{R}^3\backslash\overline{D}\right) \oplus \mathcal{N}_z$, with $z \in \mathbb{C}\backslash\mathbb{R}_+$.

3.2.2. Extensions of Q and point interaction operators in the exterior domain. Next we focus on point perturbations of the Dirichlet Laplacian in $\mathbb{R}^3\backslash D$ describing physical interactions localized in the points $y_j \in Y$. To this aim we consider the extensions of the symmetric operator Q. These are parametrized through boundary conditions, occurring in the interaction points and generalizing those given in (2.13). Let us introduce the maps $\Gamma_{i=1,2} : D(Q^*) \to \mathbb{C}^N$,

$$(\Gamma_1 u)_j = \lim_{x \to y_j} 4\pi |x - y_j| \, u(x) ; \qquad (\Gamma_2 u)_j = \lim_{x \to y_j} \left(u(x) - \frac{(\Gamma_1 u)_j}{4\pi |x - y_j|} \right) , \tag{3.28}$$

and

$$\gamma^{-1}(z) = \Gamma_1|_{\mathcal{N}_z} \ , \qquad q(z) = \Gamma_2 \circ \gamma(z) , \tag{3.29}$$

where $\Gamma_1|_{\mathcal{N}_z}$ is the restriction of Γ_1 to the defect space $\mathcal{N}_z$. According to the results in [8], the functions $z \to \gamma(z)$ and $z \to q(z)$ exist as holomorphic bounded operators valued maps: $\mathbb{C}\backslash\mathbb{R} \to \mathcal{L}\left(\mathbb{C}^N, L^2\left(\mathbb{R}^3 \setminus \overline{D}\right)\right)$ and $\mathbb{C}\backslash\mathbb{R} \to \mathcal{L}\left(\mathbb{C}^N, \mathbb{C}^N\right)$, respectively. Moreover, they allow analytic continuations to the resolvent set $res(Q_0)$ of the Dirichlet Laplacian in the exterior domain (3.18). In this framework, the Dirichlet Laplacian Q_0, introduced in (3.18), plays the role of a "reference extension" of Q; since $u \in D(Q_0)$ is continuous in $\mathbb{R}^3\backslash\overline{D}$ (being $D(Q_0) \subset H^2\left(\mathbb{R}^3\backslash\overline{D}\right)$), Q_0 can be equivalently defined as the restriction of Q^* to the functions in $H^2\left(\mathbb{R}^3\backslash\overline{D} \cup Y\right)$ fulfilling the boundary condition: $\Gamma_1 u = 0$. Let us consider the couples $(A, B) \in \mathcal{L}\left(\mathbb{C}^N, \mathbb{C}^N\right)$ such that

$$AB^* = BA^* \ , \tag{3.30}$$

$$\text{the } N \times N \text{ matrix } (AB) \text{ has maximal rank.} \tag{3.31}$$

The next theorem characterizes all the self-adjoint extensions of Q and provides us with a Krein-like resolvent formula.

THEOREM 3.4. *Given any self-adjoint extension $\tilde{Q}$ of the operator Q, there exists a couple $(A, B) \in \mathcal{L}\left(\mathbb{C}^N, \mathbb{C}^N\right)$, fulfilling the conditions* (3.30)–(3.31), *such that*

$$D\left(\tilde{Q}\right) = \{u \in D(Q^*) \mid A\Gamma_1 u = B\Gamma_2 u\} . \tag{3.32}$$

For all $z \in res(\tilde{Q}) \cap res(Q_0)$, the resolvent $(\tilde{Q} - z)^{-1}$ satisfies the relation

$$(\tilde{Q} - z)^{-1} - (Q_0 - z)^{-1} = -\gamma(z)\left[(Bq(z) - A)^{-1} B\right]\gamma^*(\bar{z}). \tag{3.33}$$

In particular, $\sigma_{ess}(\tilde{Q})$ coincides with $[0, +\infty)$, while $z \in \mathbb{C}\backslash\mathbb{R}_+$ belongs to $\sigma(\tilde{Q})$ iff $0 \in \sigma(Bq(z) - A)$.

Sketch of the proof. The representation (3.32) and the formula (3.33) follow from Propositions 4 and 5 and Theorem 1 in [26], once it is observed that the triple $\{\mathbb{C}^N, \Gamma_1, \Gamma_2\}$ defined by the maps (3.28) forms a *boundary triple* for the adjoint operator Q^* (see the definition in [26]). Since $\sigma(Q_0) = [0, +\infty)$ and the left-hand side (l.h.s.) of (3.33) is of finite rank, the spectral properties of $\tilde{Q}$ follow by using this generalized Krein formula and Weyl's essential spectrum theorem (see Theorem XIII.14 in [28]). ▯

Explicit expressions for the operators $\gamma(\cdot, z)$ and $q(z)$ appearing on the right-hand side (r.h.s.) of (3.33) are obtained by fixing a particular basis of the defect spaces. As has been shown in Lemma 3.2, a possible representation of $\mathcal{N}_z$ is given in terms of the Green's functions of the operator $(Q_0 - z)$. Let $\{e_j\}_{j=1}^N$ denote the standard basis in $\mathbb{C}^N$, and consider the action of the linear map $\gamma(z)$ on e_j. Setting $z = \zeta^2$, $\zeta \in \mathbb{C}^+$, a direct computation gives $\gamma(\zeta^2)(e_j) = \Phi_D^\zeta(\cdot, y_j)$, and

$$\left(q(\zeta^2)\right)_{n,j} = \begin{cases} \Phi_D^\zeta(y_n, y_j), & n \neq j, \\ \Phi_D^{sc}(y_j, y_j, \zeta) + \frac{i\zeta}{4\pi}, & n = j. \end{cases} \tag{3.34}$$

As already noticed in section 3.1, the notion of point scatterers, given according to Foldy's definition, basically describes systems of *independent scatterers*, in the sense that the strength of the scattered wave from the jth scatterer depends only on the value of the external field in y_j. In the mathematical modeling, the independence of point interactions corresponds to assuming separated boundary conditions at each point y_j (cf. [1, Appendix G]). Using the parametrization (3.32), this is equivalent to taking

$$B_{n,j} = \delta_{n,j}, \qquad A_{n,j} = \alpha_n \delta_{n,j}, \qquad (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N, \tag{3.35}$$

where the diagonal coefficients α_j are related to the inverse of the scattering length of the jth point interaction.

3.2.3. Direct acoustic scattering by extended and point scatterers. We consider the scattering problem for a self-adjoint extension of Q modeling independent point interactions. Let $\alpha \in \mathbb{R}^N$ and consider the operator Q_α obtained from (3.32) by fixing A and B as in (3.35). Denoting with $\Phi_{\alpha,D}^\zeta$ the Green's kernel of $(Q_\alpha - \zeta^2)^{-1}$, the resolvent formula (3.33) can be explicitly formulated as

$$\Phi_{\alpha,D}^\zeta(x, y) = \Phi_D^\zeta(x, y) - \sum_{n,j=1}^N (q(\zeta^2) - \mathrm{diag}(\alpha))_{n,j}^{-1} \Phi_D^\zeta(x, y_n) \Phi_D^\zeta(y, y_j), \tag{3.36}$$

where $(\mathrm{diag}(\alpha))_{n,j} = \alpha_n \delta_{n,j}$. According to the result of Lemma 3.1, $\Phi_{\alpha,D}^\zeta$ is holomorphic with respect to (w.r.t.) ζ in $\mathbb{C}^+$ and continuously extends to $\overline{\mathbb{C}^+}$, provided that the limits of $(q(\zeta^2) - \mathrm{diag}(\alpha))^{-1}$ exist as $\zeta^2 \to \kappa^2 \in \mathbb{R}_+^*$. Formula (3.36) models the total field corresponding to point sources $\Phi^{\kappa}(\cdot, y)$ as incident waves. To derive the total field corresponding to incident plane waves, we need only take the source point y tending to infinity in the following way. The far-field pattern of the point source $\Phi^{\kappa}(x, y)$, with respect to the second argument, is given by $e^{-i\kappa\hat{y}\cdot x}$, $\hat{y} := \frac{y}{|y|}$. Precisely, we have

$$\lim_{|y|\to\infty} 4\pi|y| \exp(-i\kappa|y|)\Phi^{\kappa}(x, y) = e^{i\kappa x\cdot d} =: u^{in}(x, \kappa, d), \qquad d := -\hat{y}\,. \tag{3.37}$$

Correspondingly, from (3.36) we obtain the relation

$$u(x, d) = u_D(x, d) - \sum_{i,j=1}^{N} (\mathcal{M}(\kappa, \alpha))^{-1}_{i,j}\, \Phi^{\kappa}_{D}(x, y_i)\, u_D(y_j, d)\,, \tag{3.38}$$

where the matrix-valued function $\mathcal{M}(\kappa, \alpha)$ is defined on $\mathbb{R}_+ \times \mathbb{R}^N$ according to

$$\lim_{z\to\kappa^2+i0} (q(z) - \mathrm{diag}\,(\alpha)) = \mathcal{M}(\kappa, \alpha)\ . \tag{3.39}$$

In this setting, $u_D(\cdot, d)$ is the total field for the scattering of the incident plan wave $e^{i\kappa x\cdot d}$ by the obstacle D, while the function $u(\cdot, d)$ denotes the total field corresponding to the incident plan wave $e^{i\kappa x\cdot d}$ scattered by $D \cup Y$. The far-field patterns are

$$u^{\infty}_{D}(\hat{x}, d) = \lim_{|x|\to\infty} 4\pi|x|e^{-i\kappa|x|}\left(u_D(x, d) - e^{i\kappa x\cdot d}\right)\,,$$

$$u^{\infty}(\hat{x}, d) = \lim_{|x|\to\infty} 4\pi|x|e^{-i\kappa|x|}\left(u(x, d) - e^{i\kappa x\cdot d}\right)\,.$$

Using (3.15) and (3.38), the second identity rephrases as

$$\begin{aligned} u^{\infty}(\hat{x}, d) &= u^{\infty}_{D}(\hat{x}, d) \\ &\quad - \lim_{|x|\to\infty} 4\pi|x|e^{-i\kappa|x|}\left(\sum_{i,j=1}^{N} (\mathcal{M}(\kappa, \alpha))^{-1}_{i,j}\ \left(\Phi^{\kappa}(x, y_i) + \Phi^{sc}_{D}(x, y_i, \kappa)\right) u_D(y_j, d)\right)\,. \end{aligned} \tag{3.40}$$

