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Elastic scattering by unbounded rough surfaces: solvability in weighted Sobolev spaces

Johannes Elschner* and Guanghui Hu

Weierstrass Institute, Mohrenstr. 39, 10117 Berlin, Germany

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This paper is concerned with the variational approach in weighted Sobolev spaces to time-harmonic elastic wave scattering by one-dimensional unbounded rough surfaces. The rough surface is supposed to be the graph of a bounded and uniformly Lipschitz continuous function, on which the total elastic displacement satisfies either the Dirichlet or impedance boundary condition. We establish uniqueness and existence results at arbitrary frequency for both elastic plane wave and point source (spherical) wave incidence in the two-dimensional case. In particular, our approach covers the elastic scattering from periodic structures (diffraction gratings), and we prove quasiperiodicity of the scattered field whenever the incident field is quasiperiodic. Moreover, the diffraction grating problem is also uniquely solvable in the presented weighted Sobolev spaces for a broad class of non-quasiperiodic incident waves.

Keywords: non-smooth rough surface; linear elasticity; radiation condition; variational formulation; weighted Sobolev spaces; Navier equation

AMS Subject Classifications: 74B05; 35J05; 35J20; 35J25; 42B10; 78A45; 74J20; 35J57; 35Q74

1. Introduction

Rough surface scattering problems for acoustic, electromagnetic, and elastic waves have been of interest to physicists, engineers, and applied mathematicians for many years due to their wide range of applications in optics, acoustics, radio-wave propagation, seismology, and radar techniques (see, e.g. [1–6]). Diffraction phenomena for elastic waves propagating through unbounded interfaces have many applications, particularly in geophysics and seismology. For instance, the problem of elastic pulse transmission and reflection through the earth is fundamental to the investigation of earthquakes and the utility of controlled explosions in search for oil and ore bodies; see, e.g. [1,7–9] and the references therein.

This paper is concerned with uniqueness and existence results in weighted Sobolev spaces for the two-dimensional problem of time-harmonic scattering of incident elastic plane and point source waves from unbounded rough surfaces. We suppose the scattering surface is given by the graph of a bounded and uniformly Lipschitz continuous function,

*Corresponding author. Email: hu@wias-berlin.de

on which the total elastic displacement satisfies either the Dirichlet or impedance boundary condition.

There is already a vast literature on the mathematical analysis of acoustic and electromagnetic scattering by rough surfaces modeled by the Helmholtz equation. We refer the reader to [10–12,38] and [13, Chapter 5] for the integral equation method applied to the Dirichlet boundary value problem with smooth ($C^{1,\alpha}$) surfaces in $\mathbb{R}^n$ ($n = 2, 3$) and to [14–16] for the scattering by penetrable interfaces and inhomogeneous layers. The variational approach proposed in [17] by Chandler-Wilde and Monk gives rises to existence and uniqueness results in non-weighted Sobolev spaces, allowing to treat the scattering problem due to an inhomogeneous source term whose support lies within a finite distance above rather general sound-soft surfaces in $\mathbb{R}^n$ ($n = 2, 3$). Moreover, this approach leads to explicit bounds on solutions in terms of the data and applies to acoustic scattering by impedance surfaces as well as by inhomogeneous rough layers; see, e.g. [13,18,19].

A rigorous analysis on the two-dimensional elastic scattering of plane waves is given by Arens [20,21] for smooth ($C^{1,\alpha}$) rigid surfaces, where the solution is sought in $C^2(D) \cap C(\overline{D})$ (the region D denotes the unbounded domain above the scattering surface) via integral equation methods. This generalizes the solvability results in [11,14,15] from the Helmholtz equation to the Navier equation. Moreover, an upward propagating radiation condition (UPRC) is proposed in [20] based on the elastic Green's tensor of the Dirichlet boundary value problem in a half plane. The UPRC is proved to be equivalent to the so-called angular spectrum representation for solutions of the Navier equation established in [22]. The latter has been used to prove well-posedness of the Dirichlet boundary value problem in non-weighted Sobolev spaces via a variational approach and perturbation arguments for semi-Fredholm operators (see [22]). A different radiation condition is used in the work of Duran et al. [23], with an emphasis placed on treating surface waves arising from local normal stress excitations on the free boundary of a half plane. This new radiation condition is inspired by the asymptotic behavior of the half-space elastic Green's tensor with the Neumann boundary condition. It leads to well-posedness of the Neumann boundary value problem in suitable weighted Sobolev spaces, but the weights there (see also [24] in the case of the Helmholtz equation) are different from ours presented in this paper.

We investigate the variational approach in appropriate weighted Sobolev spaces for both the Dirichlet and impedance boundary value problems, where the time-harmonic incident elastic plane pressure and shear waves as well as incident elastic point source waves are all covered. Our methods are closest to the recently developed variational approach of Chandler-Wilde and Elschner [25] to acoustic scattering by rough surfaces. The well-posedness there is established by using the results of [17] in the non-weighted setting and a perturbation argument. In this paper, the solvability of the impedance boundary value problem in the standard Sobolev space is established by investigating an auxiliary Dirichlet boundary value problem with an inhomogeneous source term; see Section 3.2. This novel idea for treating the impedance boundary value problem comes from [22,26] where the a priori estimates for solutions of the Helmholtz equation in unbounded periodic and non-periodic structures have been established via Rellich-type identities. It also provides a shorter and simpler proof of the well-posedness of acoustic scattering from impedance rough surfaces in standard Sobolev spaces (see [13, Chapter 3.4]) at arbitrary wavenumber.

The grating diffraction problem can be viewed as a special case of scattering by a rough surface. Existing solvability results for diffraction gratings (periodic structures) in the literature all rest on the essential assumption of quasiperiodicity of solutions. Such an assumption leads to an outgoing Rayleigh expansion of the scattered field and has considerably simplified the mathematical analysis of periodic scattering problems. We refer to [27] for uniqueness and existence proofs via integral equation methods and to [28,29] for the variational approach applied to boundary value problems of the first, second, third, or fourth kind as well as to transmission problems with non-smooth interfaces in $\mathbb{R}^n$ ($n = 2, 3$). As a consequence of the solvability in weighted spaces, we provide a theoretical justification of the quasiperiodicity of solutions for elastic diffraction grating problems, whenever the incident wave is quasiperiodic; see Section 4.1. In addition, our weighted Sobolev space for rough surface scattering problems can be regarded as the solution space for the unique solvability of diffraction of non-quasiperiodic incident waves from periodic structures, including incident elastic point-source waves generated by the free space elastic Green's tensor and a linear combination of incident plane pressure and shear waves.

The paper is organized as follows. In Section 2, we rigorously formulate the Dirichlet and impedance boundary value problems in weighted Sobolev spaces and propose their equivalent variational formulations. As in [25], the radiation condition is to be understood as a bounded linear functional on a weighted Sobolev space. We adopt the idea of [25, Remark 5.4] to formulate the boundary value problems as equivalent variational equations in a straightforward way. The right-hand sides of these equations are given explicitly in terms of the incident elastic plane waves, and they actually take a form analogously to that arising from diffraction grating problems; cf. Section 2.4 and [28,29]. In Section 3, we prove existence and uniqueness of solutions to the equivalent variational problems, following the perturbation argument of [25] that relies on commutator estimates for the Dirichlet-to-Neumann map. Section 4 concerns applications of our solvability results to the elastic scattering from periodic structures (diffraction gratings) as well as to the scattering of elastic point source (spherical) waves.

Section 5 is devoted to the proof of the crucial commutator estimates of Section 3. In contrast to the Helmholtz case [25] where a square-root symbol with two singularities is involved, we have to investigate properties of a non-smooth symbol in the form of a 2-by-2 matrix with four singularities. Therefore, additional arguments are needed in order to generalize the results of [25] to the case of elastic scattering; see Section 5 for the details. These commutator estimates play an essential rule not only in verifying the main Theorems 2.2 and 3.1, but also in establishing equivalent variational formulations in the weighted spaces (see Lemma 2.5). In particular, Lemma 5.4 (i) provides a proof of [22, Lemma 1] in the non-weighted Sobolev spaces.

We further note that the commutator estimates and the solvability results obtained in this paper can be extended to three-dimensional elastic rough surface scattering problems. Consequently, the Dirichlet and impedance problems for incident spherical and cylindrical elastic waves in three-dimension can be treated analogously.