Let $u_D = u^{sc}_D + u^{in}$ and $\Phi^{\infty}_{D}(\hat{x}, y, \kappa) = \lim_{|x|\to\infty} 4\pi|x|e^{-i\kappa|x|}\Phi^{sc}_{D}(x, y, \kappa)$; using (3.37) we have

$$\begin{aligned} u^{\infty}(\hat{x}, d) &= u^{\infty}_{D}(\hat{x}, d) \\ &- \left(\sum_{i,j=1}^{N} (\mathcal{M}(\kappa, \alpha))^{-1}_{i,j}\ \left(e^{-i\kappa y_i\cdot\hat{x}} + \Phi^{\infty}_{D}(\hat{x}, y_i, \kappa)\right)\ \left(e^{i\kappa y_j\cdot d} + u^{sc}_{D}(y_j, d)\right)\right)\,. \end{aligned} \tag{3.41}$$

The total field u_D satisfies the scattering problem

$$\begin{cases} \left(\Delta + \kappa^2\right) u_D = 0 & \text{in } \mathbb{R}^3\backslash\overline{D}, \\ u_D = 0 & \text{on} \quad \partial D, \\ u_D := u^{in} + u^{sc}_{D}, & (\partial_r - i\kappa)u^{sc}_{D}(x, d) = o(1/r)\ \text{ for } r = |x| \to \infty\,. \end{cases} \tag{3.42}$$

It is well known that (3.42) admits a unique solution which is $C^{\infty}\left(\mathbb{R}^3 \setminus \overline{D}\right)$ w.r.t. x (e.g., in [7]). In the next proposition, we show that u is the solution of the scattering problem

$$\begin{cases} \left(\Delta + \kappa^2\right) u = 0 & \text{in } \mathbb{R}^3\backslash\overline{D \cup Y}\,, \\ u|_{\partial D} = 0\,, & \Gamma_2 u = \mathrm{diag}\,(\alpha)\,\Gamma_1 u\,, \\ u := u^{in} + u^{sc}, & (\partial_r - i\kappa)u^{sc}(x, d) = o(1/r)\ \text{ for } r = |x| \to \infty\,. \end{cases} \tag{3.43}$$

PROPOSITION 3.5. *Let S_α denote the subset*

$$S_\alpha = \{\kappa \in \mathbb{R} \mid \det \mathcal{M}(\kappa, \alpha) = 0\} , \tag{3.44}$$

and let $u_D(\cdot, d)$ be the solution of problem (3.42). *Then, for $\kappa \in \mathbb{R}\backslash S_\alpha$, problem* (3.43) *has one and only one solution, and it is given by* (3.38).

Proof. Since the direction of the incident wave does not play a role in this proof, the dependence of the scattering functions from d is omitted. From Rellich's lemma and the weak unique continuation property, it follows that the solution of (3.43) is unique. Regarding the existence issue, we next show that the function (3.38) solves problem (3.43). The terms $u_D(y_j, d)$, $\Phi_D^\kappa(\cdot, y_i)$ on the r.h.s. of (3.38) are well-defined smooth functions of x. Then, for any $\kappa \in \mathbb{R}\backslash S_\alpha$, the r.h.s. of (3.38) exists. According to the definition of u_D and $\Phi_D^\kappa(\cdot, y_j)$, the function on the r.h.s. of (3.38) solves the equation $(-\Delta - \kappa^2) u = 0$ in $\mathbb{R}^3 \backslash \left(\overline{D \cup Y}\right)$, fulfilling the Dirichlet condition on ∂D. Moreover, the scattered field

$$u^{sc}(x, d) = u_D^{sc}(x, d) - \sum_{i,j=1}^{N} \left(\mathcal{M}^{-1}(k, \alpha)\right)_{ij} u_D(y_j, d)\, \Phi_D^\kappa(\cdot, y_i) \tag{3.45}$$

satisfies the Sommerfeld radiation condition since both $u_D^{sc}(x, d)$ and $\Phi_D^\kappa(x, y_i)$ do. Let us now show that $\Gamma_2 u = \operatorname{diag}(\alpha)\, \Gamma_1 u$. To this aim, we notice that the regularity of the unperturbed total field implies $u_D(y_j) = (\Gamma_2 u_D)_j$. Thus, we can also write (3.38) as

$$u(\cdot, d) = u_D(\cdot, d) - \sum_{i,j=1}^{N} \left(\mathcal{M}^{-1}(\kappa, \alpha)\right)_{i,j} (\Gamma_2 u_D(\cdot, d))_j\, \Phi_D^\kappa(\cdot, y_i) . \tag{3.46}$$

Set $u(\cdot, d) = \phi - \psi$ with

$$\phi = u_D(\cdot, d) , \tag{3.47}$$

$$\psi = \sum_{i,j=1}^{N} \left(\mathcal{M}^{-1}(\kappa, \alpha)\right)_{i,j} (\Gamma_2 u_D(\cdot, d))_j\, \Phi_D^\kappa(\cdot, y_i) . \tag{3.48}$$

The function ψ can be pointwisely approximated by elements of the defect spaces $\mathcal{N}_z$ as $z \to \kappa^2 + i0$. Let ψ_z be given by

$$\psi_z = \sum_{i,j=1}^{N} \left((q(z) - \operatorname{diag}(\alpha))^{-1}\right)_{i,j} (\Gamma_2 \phi)_j\, \Phi_D^\zeta(\cdot, y_i) . \tag{3.49}$$

For $\zeta \in \mathbb{C}^+$, $z = \zeta^2$, this function is well defined and belongs to $\mathcal{N}_z$, while it results in

$$\lim_{z \to \kappa^2 + i0} \psi_z = \psi , \tag{3.50}$$

provided that $\mathcal{M}^{-1}(\kappa, \alpha)$ exists. Since u_D is $\mathbb{C}_x^1$-continuous in $\mathbb{R}$, we have $\Gamma_1 \phi = 0$ and the following relation holds:

$$\begin{aligned}
&\mathcal{M}(\kappa, \alpha)\, \Gamma_1 (\phi - \psi) = -\mathcal{M}(\kappa, \alpha)\, \Gamma_1 \psi = - \lim_{z \to \kappa^2 + i0} (q(z) - \operatorname{diag}(\alpha))\, \Gamma_1 \psi_z \\
&= - \lim_{z \to \kappa^2 + i0} (\Gamma_2 \gamma(\cdot, z) - \operatorname{diag}(\alpha))\, \Gamma_1 \psi_z = - \lim_{z \to \kappa^2 + i0} (\Gamma_2 - \operatorname{diag}(\alpha) \circ \Gamma_1)\, \psi_z \\
&= (-\Gamma_2 + \operatorname{diag}(\alpha) \circ \Gamma_1)\, \psi .
\end{aligned} \tag{3.51}$$

The nth component of the vector on the l.h.s. of (3.51) writes as

$$\begin{aligned}\left(\mathcal{M}\left(\kappa,\alpha\right)\Gamma_1\left(\phi-\psi\right)\right)_n &= \left(-\mathcal{M}\left(\kappa,\alpha\right)\Gamma_1\psi\right)_n \\ &= -\sum_{i,j=1}^{N}\left(\mathcal{M}^{-1}\left(\kappa,\alpha\right)\right)_{ij}\left(\Gamma_2 u_D\left(\cdot,d\right)\right)_j\left(\mathcal{M}\left(\kappa,\alpha\right)\Gamma_1\Phi_D^{\kappa}\left(\cdot,y_i\right)\right)_n .\end{aligned}$$

Recalling that $\Gamma_1\Phi_D^{\kappa}\left(\cdot,x_i\right)=e_i$, we get

$$\left(\mathcal{M}\left(\kappa,\alpha\right)\Gamma_1\left(\phi-\psi\right)\right)_n = -\sum_{i,j=1}^{N}\left(\mathcal{M}\left(\kappa,\alpha\right)\right)_{ni}\left(\mathcal{M}^{-1}\left(\kappa,\alpha\right)\right)_{ij}\left(\Gamma_2\phi\right)_j = -\left(\Gamma_2\phi\right)_n ,$$

which implies

$$\mathcal{M}\left(\kappa,\alpha\right)\,\Gamma_1\left(\phi-\psi\right) = -\Gamma_2\phi\,. \tag{3.52}$$

From (3.51), (3.52), and (3.46), the interface condition $\Gamma_2 u = \operatorname{diag}\left(\alpha\right)\circ\Gamma_1 u$ follows. ∎

Remark 3.6. The explicit form of $\mathcal{M}\left(\kappa,\alpha\right)$, obtained by using (3.34) and the definition (3.39), is given by

$$\left(\mathcal{M}\left(\kappa,\alpha\right)\right)_{n,j} = \begin{cases}\Phi_D^{\kappa}\left(y_n,y_j\right), & n\neq j\,,\\[2ex] \Phi_D^{sc}(y_j,y_j,\kappa)+\frac{i\kappa}{4\pi}-\alpha_j\,, & n=j\,.\end{cases} \tag{3.53}$$

Let us consider the expression of the total field obtained in (3.10) with Foldy's approach. With the above notation, this can be written as

$$u(x,d) = u_D(x,d)+\sum_{i,j=1}^{N}\left([\tilde{\Gamma}^{-1}]\operatorname{diag}\left(g\right)\right)_{i,j}\Phi_D^{\kappa}(x,y_i)\,u_D(y_j,d)\,, \tag{3.54}$$

where the vector $g=(g_1,\ldots,g_n)$ fixes Foldy's scattering coefficients. Comparing this formula with (3.38), we deduce a condition for the equivalence of the two models:

$$\operatorname{diag}\left(\frac{1}{g}\right)\tilde{\Gamma} = \mathcal{M}\left(\kappa,\alpha\right)\,. \tag{3.55}$$

Then, using (3.9) and (3.53), it follows that

$$g_j = \left(\Phi_D^{sc}(y_j,y_j,\kappa)+\frac{i\kappa}{4\pi}-\alpha_j\right)^{-1}, \qquad j=1,\ldots,N. \tag{3.56}$$

Under this condition, the representation of the total field, due to the obstacles $D\cup Y$ using the Foldy approach with scattering coefficients g_j, is nothing but the one obtained using a multiple-point interaction model with independent points and choosing frequency-dependent parameters α_j according to (3.56).

In order to use formula (3.38) it is important to characterize the set S_α where this representation fails. We next show that the inverse matrix $\mathcal{M}^{-1}\left(\kappa,\alpha\right)$ is defined on $\mathbb{R}$ outside a discrete set.