We end up this section by introducing some notation to be used later. Denote by $(\cdot)^\top$ the transpose of a vector or a matrix. For $a \in \mathbb{C}$, let $|a|$ denote its modulus, and for $\mathbf{a} \in \mathbb{C}^2$, let $|\mathbf{a}|$ denote its Euclidean norm. For a matrix $M = (m_{ij}) \in \mathbb{C}^{2\times 2}$, $||M||$ denotes the norm defined by $||M|| := \max_{i,j} |m_{ij}|$. The symbol $\mathbf{a} \cdot \mathbf{b}$ stands for the inner product $a_1 b_1 + a_2 b_2$ of $\mathbf{a} = (a_1, a_2)^\top$, $\mathbf{b} = (b_1, b_2)^\top \in \mathbb{C}^2$. Standard L^2-based scalar Sobolev spaces defined in a domain Ω or on a surface Γ are denoted by $H^s(\Omega)$ or $H^s(\Gamma)$ for $s \in \mathbb{R}$. Throughout the paper, the branch cut of a complex square root is always chosen such that its imaginary part is non-negative. Unless otherwise stated, we always use c, C to denote generic positive constants which may vary from line to line.

2. Boundary value problems and equivalent variational formulations

2.1. *The basic model*

We precisely formulate the scattering problems as follows. Let $D \subset \mathbb{R}^2$ be an unbounded connected open set such that for some constants $f_- < f_+$ it holds that

$$U_{f_+} \subset D \subset U_{f_-}, \quad U_{f_+} := \{x = (x_1, x_2) : x_2 > f_+\}. \tag{2.1}$$

As in our previous paper [22], the boundary $\Gamma := \partial D$ of D is supposed to be the graph of a uniformly Lipschitz continuous function f, i.e.

$$\Gamma = \{x \in \mathbb{R}^2 : x_2 = f(x_1), x_1 \in \mathbb{R}\}, \tag{2.2}$$

and there is a constant $L > 0$ such that

$$|f(x_1) - f(x_2)| \leq L\,|x_1 - x_2|, \quad \text{for all} \quad x_1, x_2 \in \mathbb{R}. \tag{2.3}$$

Such a geometric assumption on Γ is weaker than the condition used in [20,21] but stronger than that in [17,25]. Our a priori estimates of solutions derived in Section 3 always depend on the global Lipschitz constant L. Assume the region D is filled with an isotropic homogeneous elastic medium characterized by the Lamé constants λ, μ satisfying $\mu > 0$, $\lambda + \mu > 0$. Let u^{in} be a time-harmonic elastic plane wave (with time variation of the form $\exp(-i\omega t)$, $\omega > 0$) incident on the rough surface Γ from above. The incident wave is assumed to be a linear combination of plane pressure and shear waves having the same incident angle $\theta \in (-\pi/2, \pi/2)$, i.e.

$$u^{in} = c_1 u_p^{in} + c_2 u_s^{in}, \quad c_j \in \mathbb{C},\ j = 1, 2, \tag{2.4}$$

where

$$u_p^{in} := \hat{\theta} \exp(ik_p\hat{\theta} \cdot x), \qquad \hat{\theta} := (\sin\theta, -\cos\theta), \quad k_p := \omega/\sqrt{2\mu + \lambda},$$
$$u_s^{in} := \hat{\theta}^{\perp} \exp(ik_s\hat{\theta} \cdot x), \quad \hat{\theta}^{\perp} := (\cos\theta, \sin\theta), \quad k_s := \omega/\sqrt{\mu}.$$

Note that k_p and k_s are called the compressional and shear wave numbers, respectively. The case of incident elastic point source (spherical) waves will be treated in Section 4.2, following the approach for plane wave incidence.