LEMMA 3.7. *The r.h.s. of* (3.38) *is well defined a.e. w.r.t. $\kappa\in\mathbb{R}$, with the only possible exception being a discrete set of points.*

Proof. Let us introduce the matrix-valued function $Q(\zeta)$:

$$(Q(\zeta))_{n,j} = \begin{cases} \Phi_D^{\zeta}(y_n, y_j)\,, & n \neq j\,, \\ \Phi_D^{sc}(y_j, y_j, \zeta) + \frac{i\zeta}{4\pi}\,, & n = j\,. \end{cases} \tag{3.57}$$

Taking into account the definition of Φ_D^{ζ} (equation (3.15)) and the properties of its regular part $\Phi_D^{sc}(\cdot, y, \zeta)$ (see the proof of Lemma 3.1), it follows that $Q(\zeta)$ is analytic in $\mathbb{C}^+$ and meromorphic in $\mathbb{C}$, while, according to the definition (3.34), the identity

$$q(\zeta^2) = Q(\zeta)\,, \tag{3.58}$$

holds for $\zeta \in \mathbb{C}^+$. Thus, the map

$$F_\alpha(\zeta) = \det\left(Q(\zeta) - \operatorname{diag}(\alpha)\right)^{-1}$$

is analytic in $\mathbb{C}^+$ and meromorphic in the whole complex plane. Moreover, from the definition (3.39) and the identity (3.58), we get

$$\mathcal{M}(\kappa, \alpha) = \lim_{\substack{\zeta\to\kappa,\\ \operatorname{Im}\zeta>0}} \left(Q(\zeta) - \operatorname{diag}(\alpha)\right). \tag{3.59}$$

Therefore, S_α (see the definition (3.44)) identifies with the discrete set of the possible poles of the meromorphic function $F_\alpha(\zeta)$ on the real axis. ∎

Remark 3.8. The previous lemma says that for a fixed configuration of the extended as well as the point-like scatterers, the set of the singular points of the matrix $\mathcal{M}(\kappa, \alpha)$ is at most discrete. In this remark, we provide a condition linking all the parameters of the scattering model, namely, the configuration of the scatterers, the frequency κ, as well as the coefficients α, under which the matrix $\mathcal{M}(\kappa, \alpha)$ is diagonally dominant and hence invertible. This condition is

$$C\frac{N-1}{d}\max_{j=1,\dots,N}\left|\Phi_D^{sc}(y_j, y_j, \kappa) + \frac{i\kappa}{4\pi} - \alpha_j\right|^{-1} < 1\,, \tag{3.60}$$

where $d := \min_{j\neq m}|y_j - y_m|$ and C is the constant (depending on D) appearing in the known estimates $|\Phi_D(x,y)| \leq C|x-y|^{-1}$, $x, y \in \mathbb{R}^3 \setminus \overline{D}$. Let us finally mention the following behavior of the function $\Phi_D^{sc}(y_j, y_j)$ in terms of y_j in the two regimes:

1. y_j's are far away from the extended scatterer D. In this case we have $|\Phi_D^{sc}(y_j, y_j, \kappa)| \leq Cd^{-1}(y_j, D)$, where the constant C depends on D. Here $d(y_j, D)$ is the Euclidean distance between y_j and D. In this case, the condition (3.60) behaves like $C\frac{N-1}{d}\max_{j=1,\dots,N}|\frac{i\kappa}{4\pi} - \alpha_j|^{-1} < 1$, which reflects only the multiple scattering between the point-like scatterers; i.e., the interaction between the point-like scatterers and the extended one is neglected.
2. y_j's are close to the extended scatterer D. In this case we have $\Phi_D^{sc}(y_j, y_j, \kappa) = \frac{1}{4\pi\ d(y_j, D)} + O(\ln(d(y_j, D)))$, where the constant appearing in the second term of the expansion depends on D; see, for instance, Proposition 3.2 of [25]. In this case, the condition (3.60) is mostly satisfied. Here, the effect of the interaction between each point-like scatterer and the extended one dominates the effect of the interaction between the point-like scatterers.

4. The inverse scattering by extended and point-like scatterers. In this section we turn to studying the inverse problem of detecting the shape of the extended sound-soft obstacle and positions of the point-like scatterers from the far-field data corresponding to all incident plane waves at a fixed frequency. Our goal is to establish the factorization method by Kirsch for the two-scale model under consideration.

Let u^{in} be a time-harmonic incident wave. The classical scattering theory in the absence of the point-like scatterers is devoted to finding the scattered field $u_D^{sc} \in H^1_{loc}(\mathbb{R}^3\backslash\overline{D})$ satisfying

$$\text{(I)} \begin{cases} (\Delta+\kappa^2)u_D^{sc} = 0 & \text{in} \quad \mathbb{R}^3\backslash\overline{D}, \\ u_D^{sc} = -u^{in} & \text{on} \quad \partial D, \\ \partial u_D^{sc}/\partial r - i\kappa u_D^{sc} = o(1/r) & \text{as} \quad r\to\infty, r=|x|. \end{cases}$$

If the obstacle consists of both extended and point-like scatterers, we have seen in the previous sections that the corresponding model is to look for the scattered field $u^{sc} \in H^1_{loc}(\mathbb{R}^3\backslash\overline{D})$ such that

$$\text{(II)} \begin{cases} (\Delta+\kappa^2)u^{sc} = 0 & \text{in} \quad \mathbb{R}^3\backslash\{\overline{D}\cup Y\}, \\ u^{sc} = -u^{in} & \text{on} \quad \partial D, \\ (\Gamma_2 u)_j = \alpha_j(\Gamma_1 u)_j, & \alpha_j \in \mathbb{C},\ j=1,2,\ldots,N, \\ \partial u^{sc}/\partial r - i\kappa u^{sc} = o(1/r) & \text{as} \quad r\to\infty,\ r=|x|, \end{cases}$$

where $u = u^{sc}+u^{in}$ denotes the total field and the operators Γ_1, Γ_2 are defined as in (3.28).

We review several symbols employed in sections 2 and 3. If u^{in} is a plane wave with the incident angle $d \in \mathbb{S}^2 = \{x : |x| = 1\}$, i.e., $u^{in}(x) = \exp(i\kappa x\cdot d)$, we denote by $u_D^{sc}(x,d), u_D(x,d), u_D^\infty(\hat{x},d)$, respectively, $u^{sc}(x,d), u(x,d), u^\infty(\hat{x},d)$, the scattered field, total field, and far-field pattern to problem (I), respectively, (II). Analogously, if the incident wave is a point source, i.e., $u^{in} = \Phi^\kappa(x,y)$ for some $y \in \mathbb{R}^3\setminus\overline{D}$, we employ the symbols $\Phi_D^{sc}(x,y), \Phi_D^\kappa(x,y), \Phi_D^\infty(\hat{x},y)$, respectively, $\Phi_{\alpha,D}^{sc}(x,y), \Phi_{\alpha,D}^\kappa(x,y), \Phi_{\alpha,D}^\infty(\hat{x},y)$ to denote the corresponding quantities.

The well-posedness of (II) is described in the following lemma, where u^{in} is allowed to be either a plane wave or a point source wave.

LEMMA 4.1. *Let $u_D = u^{in}+u_D^{sc}$, where u_D^{sc} is the unique solution to problem* (I). *Then the unique solution to problem* (II) *is given by*

$$u^{sc}(x) = u_D^{sc}(x) - \sum_{m,j=1}^{N}\left\{[\mathcal{M}^{-1}(\kappa,\alpha)]_{m,j}\left[(\Gamma_2 u_D)_j - \alpha_j(\Gamma_1 u_D)_j\right]\Phi_D^\kappa(x,y_m)\right\}, \tag{4.1}$$

where the matrix $\mathcal{M}(\kappa,\alpha)$ is defined as in (3.53). *In particular, if $\Gamma_1 u_D = 0$, then the expression* (4.1) *reduces to*

$$u^{sc}(x) = u_D^{sc}(x) - \sum_{m,j=1}^{N}\left\{[\mathcal{M}^{-1}(\kappa,\alpha)]_{m,j}\,(\Gamma_2 u_D)_j\,\Phi_D^\kappa(x,y_m)\right\}. \tag{4.2}$$

The proof of (4.1) can be carried out analogously to the proof of (3.46) in section 3.2.3, where (4.2) is justified under the assumption $\Gamma_1 u_D = 0$. Note that $\Gamma_1 u_D = 0$ if u^{in} is continuous at y_j, e.g., u^{in} is a plane wave. However, $\Gamma_1 u_D \neq 0$ if $u^{in} = \Phi_D^\kappa(x,y_j)$ for some $y_j \in Y$. By the definition of the far-field pattern, we get from (4.2) the far field of u^{sc} for incident plane waves.

COROLLARY 4.2. *If* $u^{in}(x) = \exp(i\kappa x\cdot d)$, *then the far-field pattern of the scattered field corresponding to* (II) *can be formulated as*

$$\begin{aligned} u^\infty(\hat{x}, d) &= u_D^\infty(\hat{x}, d) \\ &\quad - \sum_{m,j=1}^{N} \left\{ [\mathcal{M}^{-1}(\kappa,\alpha)]_{m,j} \, (e^{i\kappa y_j\cdot d} + u_D^{sc}(y_j, d)) \, (e^{-i\kappa\hat{x}\cdot y_m} + \Phi_D^\infty(\hat{x}, y_m)) \right\}. \end{aligned} \tag{4.3}$$

We are interested in the following inverse problem.

(IP): Recover ∂D and y_j $(j = 1, 2, \ldots, N)$ from the far-field data $u^\infty(\hat{x}, d)$ over all observation points $\hat{x} \in \mathbb{S}^2$ corresponding to all incident directions $d \in \mathbb{S}^2$.

The following lemma generalizes the reciprocity relation for an extended obstacle to the two-scale model.

LEMMA 4.3. *It holds that*

(i) $u^\infty(\hat{x}, d) = u^\infty(-d, -\hat{x})$ *for all* $\hat{x}, d \in \mathbb{S}^2$;

(ii) $u^{sc}(y, -d) = \Phi_{\alpha,D}^\infty(d, y)$ *for all* $d \in \mathbb{S}^2$, $y \notin \overline{D} \cup Y$.

Proof. We have

$$\begin{aligned} u^\infty(-d, -\hat{x}) &= u_D^\infty(-d, -\hat{x}) \\ &\quad - \sum_{m,j=1}^{N} \left\{ [\mathcal{M}^{-1}(\kappa,\alpha)]_{m,j} \, (e^{-i\kappa y_j\cdot \hat{x}} + u_D^{sc}(y_j, -\hat{x})) \, (e^{i\kappa d\cdot y_m} + \Phi_D^\infty(-d, y_m)) \right\}. \end{aligned}$$

Comparing this identity with (4.3) and making use of the reciprocity relations for the extended obstacle D,

$$u_D^\infty(\hat{x}, d) = u_D^\infty(-d, -\hat{x}), \quad u_D^{sc}(x, d) = \Phi_D^\infty(-d, x), \quad \Phi_D^\infty(x, y) = \Phi_D^\infty(y, x), \tag{4.4}$$

for all $x \neq y$, $x, y \in \mathbb{R}^3\backslash\overline{D}$, we finish the proof of the first assertion. The second assertion follows from the equations

$$u^{sc}(y, -d) = u_D^{sc}(y, -d) - \sum_{m,j=1}^{N} \left\{ [\mathcal{M}^{-1}(\kappa,\alpha)]_{m,j} \, (e^{-i\kappa y_j\cdot d} + u_D^{sc}(y_j, -d)) \, \Phi_D^\kappa(y, y_m) \right\},$$

$$\Phi_{\alpha,D}^\infty(d, y) = \Phi_D^\infty(d, y) - \sum_{m,j=1}^{N} \left\{ [\mathcal{M}^{-1}(\kappa,\alpha)]_{m,j} \, \Phi_D^\kappa(y_j, y) (e^{-i\kappa y_m\cdot d} + \Phi_D^\infty(d, y_m)) \right\}$$

and the last two identities in (4.4). ∎

4.1. Data-to-pattern operator. Given $f \in H^{1/2}(\partial D)$ and $\mathbf{c} = (c_1, c_2, \ldots, c_N) \in \mathbb{C}^N$, we consider the problem of finding $v \in H^1_{loc}(\mathbb{R}^3\backslash(\overline{D} \cup Y))$ such that

$$\text{(III)} \begin{cases} (\Delta + \kappa^2)v = 0 & \text{in} \quad \mathbb{R}^3\backslash\{\overline{D} \cup Y\}, \\ v = f & \text{on} \quad \partial D, \\ (\Gamma_2 v)_j - \alpha_j(\Gamma_1 v)_j = c_j, & j = 1, 2, \ldots, N, \\ \partial v/\partial r - i\kappa v = o(1/r) & \text{as} \quad r \to \infty, r = |x|. \end{cases}$$

By Lemma 4.1, problem (III) is uniquely solvable with the solution taking the form

$$v(x) = u_f^{sc}(x) - \sum_{m,j=1}^{N} \left\{ [\mathcal{M}^{-1}(\kappa,\alpha)]_{m,j} \, [(\Gamma_2 u_f^{sc})_j - c_j] \, \Phi_D^\kappa(x, y_m) \right\},$$

where u_f^{sc} denotes the unique solution to (I) with the Dirichlet data $u_f^{sc} = f$ on ∂D. The far-field data v^∞ of v defines the data-to-pattern operator $\tilde{G} : H^{1/2}(\partial D) \times \mathbb{C}^N \to L^2(\mathbb{S}^2)$ as

$$
\begin{aligned}
(4.5)\quad \tilde{G}(f, \mathbf{c}) &:= v^\infty(\hat{x}) \\
&= u_f^\infty(\hat{x}) - \sum_{m,j=1}^{N} \left\{ [\mathcal{M}^{-1}(\kappa,\alpha)]_{m,j} \left[(\Gamma_2 u_f^{sc})_j - c_j \right] \left(e^{-i\kappa\hat{x}\cdot y_m} + \Phi_D^\infty(\hat{x}, y_m) \right) \right\},
\end{aligned}
$$

with u_f^∞ being the far-field pattern of u_f^{sc}.

Remark 4.4. If $f = -u^{in}|_{\partial D}, c_j = -(\Gamma_2 u^{in})_j + \alpha_j(\Gamma_1 u^{in})_j$, then v coincides with u^{sc} given in (4.1).

The set $D \cup Y$ of extended and point-like obstacles can be characterized by the ranges of $\tilde{G}$. Recall that the far-field pattern of the free-space fundamental solution is given by $\phi_y(\hat{x}) := e^{-i\kappa\hat{x}\cdot y}$.

LEMMA 4.5. *The function ϕ_y belongs to the range $\mathcal{R}(\tilde{G})$ of $\tilde{G}$ iff $y \in D \cup Y$.*

Proof. Assume first that $y \in D \cup Y$. Set $f = \Phi^\kappa(x, y)|_{\partial D}$ and $\mathbf{c} = \{c_j\} \in \mathbb{C}^N$ with

$$
(4.6)\qquad
\begin{aligned}
c_j &:= (\Gamma_2 \Phi^\kappa(x, y))_j - \alpha_j(\Gamma_1 \Phi^\kappa(x, y))_j \\
&= \begin{cases} \Phi^\kappa(y_j, y) & \text{if } y \in D, \\ \Phi^\kappa(y_j, y_m) & \text{if } y = y_m \in Y, m \neq j, \\ i\kappa/(4\pi) - \alpha_j & \text{if } y = y_j \in Y. \end{cases}
\end{aligned}
$$

Then we see the unique solution to problem (III) is $v = \Phi^\kappa(x, y)$ and hence $\tilde{G}(f, \mathbf{c}) = v^\infty = \phi_y$.

Now suppose that $\phi_y = \tilde{G}(\tilde{f}, \tilde{\mathbf{c}})$ for some $\tilde{f} \in H^{1/2}(\partial D)$ and $\tilde{\mathbf{c}} \in \mathbb{C}^N$. Let $\tilde{v}$ be the solution to (III) with $f = \tilde{f}$ and $\mathbf{c} = \tilde{\mathbf{c}}$ so that $\tilde{v}^\infty = \phi_y$. Applying Rellich's identity and the unique continuation of solutions to the Helmholtz equation, we get $\tilde{v}(x) = \Phi^\kappa(x, y)$ for all $x \notin \overline{D} \cup Y$. If $y \in \partial D$ or $y \in \mathbb{R}^3 \backslash (\overline{D} \cup Y)$, one can readily derive a contraction from the boundedness of the limit $\tilde{v}(x) \to \tilde{v}(y)$ as $x \to y$ and the singularity of $\Phi^\kappa(x, y)$ at $x = y$. This implies $y \in D \cup Y$. ∎

However, the above characterization cannot be numerically implemented, since knowledge of the data-to-pattern operator is not available from our measurement data. The essence of the factorization method is to connect the ranges of $\tilde{G}$ with the far-field operator $\tilde{F} : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ defined as

$$
(4.7)\qquad (\tilde{F}g)(\hat{x}) = \int_{\mathbb{S}^2} u^\infty(\hat{x}, d)\, g(d)\, ds(d), \quad g \in L^2(\mathbb{S}^2).
$$

To do this we shall factorize $\tilde{F}$ in terms of $\tilde{G}$ in the subsequent section. It is obvious that the spectrum of $\tilde{F}$ can be straightforwardly extracted from $u^\infty(\hat{x}, d)$ for all $\hat{x}, d \in \mathbb{S}^2$. Below we show some properties of $\tilde{G}$ which will be used in section 4.4 for establishing the relation between $\mathcal{R}(\tilde{G})$ and $\mathcal{R}(\tilde{F})$.

LEMMA 4.6. *Assume that κ^2 is not a Dirichlet eigenvalue of $-\Delta$ in D. Then the data-to-pattern operator $\tilde{G}$ is one-to-one, compact with dense range in $L^2(\mathbb{S}^2)$.*

Proof. Define $G : H^{1/2}(\partial D) \to L^2(\mathbb{S}^2)$ as the data-to-pattern operator in the absence of the point-like scatterers. It is seen from (4.5) that the compactness of $\tilde{G}$ follows from the compactness of G, shown in [19, Lemma 1.13], since $\Gamma_2 u_f^{sc}$ is smoothing. The injectivity and denseness can be proved in the same way as in [19, Lemma 1.13] for G. ∎

4.2. Factorization of the far-field operator. Introduce the Herglotz wave function

$$(Hg)(x) := \int_{\mathbb{S}^2} e^{i\kappa x\cdot d} g(d)\, ds(d), \quad g \in L^2(\mathbb{S}^2).$$

The far-field operator $\tilde{F}$ defined in (4.7) is nothing else but the far-field pattern of the scattered field to problem (II) with the incident wave $u^{in}(x) = (Hg)(x)$. Define the bounded operator $\tilde{H} : L^2(\mathbb{S}^2) \to H^{1/2}(\partial D) \times \mathbb{C}^N$ as

$$\tilde{H}g := \big(Hg|_{\partial D},\, (c_1, \ldots, c_N)\big), \quad c_j := (\Gamma_2 Hg)_j - \alpha_j(\Gamma_1 Hg)_j = (Hg)(y_j).$$

Then the factorization

$$\tilde{F} = -\tilde{G}\tilde{H}$$

holds. Introduce the single-layer potentials

$$\begin{aligned} S(\varphi)(x) &:= \int_{\partial D} \Phi^\kappa(x,y)\varphi(y) ds(y), \quad x \in \mathbb{R}^3, \\ J(\varphi)(x) &:= \int_{\partial D} \Phi^\kappa(x,y)\varphi(y) ds(y), \quad x \in \partial D, \end{aligned} \tag{4.8}$$

for $\varphi \in H^{-1/2}(\partial D)$ and the function

$$(K(\mathbf{b}))(x) := \sum_{j=1}^N b_j\, \Phi^\kappa(x, y_j), \quad x \neq y_j, \quad \mathbf{b} = (b_1, b_2, \ldots, b_N) \in \mathbb{C}^N.$$

Clearly, the sum $S(\varphi) + K(\mathbf{b}) =: \tilde{S}(\varphi, \mathbf{b})$ is a radiating solution to the Helmholtz equation in $\mathbb{R}^3 \backslash (\overline{D} \cup Y)$. Define the operator $\mathcal{S} : H^{-1/2}(\partial D) \times \mathbb{C}^N \to H^{1/2}(\partial D) \times \mathbb{C}^N$ as

$$\mathcal{S}(\varphi, \mathbf{b}) := \big(\tilde{S}(\varphi, \mathbf{b}))|_{\partial D},\, \{c_j\}_{j=1}^N\big), \quad c_j := (\Gamma_2 \tilde{S}(\varphi, \mathbf{b}))_j - \alpha_j(\Gamma_1 \tilde{S}(\varphi, \mathbf{b}))_j. \tag{4.9}$$