We look for the total elastic displacement $u = (u_1, u_2)^\top$ such that the Navier equation

$$(\Delta^* + \omega^2)\, u = 0 \quad \text{in} \quad D\,, \quad \Delta^* := \mu\Delta + (\lambda + \mu)\, \text{grad div}\,, \tag{2.5}$$

together with one of the following boundary conditions on Γ:

$$\text{Dirichlet boundary condition}: \quad u = 0, \tag{2.6}$$

$$\text{impedance boundary condition}: \quad Tu - i\eta u = 0, \quad \eta > 0, \tag{2.7}$$

holds in a distributional sense, and that the scattered field $u^{sc} := u - u^{in}$ satisfies an appropriate radiation condition as $x_2 \to +\infty$. Note that in (2.5), we have assumed for simplicity that the mass density of the elastic medium in D is equal to one. The operator T in (2.7) stands for the *stress vector* or *traction* having the form

$$Tu = 2\mu\, \partial_{\mathbf{n}} u + \lambda\, \mathbf{n}\, \text{div}\, u + \mu \begin{pmatrix} n_2\,(\partial_1 u_2 - \partial_2 u_1) \\ n_1\,(\partial_2 u_1 - \partial_1 u_2) \end{pmatrix} \quad \text{on } \Gamma \tag{2.8}$$

where $\mathbf{n} = (n_1, n_2)^\top$ denotes the unit normal pointing into the exterior of D.

2.2. *Weighted Sobolev spaces*

For $h > f^+ := \sup_{x_1\in\mathbb{R}}\{f(x_1)\}$, let $\Gamma_h := \{x = (x_1, x_2) : x_2 = h\}$ and $S_h := D\backslash\overline{U}_h$. Our variational formulation will be posed on the infinite strip S_h; see Figure 1. Let $\mathcal{F}v$ denote the Fourier transform of v defined by

$$\mathcal{F}v(\xi) = (2\pi)^{-1/2}\int_{\mathbb{R}} \exp(-it\,\xi)v(t)\,dt, \quad \xi \in \mathbb{R},$$

with the inverse transform given by

$$\mathcal{F}^{-1}w(t) = (2\pi)^{-1/2}\int_{\mathbb{R}} \exp(it\,\xi)w(\xi)\,d\xi, \quad t \in \mathbb{R}.$$

We first introduce weighted Sobolev spaces. For $\varrho \in \mathbb{R}$, $l \in \mathbb{N}$ and a domain $G \subset \mathbb{R}$, define the Hilbert spaces

$$L^2_\varrho(G) := \left(1 + x_1^2\right)^{-\varrho/2} L^2(G)\,, \quad H^l_\varrho(G) := \left(1 + x_1^2\right)^{-\varrho/2} H^l(G)\,,$$

equipped with the corresponding canonical norm and scalar product. The space $V_{h,\varrho}$ is then defined as the closure of $\{u|_{S_h} : u \in C_0^\infty(D)\}$ in the norm

$$\|u\|_{V_{h,\varrho}} = \|u\|_{H^1_\varrho(S_h)} = \left(\int_{S_h}\left(\left|\left(1+x_1^2\right)^{\varrho/2}u\right|^2 + \left|\nabla\left(1+x_1^2\right)^{\varrho/2}u\right|^2\right)dx\right)^{1/2}. \quad (2.9)$$

From time to time, we employ the following equivalent norm to $||\cdot||_{V_{h,\varrho}}$:

$$||u||' := \left(\int_{S_h}\left(1+x_1^2\right)^{\varrho}\left(|u|^2 + |\nabla u|^2\right)dx\right)^{1/2}, \quad u \in V_{h,\varrho}. \quad (2.10)$$

Moreover, we introduce

$$H^s_\varrho(\Gamma_h) := \left(1+x_1^2\right)^{-\varrho/2} H^s(\Gamma_h), \quad \varrho \in \mathbb{R}\,,$$

where $H^s(\Gamma_h)$ is identified with the Sobolev space $H^s(\mathbb{R})$ with norm

$$\|v\|_{H^s(\mathbb{R})} = \left(\int_{\mathbb{R}}\left(1+\xi^2\right)^s |\mathcal{F}v|^2 d\xi\right)^{1/2}.$$

The weighted space $H^s_\varrho(\mathbb{R})$ will be endowed with the norm

$$||v||_{H^s_\varrho(\mathbb{R})} := ||\left(1+x_1^2\right)^{\varrho/2} v(x_1)||_{H^s(\mathbb{R})}. \quad (2.11)$$

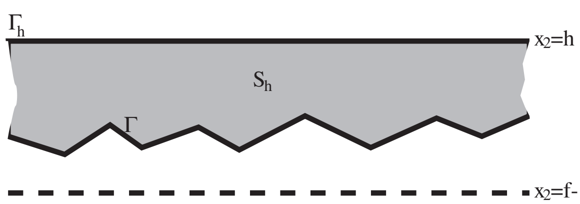

Figure 1. Geometrical setting of the scattering problem.