Then the unique solution to problem (III) with $(f, \mathbf{c}) = \mathcal{S}(\varphi, \mathbf{b})$ is given by

$$v(x) = \tilde{S}(\varphi, \mathbf{b})(x), \quad x \in \mathbb{R}^3 \backslash (\overline{D} \cup Y).$$

This implies that, for any $(\varphi, \mathbf{b}) \in H^{-1/2}(\partial D) \times \mathbb{C}^N$,

$$\tilde{G}\mathcal{S}(\varphi, \mathbf{b}) = v^\infty(\hat{x}) = \int_{\partial D} e^{-i\kappa \hat{x}\cdot y} \varphi(y) ds(y) + \sum_{j=1}^N e^{-i\kappa\hat{x}\cdot y_j} b_j = \tilde{H}^*(\varphi, \mathbf{b}),$$

where the last equality is derived from the definition of $\tilde{H}$. Hence, we get $\tilde{H} = \mathcal{S}^* \tilde{G}^*$ and

$$\tilde{F} = -\tilde{G}\, \mathcal{S}^*\, \tilde{G}^*. \tag{4.10}$$

Remark 4.7. In the absence of the point-like scatterers, i.e., $Y = \emptyset$, there holds $\tilde{H} = H$, $\tilde{G} = G$, and $\mathcal{S} = J$; see [19]. If there is no extended obstacle, i.e., $D = \emptyset$, then the factorization (4.10) can be reduced to the case of the MUSIC algorithm as considered in [6] or [19, Chapter 4].

4.3. Properties of the middle operator. Define the entries of the matrix $\Theta(\kappa,\alpha) = [\Theta]_{m,j}$ in the following way:

$$\Theta_{m,j} = \Theta_{m,j}(\kappa,\alpha) := \begin{cases} \Phi^\kappa(y_m, y_j) & \text{if} \quad m \neq j, \\ i\kappa/4\pi - \alpha_j & \text{if} \quad m = j. \end{cases} \tag{4.11}$$

Note that $\Theta(\kappa,\alpha)$ differs from $\mathcal{M}(\kappa,\alpha)$ only in the diagonal terms. Let $B_\epsilon(y_j) := \{x : |x-y_j| = \epsilon\}$ for some $\epsilon > 0$. Assume that $f(x)$ is a continuous function at $x = y_j$. Using the mean value theorem, one can easily prove that

$$\lim_{\epsilon\to 0} \int_{B_\epsilon(y_j)} \partial_\nu \Phi^\kappa(x, y_m) f(x) ds(x) = \begin{cases} -f(y_j) & \text{if} \quad m = j, \\ 0 & \text{if} \quad m \neq j, \end{cases} \tag{4.12}$$

where the normal ν on $B_\epsilon(y_j)$ is directed into the region $|x-y_j| > \epsilon$, and

$$\begin{aligned} A_{j,m,l}(\epsilon) &:= \int_{B_\epsilon(y_j)} \overline{\Phi}^\kappa(x, y_l) \partial_\nu \Phi^\kappa(x, y_m) ds(x) \\ &= \begin{cases} o(\epsilon) & \text{if} \quad m \neq j, \\ -\overline{\Phi}^\kappa(y_j, y_l) + o(\epsilon) & \text{if} \quad m = j \neq l, \\ (i\kappa - 1/\epsilon)/(4\pi) & \text{if} \quad m = j = l, \end{cases} \end{aligned} \tag{4.13}$$

where $o(\epsilon) \to 0$ as $\epsilon \to 0$.

The properties of the middle operator $\mathcal{S}$ are shown below.

LEMMA 4.8. *Assume that κ^2 is not a Dirichlet eigenvalue of $-\Delta$ in D and that the matrix Θ is invertible. Then the following hold:*

(i) *The operator $\mathcal{S} : H^{-1/2}(\partial D) \times \mathbb{C}^N \to H^{1/2}(\partial D) \times \mathbb{C}^N$ is an isomorphism.*

(ii) *We have*

$$\operatorname{Im} \langle (\varphi, \mathbf{b}), \mathcal{S}(\varphi, \mathbf{b}) \rangle < 0$$

for all $(\varphi, \mathbf{b}) \in H^{-1/2}(\partial D) \times \mathbb{C}^N$, $\varphi \neq 0$, $|\mathbf{b}| \neq 0$, provided $\operatorname{Im} \alpha_j \leq 0$ for all $j = 1, 2, \ldots, N$.

(iii) *There exists a self-adjoint and coercive operator $\mathcal{S}_0 : H^{-1/2}(\partial D) \times \mathbb{C}^N \to H^{1/2}(\partial D) \times \mathbb{C}^N$ such that $\mathcal{S} - \mathcal{S}_0 : H^{-1/2}(\partial D) \times \mathbb{C}^N \to H^{1/2}(\partial D) \times \mathbb{C}^N$ is compact.*

Proof. (i) From (4.6) and the definitions of $\Gamma_m (m = 1, 2)$, $K(\mathbf{b})$, and Θ, we see

$$(\Gamma_2 S(\varphi))_j - \alpha_j (\Gamma_1 S(\varphi))_j = (\Gamma_2 S(\varphi))_j, \quad \{(\Gamma_2 K(\mathbf{b}))_j - \alpha_j (\Gamma_1 K(\mathbf{b}))_j\}_{j=1}^{j=N} = \Theta\, \mathbf{b}.$$

This enables us to rewrite the operator $\mathcal{S}$ in (4.9) as the matrix form

$$\mathcal{S}(\varphi, \mathbf{b}) = \begin{pmatrix} J & K \\ \Gamma_2 S & \Theta \end{pmatrix} \begin{pmatrix} \varphi \\ \mathbf{b} \end{pmatrix}. \tag{4.14}$$

If κ^2 is not a Dirichlet eigenvalue of $-\Delta$ in D, the single-layer operator J is an isomorphism from $H^{-1/2}(\partial D)$ to $H^{1/2}(\partial D)$. Since $\Theta : \mathbb{C}^N \to \mathbb{C}^N$ is invertible and $K : \mathbb{C}^N \to H^{1/2}(\partial D)$, $\Gamma_2 S : H^{-1/2}(\partial D) \to \mathbb{C}^N$ are compact operators, it can be concluded that $\mathcal{S}$ is a Fredholm operator with index zero. Consequently, $\mathcal{S}$ is an isomorphism, provided it is injective.

Suppose that $\mathcal{S}(\varphi, \mathbf{b}) = 0$ and that $\tilde{S}(\varphi, \mathbf{b})$ is the unique solution to problem (III) with $(f, \mathbf{c}) = \mathcal{S}(\varphi, \mathbf{b})$. The uniqueness of solutions to (III) gives $\tilde{S}(\varphi, \mathbf{b}) = 0$ in

$\mathbb{R}^3\backslash(\overline{D}\cup Y)$. Again using the fact that κ^2 is not a Dirichlet eigenvalue of $-\Delta$ in D, we see that $\tilde{S}(\varphi,\mathbf{b})=0$ in D. Then $\varphi=0$ follows from the jump relations of $\tilde{S}(\varphi,\mathbf{b})$ over ∂D. Inserting $\varphi=0$ into (4.14) and using $\mathcal{S}(\varphi,\mathbf{b})=0$, we arrive at $\Theta\,\mathbf{b}=0$. Finally we get $|\mathbf{b}|=0$ as a consequence of the invertibility of Θ. This together with the Fredholm alternative yields the unique solvability of $\mathcal{S}(\varphi,\mathbf{b})=(\phi,\mathbf{c})$ for any $(\phi,\mathbf{c})\in H^{1/2}(\partial D)\times\mathbb{C}^N$.

(ii) From (4.14) and (4.9),

$$\operatorname{Im}\langle(\varphi,\mathbf{b}),\mathcal{S}(\varphi,\mathbf{b})\rangle=\operatorname{Im}\langle\varphi,\tilde{S}(\varphi,\mathbf{b})|_{\partial D}\rangle+\operatorname{Im}\langle\mathbf{b},(\Gamma_2 S)\varphi+\Theta\mathbf{b}\rangle. \tag{4.15}$$

The first term on the r.h.s. (4.15) will be calculated as follows. Set $w(x)=\tilde{S}(\varphi,\mathbf{b})(x)$, $x\in\mathbb{R}^3\backslash(\overline{D}\cup Y)$. Choose $R>0$ sufficiently large and $\epsilon>0$ sufficiently small. Using the jump relations of w over ∂D and integration by parts, we get

$$\begin{aligned}\langle\varphi,\tilde{S}\,(\varphi,\mathbf{b})\,|_{\partial D}\rangle&=\int_{\partial D}\left(\frac{\partial w^-}{\partial\nu}-\frac{\partial w^+}{\partial\nu}\right)\overline{w}\,ds\\ &=\int_{D\cup D_{R,\epsilon}}|\nabla w|^2-\kappa^2|w|^2dx-\int_{B_R(O)}\overline{w}\frac{\partial w}{\partial\nu}\,ds+\sum_{j=1}^N\int_{B_\epsilon(y_j)}\overline{w}\frac{\partial w}{\partial\nu}\,ds,\end{aligned} \tag{4.16}$$

where $D_{R,\epsilon}:=\{x:x\in\mathbb{R}^3\backslash\overline{D},|x|<R,|x-y_j|>\epsilon,j=1,2,\ldots,N\}$. In view that $w(x)=S(\varphi)+K(\mathbf{b})$, using (4.12) we find that

$$\int_{B_\epsilon(y_j)}\overline{w}\frac{\partial w}{\partial\nu}\,ds=\int_{B_\epsilon(y_j)}\left\{\overline{S\varphi}\,\frac{\partial K(\mathbf{b})}{\partial\nu}+\overline{K(\mathbf{b})}\,\frac{\partial K(\mathbf{b})}{\partial\nu}\right\}ds+o(\epsilon)\quad\text{as }\epsilon\to0,$$

since $S(\varphi)$ is continuous at each y_j. From (4.12) and the definition of $K(\mathbf{b})$, it follows that

$$\int_{B_\epsilon(y_j)}\overline{S\varphi}\,\frac{\partial K(\mathbf{b})}{\partial\nu}\,ds=\sum_{m=1}^N b_m\int_{B_\epsilon(y_j)}\overline{S\varphi}\,\frac{\partial\Phi^\kappa(x,y_m)}{\partial\nu}\,ds=-b_j\,\overline{S\varphi}(y_j)+o(\epsilon)$$

as $\epsilon\to0$. By (4.13), it holds that

$$\int_{B_\epsilon(y_j)}\overline{K(\mathbf{b})}\,\frac{\partial K(\mathbf{b})}{\partial\nu}\,ds=\sum_{l,m=1}^N\overline{b}_l\,b_mA_{j,m,l}(\epsilon)=\frac{|b_j|^2}{4\pi}\left(i\kappa-\frac{1}{\epsilon}\right)-\sum_{l=1,l\neq j}^N\overline{b}_l\,b_j\,\overline{\Phi}(y_j,y_l).$$

Hence, summarizing over j and recalling the definitions of Γ_2 and Θ,

$$\sum_{j=1}^N\int_{B_\epsilon(y_j)}\overline{w}\frac{\partial w}{\partial\nu}ds=-\langle\mathbf{b},\Gamma_2S\varphi\rangle-\langle\mathbf{b},\Theta\mathbf{b}\rangle-|\mathbf{b}|^2/(4\pi\epsilon)-\sum_{j=1}^N\overline{\alpha}_j|b_j|^2+o(\epsilon). \tag{4.17}$$

Since w satisfies the Sommerfeld radiation condition, letting $R\to\infty$ we get

$$\int_{B_R(O)}\overline{w}\frac{\partial w}{\partial\nu}ds=i\kappa\int_{|x|=R}|w|^2ds+o(R)=\frac{i\kappa}{(4\pi)^2}||w^\infty||^2_{L^2(\mathbb{S}^2)}+o(1/R), \tag{4.18}$$

where w^∞ denotes the far-field pattern of w. Inserting (4.17), (4.18) back to (4.16), taking the imaginary part, and letting $R\to\infty,\epsilon\to0$, we get

$$\operatorname{Im}\langle\varphi,\tilde{S}(\varphi,\mathbf{b})|_{\partial D}\rangle=-\operatorname{Im}\langle\mathbf{b},(\Gamma_2S)\varphi+\Theta\mathbf{b}\rangle-\kappa/(4\pi)^2||w^\infty||^2_{L^2(\mathbb{S}^2)}+\sum_{j=1}^N|b_j|^2\operatorname{Im}\alpha_j. \tag{4.19}$$

Now, it is seen from (4.15) and (4.19) that

$$\operatorname{Im}\langle(\varphi,\mathbf{b}),\mathcal{S}(\varphi,\mathbf{b})\rangle = -\kappa/(4\pi)^2||w^\infty||^2_{L^2(\mathbb{S}^2)} + \sum_{j=1}^{N}|b_j|^2\operatorname{Im}\alpha_j \le 0$$

under the assumption that $\operatorname{Im}\alpha_j \le 0$ for $j = 1, 2, \dots, N$.

If $\operatorname{Im}\langle(\varphi,\mathbf{b}),\mathcal{S}(\varphi,\mathbf{b})\rangle = 0$, then we have $w^\infty = 0$, and by Rellich's lemma, $w(x) = \tilde{S}(\varphi,\mathbf{b})(x) = 0$ for $x \in \mathbb{R}^3\backslash(\overline{D}\cup Y)$. Arguing the same as in the proof of assertion (i) we obtain $\varphi = 0, \mathbf{b} = 0$. This implies that $\operatorname{Im}\langle(\varphi,\mathbf{b}),\mathcal{S}(\varphi,\mathbf{b})\rangle < 0$ for all $\varphi \neq 0$ and $|\mathbf{b}| \neq 0$.

(iii) Denote by J_0 the single-layer operator defined as in (4.8) with the $\kappa = i$. It was proved in [19, Lemma 1.14] that J_0 is a self-adjoint and coercive operator from $H^{-1/2}(\partial D)$ to $H^{1/2}(\partial D)$, i.e.,

$$\langle\varphi, J_0\varphi\rangle \ge c_0||\varphi||^2_{H^{-1/2}(\partial D)}, \quad \varphi \in H^{-1/2}(\partial D),$$

for some positive constant c_0. Moreover, $J - J_0$ is compact from $H^{-1/2}(\partial D)$ to $H^{1/2}(\partial D)$. To prove (iii), we define the operator $\mathcal{S}_0 : H^{-1/2}(\partial D)\times\mathbb{C}^N \to H^{1/2}(\partial D)\times\mathbb{C}^N$ as

$$\mathcal{S}_0(\varphi,\mathbf{b}) = (J_0\varphi, c_0\,\mathbf{b}), \quad \varphi \in H^{1/2}(\partial D),\ \mathbf{b}\in\mathbb{C}^N.$$

Then $\mathcal{S}_0$ is coercive, i.e.,

$$\langle(\varphi,\mathbf{b}),\mathcal{S}_0(\varphi,\mathbf{b})\rangle \ge c_0\,(||\varphi||^2_{H^{-1/2}(\partial D)} + |\mathbf{b}|^2).$$

With such a choice, the difference

$$(\mathcal{S}-\mathcal{S}_0)(\varphi,\mathbf{b}) = \big((J-J_0)\varphi + K(\mathbf{b})|_{\partial D},\ \Gamma_2 S\varphi + \Theta\mathbf{b} - c_0\,\mathbf{b}\big)$$

is compact, since $J - J_0, K, \Gamma_2 S$, and the multiplication operator by Θ or c_0 are all compact. ∎

4.4. Uniqueness and inversion algorithms. The properties of the data-to-pattern operator $\tilde{G}$ and the operator $\mathcal{S}$ (see Lemmas 4.6 and 4.8) put us in a position where we can directly apply the following range identity (see [19, Theorem 2.5.1]) to the factorization of the far-field operator established in (4.10). Recall that the real and imaginary parts of the operator F over a Hilbert space are given by

$$\operatorname{Re}F := (F+F^*)/2, \quad \operatorname{Im}F := (F-F^*)/(2i).$$

Obviously, both $\operatorname{Re}F$ and $\operatorname{Im}F$ are self-adjoint operators.

LEMMA 4.9 (range identity). *Let $X \subset Y \subset X^*$ be a Gelfand triple with Hilbert space Y and reflexive Banach space X such that the embedding is dense. Furthermore, let Y be a second Hilbert space, and let $F : Y \to Y$, $G : X \to Y$, and $T : X^* \to X$ be linear and bounded operators with $F = GTG^*$. Suppose further that the following hold:*

(a) *G is compact and has dense range.*

(b) *There exists $t \in [0, 2\pi]$ such that* $\operatorname{Re}[\exp(it)T]$ *has the form* $\operatorname{Re}[\exp(it)T] = T_0 + T_1$ *with some compact operator T_1 and some coercive operator $T_0 : X^* \to X$; i.e., there exists $c > 0$ with*

$$\langle\varphi, T_0\varphi\rangle \ge c\|\varphi\|^2 \quad \forall\ \varphi \in X^*. \tag{4.20}$$

(c) $\mathrm{Im}\, T$ *is nonnegative on* $\mathcal{R}(G^*) \subset X^*$, *i.e.,* $\langle \varphi, \mathrm{Im}\, T\varphi\rangle \geq 0$ *for all* $\varphi \in \overline{\mathcal{R}(G^*)}$.

(d) $\mathrm{Re}\,[\exp(it)T]$ *is one-to-one or* $\mathrm{Im}\, T$ *is strictly positive on the closure* $\overline{\mathcal{R}(G^*)}$ *of* $\mathcal{R}(G^*)$; *i.e., for all* $\varphi \in \overline{\mathcal{R}(G^*)}$ *with* $\varphi \neq 0$ *it holds that* $\langle \varphi, \mathrm{Im}\, T\varphi\rangle > 0$.

Then the operator $F_\sharp := |\mathrm{Re}\, \exp(it)F| + |\mathrm{Im}\, F|$ *is positive definite and the ranges of* $G : X \to Y$ *and* $F_\sharp^{1/2} : Y \to Y$ *coincide.*

To apply Lemma 4.9, we set

$$t = \pi,\ F = \tilde{F},\ G = \tilde{G},\ T = -\mathcal{S}^*,\ T_0 = \mathcal{S}_0,\ T_1 = \mathrm{Re}\,(\mathcal{S} - \mathcal{S}_0),$$
$$Y = L^2(\mathbb{S}^2),\ \ X = H^{-1/2}(\partial D) \times \mathbb{C}^N.$$

In our settings, all the conditions in Lemma 4.9 are satisfied. In fact, conditions (a) and (b) follow from Lemmas 4.6 and 4.8(iii), respectively. Conditions (c) and (d) are guaranteed by Lemma 4.8(i) and (ii) under the assumption that $\mathrm{Im}\,\alpha_j \leq 0$ for all $j = 1, 2, \ldots, N$. Combining Lemmas 4.5 and 4.9, we conclude the following.

THEOREM 4.10. *Assume that* κ^2 *is not a Dirichlet eigenvalue of* $-\Delta$ *in* D, *the matrix* $\Theta(\alpha)$ *is invertible, and* $\mathrm{Im}\,\alpha_j \leq 0$ *for all* $j = 1, 2, \ldots, N$. *Then*

(i) *the function* $\phi_y(\hat{x})$ *belongs to* $\mathcal{R}(\tilde{F}_\sharp^{1/2})$ *iff* $y \in D \cup Y$;

(ii) *the far-field data* $u(\hat{x}, d)$ *for all* $\hat{x}, d \in \mathbb{S}^2$ *uniquely determine the shape of the extended obstacle and positions of the point-like scatterers.*

Note that the uniqueness described in Theorem 4.10(ii) is only a corollary of the first assertion. By Picard's theorem (see, e.g., [7, Theorem 4.8]), the set $D \cup Y$ can be characterized through the eigensystem of the far-field operator as follows.

COROLLARY 4.11. *Suppose the assumptions in Theorem* 4.10 *hold. Let* (λ_j, ψ_j) *be an eigensystem of the (positive) operator* $\tilde{F}_\sharp := |\mathrm{Re}\,\tilde{F}| + |\mathrm{Im}\,\tilde{F}|$. *We have the following characterization of* $D \cup Y$:

$$y \in D \cup Y \quad \Longleftrightarrow \quad W(y) := \left[\sum_{j=1}^{\infty} \frac{|(\phi_y, \psi_j)_{L^2(\mathbb{S}^2)}|^2}{\lambda_j}\right]^{-1} > 0, \tag{4.21}$$

where $\phi_y(\hat{x}) := \exp(-i\kappa\hat{x} \cdot y)$ *for* $\hat{x} \in \mathbb{S}^2$.

Thus, the function $W(y)$ on the r.h.s. (4.21) can be regarded as an indicator function for the unknown scatterer $D \cup Y$, where the variable y is the sampling point. The values of W for $y \in D \cup Y$ should be much larger than those for $y \in \mathbb{R}^3 \backslash \{\overline{D} \cup Y\}$. We complete this section with the following remarks.

Remark 4.12.

(i) Note that the matrix $\Theta(\kappa, \alpha)$ is the same as the matrix $\mathcal{M}(\kappa, \alpha)$ in section 2 (see Remark 3.8) when the extended obstacle is absent. Hence a condition similar to that in Remark 3.8 is enough to invert $\Theta(\kappa, \alpha)$. Precisely, if

$$\frac{N-1}{d} \max_{j=1,\ldots,N} \left|\frac{i\kappa}{4\pi} - \alpha_j\right|^{-1} < 1\,, \tag{4.22}$$

then $\Theta(\kappa, \alpha)$ is diagonally dominant and hence invertible. Note that we took the constant C appearing in Remark 3.8 to be $C = (4\pi)^{-1}$ and $h = 0$ since $D = \emptyset$, and hence $|\Phi_D(x, y)| = |\Phi^\kappa(x, y)| = (4\pi)^{-1}|x - y|^{-1}$. Actually, in the case of absence of extended obstacles, we have a weaker condition than (4.22) to ensure the invertibility of $\Theta(\kappa, \alpha)$. Namely, there exists a positive constant a_0 such that if

$$\frac{\max_{j=1,\ldots,N} |\frac{i\kappa}{4\pi} - \alpha_j|^{-1}}{d} < a_0 \ \text{ and } \ \min_{j\neq m} \cos(\kappa|y_j - y_m|) \geq 0, \tag{4.23}$$

then $\Theta(\kappa,\alpha)$ is invertible. Note that this last condition is independent of the number of obstacles N. Such a condition is derived in the framework of scattering by many small obstacles in [6], where the coefficients $|\frac{i\kappa}{4\pi}-\alpha_j|^{-1}$ are replaced by the diameters of the small obstacles.

(ii) The condition $\operatorname{Im}\alpha_j=0$ is needed to have the self-adjointness of the scattering operators in section 2. Indeed, we know that $(Q_\alpha)^*=Q_\alpha$, and hence we need $\alpha^*=\alpha$, which implies that $\operatorname{Im}\alpha_j=0$ if we take $\alpha:=\operatorname{diag}(\alpha_1,\ldots,\alpha_N)$.

(iii) Our arguments apply to the case where the extended scatterer D is of sound-hard or impedance type, or is a penetrable medium. For the forward problem, the waves u_D^{sc}, u_D, and Φ_D^κ in (4.1) should be redefined according to the underlying extended scatterer D. Concerning the inverse problem, our approach in section 4 extends to these cases with an additional complexity from how to factorize the far-field operator with an appropriate middle operator and then to justify the conditions in Lemma 4.9. We believe that this can be achieved taking into account the various versions of the factorization method in [19] for extended penetrable or impenetrable scatterers only.

5. Numerical results and discussions. This section is devoted to reporting numerical examples for testing the accuracy and validity of the factorization method in $\mathbb{R}^2$. We first list the necessary changes for carrying over the mathematical analysis of the direct and inverse scattering from three to two dimensions. The free space fundamental solution to the Helmholtz equation $(\Delta+\kappa^2)u=0$ in two dimensions is given by

$$\Phi^\kappa(x,y)=\frac{i}{4}H_0^{(1)}(\kappa|x-y|),\quad x\neq y,$$

where $H_0^{(1)}(t)$ denotes the Hankel function of the first kind and of order zero. In $\mathbb{R}^2$, the Sommerfeld radiation condition has to be replaced by

$$\lim_{r\to\infty}\sqrt{r}(\partial_r u-i\kappa u)=0,\quad r=|x|,$$

uniformly for all directions $x/|x|$. With some normalization we defined the far-field pattern as

$$u(x)=\gamma\frac{e^{i\kappa|x|}}{\sqrt{|x|}}\left\{u^\infty(\hat{x})+\mathcal{O}\left(\frac{1}{|x|}\right)\right\},\quad \gamma=\frac{e^{i\pi/4}}{\sqrt{8\pi\kappa}}.$$

The operators Γ_j $(j=1,2)$ introduced in (3.28) have to be replaced by

(5.1)
$$(\Gamma_1 u)_j=\lim_{x\to x_j}-\frac{2\pi}{\ln|x-x_j|}u(x)\,,\qquad (\Gamma_2 u)_j=\lim_{x\to x_j}\left(u(x)+\frac{1}{2\pi}\ln|x-x_j|\,(\Gamma_1 u)_j\right).$$

Then, using the expansion of $\Phi^\kappa(x,y_j)$ (see, e.g., [7, Chapter 3.4]), we see

$$(\Gamma_1\Phi^\kappa(x,y_j))_j=1,\quad (\Gamma_2\Phi^\kappa(x,y_j))_j=\frac{i}{4}-\frac{1}{2\pi}\ln\frac{\kappa}{2}-\frac{C}{2\pi}=:\eta\,,$$

where C denotes Euler's constant. Consequently, we define the matrix

$$[\mathcal{M}(\kappa,\alpha)]_{m,j}=\begin{cases}\Phi_D^\kappa(y_m,y_j) & \text{if}\quad m\neq j,\\ \Phi_D^{sc}(y_j,y_j)+\eta-\alpha_j & \text{if}\quad m=j,\end{cases}$$

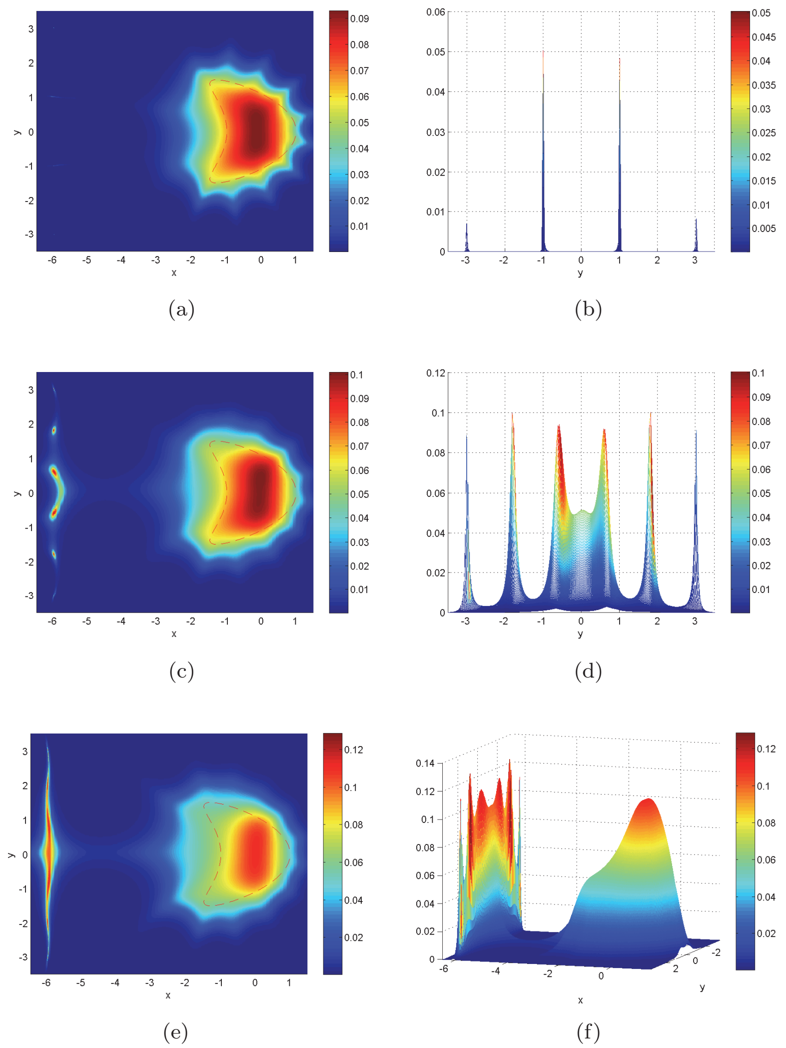

FIG. 1. *Reconstruction of a kite and N point-like obstacles lying on a line segment. In* (a) *and* (b), $N = 4$*; in* (c) *and* (d), $N = 6$*; and in* (e) *and* (f), $N = 8$.

in place of the one given in Lemma 4.1. Then the well-posedness of the forward scattering and the factorization method for the inverse scattering can be established in the same manner as for three dimensions.

In the following experiments, unless otherwise stated we always set the wave number $\kappa = 1$. The far-field operator F is discretized by 64 incident directions and 64 observation directions equivalently distributed in the unit disk.

Experiment 1. We use the inversion algorithm (4.21) to reconstruct a kite-shaped obstacle and a finite number N of point-like obstacles equivalently lying on the line segment $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = -6, x_2 \in [-3, 3]\}$. The numerical results are shown in Figure 1. We take $\alpha_j = 1$. The number N is set as $N = 4, 6, 8$ in Figures 1(a), 1(c), and 1(e), respectively, where the plots of the indicator functions are visualized from the direction $(0, 0, 1)$. Figures 1(b) and 1(d) show the recovery of the point-

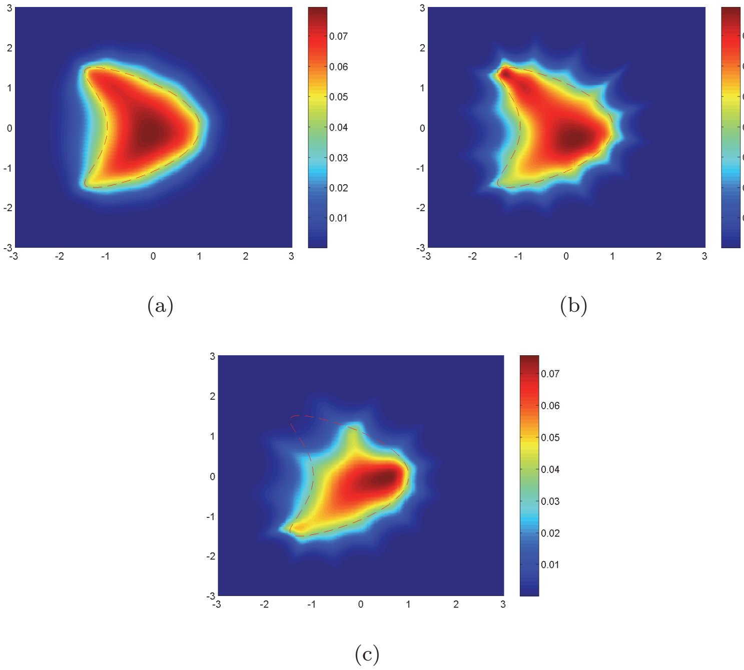

FIG. 2. *Reconstruction of a kite and one point-like obstacle located at* (a, a). *We set* $a = 8, 4, 1.5$ *in* (a), (b), *and* (c), *respectively.*

like obstacles corresponding to Figures 1(a) and 1(c) from the viewpoint $(-1, 0, 0)$, while Figure 1(f) is the view of (e) from the point $(1, -3, 0.5)$. We conclude from Figure 1 that the factorization method works well only if the point-like scatterers are well separated. When the number N increases, neither the positions of the point-like obstacles nor the shape of the extended obstacle can be precisely retrieved; see Figures 1(e) and 1(f), where the minimum distance between the pointwise scatterers is less than the wavelength. In the well-separated case, e.g., Figures 1(a) and 1(b), the point-like scatterers that are closer to the extended obstacle are less resolved than those further away from the extend obstacle (see also Figure 1(c)).

Experiment 2. We show the sensitivity of the factorization method to the distance between the point-like and extended obstacles. The recovery of a fixed kite and one point-like obstacle with different locations at $(8, 8)$, $(4, 4)$, $(1, 5, 1, 5)$ are illustrated in Figures 2(a), 2(b), and 2(c), respectively. We set $\alpha = \alpha_j = 1$ in each test. The shape of the extended obstacle can be identified only if the point-like obstacle keeps some distance from it (see Figure 2(a)). In Figure 2(c), neither of them is well reconstructed since they are getting too closed. The location of the point-like obstacle in Figure 2(a) or 2(b) can be visualized from the XY-plane rather than from the direction $(0, 0, 1)$.

Experiment 3. Figure 3 illustrates the sensitivity of the factorization method to the values of α_j. We fix the kite as in Experiment 2 and also the position of the point-like obstacle at $(2.5, 2.5)$. We set $\alpha = \alpha_j = 10, 1, 0.05$ in Figures 3(a), 3(c), and 3(e), respectively. These figures are visualized from the direction $(0, 1, 0)$ in Figures 3(b), 3(d), and 3(f), respectively. If α is big, e.g., in Figures 3(a) and 3(b), the position of the point-like obstacle cannot be located. If α is too small, e.g., in Figures 3(e) and 3(f), the reconstruction of the extend obstacle becomes distorted and unreliable. It

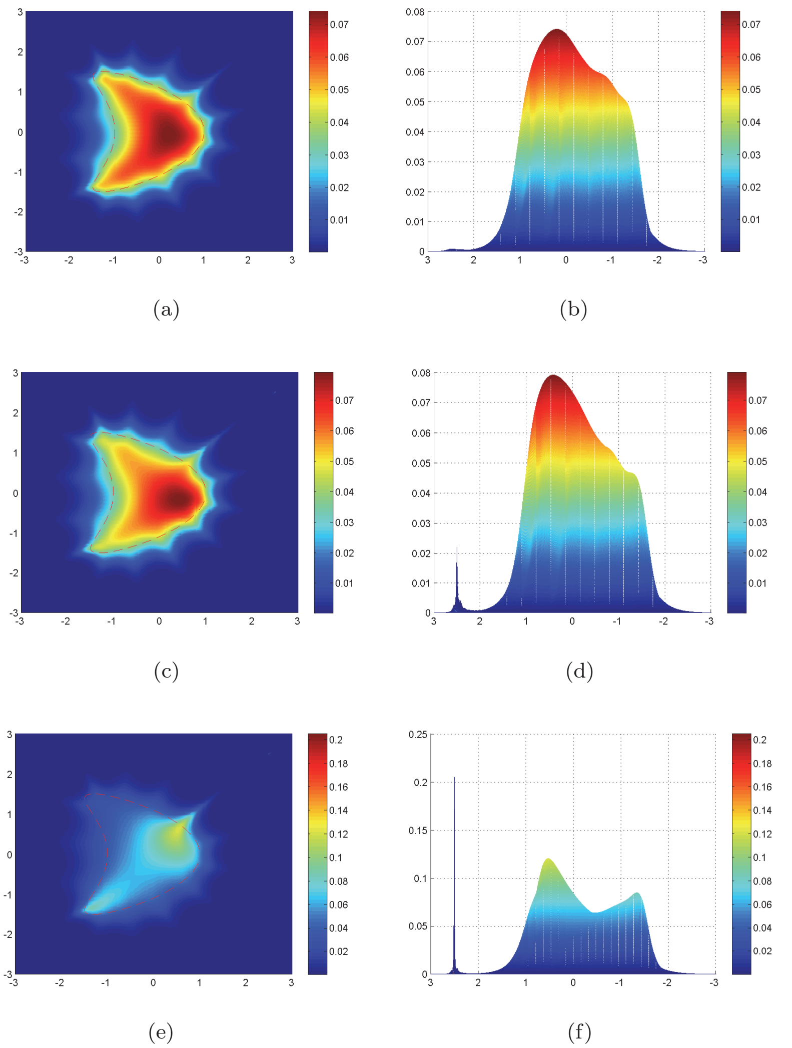

FIG. 3. *Reconstruction of a kite and one point-like obstacle located at* $(2.5, 2.5)$. *In* (a) *and* (b), $\alpha = 10$; *in* (c) *and* (d), $\alpha = 1$; *and in* (e) *and* (f), $\alpha = 0.05$.

can also be observed that the values of the indicator function around the point-like obstacle grow as the value of α decreases, i.e., the point-like obstacle is more visible for small α.

Experiment 4. The factorization method can be applied to the case where only partial far-field data are available, i.e., $u^\infty(\hat{x}, d)$ for $\hat{x}, d \in \tilde{\mathbb{S}}^2 \subset \{x : |x| = 1\}$. For details we refer the reader to [19, Chapter 2.3]. In this experiment, we consider the reconstruction from limited aperture data $\tilde{\mathbb{S}}^2 = \{(\cos\theta, \sin\theta) : \theta \in [-\pi/2, \pi/2]\}$. This implies that the obstacles are illuminated by incident plane waves only from the r.h.s., and the far-field data are measured on the same side. To make the numerical results comparable to the case where the full far-field data are used, we discretize the corresponding far-field data by 32 incident directions and 32 observation directions equivalently distributed in the right half of the unit disk. We put N point-like ob-

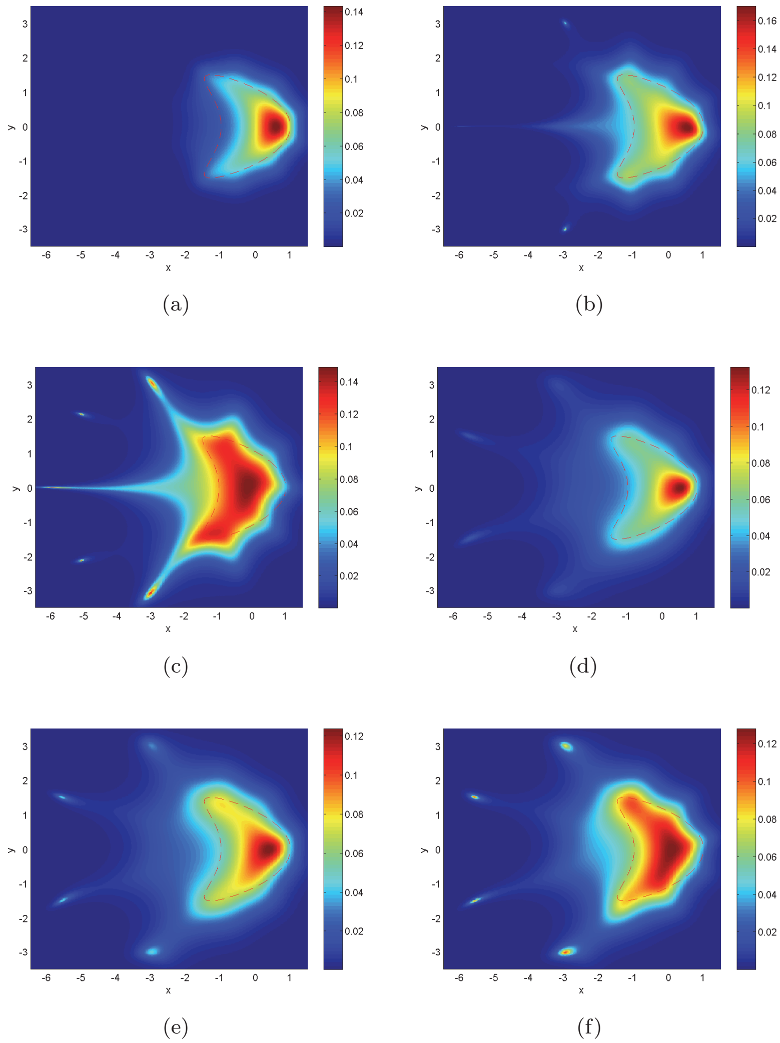

FIG. 4. *Reconstruction of a kite and N point-like obstacles lying on a half-circle using limited aperture data. In* (a), $N = 0$. *In* (b) *and* (c), $N = 3$ *and* 5, *respectively, and* $\alpha_j = 0.3$. *In* (d), (e), *and* (f), *we set* $\alpha_j = 2$, 0.75, *and* 0.3, *respectively, and* $N = 4$.

stacles equivalently lying on the half-circle $\{(x_1, x_2) : x_1 = \cos\beta - 3, x_2 = \sin\beta, \beta \in [\pi/2, 3\pi/2]\}$.

Figure 4(a) illustrates the recovery of the extended obstacle in the absence of the point-like scatterers (i.e., $N = 0$). It can be observed how the reconstruction of the unlighted (left half) part of the kite deteriorates due to the limited incident directions from the r.h.s. The shadow part can be well reconstructed in Figures 4(c) and 4(f) by virtue of the multiple scattering effect between the point-like and extend obstacles. In Figures 4(d), 4(e), and 4(f), we fix the number of point-like obstacles (i.e., $N = 4$) and change the value of the coefficients α_j. It is seen that the visibility of the unilluminated part also depends on α_j. However, numerical experiments show poor reconstructions for α_j less than 0.2.

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