Obviously, the restriction of the incident plane wave u^{in} given in (2.4) to S_h ($h > f^+$) belongs to the space $H^1_\varrho(S_h)^2$ for all $\varrho < -1/2$. Below we collect some properties of $H^s_\varrho(G)$, which will be used for our subsequent analysis.

PROPOSITION 2.1 [30,31]

(i) *F is an isometry of $L^2(\mathbb{R})$ onto itself and also an isometry of $L^2_\varrho(\mathbb{R})$ onto $H^\varrho(\mathbb{R})$. More generally, F is an isomorphism of $H^s_\varrho(\mathbb{R})$ onto $H^\varrho_s(\mathbb{R})$ for all $s, \varrho \in \mathbb{R}$.*

(ii) *The trace operators*

$$\gamma_- : H^1_\varrho(S_h) \to H^{1/2}_\varrho(\Gamma_h)\,, \quad \gamma_+ : H^1_\varrho(U_h \backslash \bar{U}_H) \to H^{1/2}_\varrho(\Gamma_h)\,, \quad H > h\,,$$

are continuous.

(iii) *The dual space of $H^s_\varrho(\mathbb{R})$ with respect to the L^2 scalar product is $H^{-s}_{-\varrho}(\mathbb{R})$, that is, $H^s_\varrho(\mathbb{R})^* = H^{-s}_{-\varrho}(\mathbb{R})$ for all $s, \varrho \in \mathbb{R}$.*

2.3. *Radiation condition and boundary value problems*

To formulate the Dirichlet and impedance boundary value problems, we need an appropriate radiation condition for the scattered field in D as $x_2 \to \infty$. Assuming that u^{sc} is a linear superposition of outgoing plane waves in D, we shall represent the scattered field in U_h in terms of the trace $u^{sc}_h := u^{sc}|_{\Gamma_h}$. Using Fourier transform, it was derived in [22] that

$$u^{sc}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(e^{i\gamma_p(\xi)\,(x_2-h)} M_p(\xi) + e^{i\gamma_s(\xi)\,(x_2-h)} M_s(\xi) \right) \hat{u}^{sc}_h(\xi)\, e^{ix_1\xi}\, d\xi \tag{2.12}$$

for $x_2 > h$, where M_p and M_s are two matrices given by

$$M_p(\xi) = \frac{1}{\xi^2 + \gamma_p\gamma_s} \begin{pmatrix} \xi^2 & \xi\gamma_s \\ \xi\gamma_p & \gamma_p\gamma_s \end{pmatrix}, \quad M_s(\xi) = \frac{1}{\xi^2 + \gamma_p\gamma_s} \begin{pmatrix} \gamma_p\gamma_s & -\xi\gamma_s \\ -\xi\gamma_p & \xi^2 \end{pmatrix}, \tag{2.13}$$

respectively, with $\gamma_p(\xi) := \sqrt{k_p^2 - \xi^2}$, $\gamma_s(\xi) := \sqrt{k_s^2 - \xi^2}$. Obviously, $M_p(\xi) + M_s(\xi) = I$ for any $\xi \in \mathbb{R}$, where I denotes the 2×2 unit matrix. The right-hand side of (2.12) can be interpreted as a superposition of upward propagating homogeneous compressional resp. shear plane waves corresponding to $|\xi| \le k_p$ resp. $|\xi| \le k_s$ and some evanescent surface waves corresponding to $|\xi| > k_p$ resp. $|\xi| > k_s$. Hence, expression (2.12) is always referred to as the angular spectral representation for solutions of the Navier equation in the literature (see e.g. [32]). Moreover, such a radiation condition can be written in an alternative form that is identical with the Upward Propagating Radiation Condition (UPRC) proposed by Arens [20] (see [22, Remark 1])

$$u^{sc}(x) = -i \int_{\Gamma_h} T_y \left[\Pi_h(x, y)\right] u^{sc}(y)\, ds(y) \quad \text{for } x_2 > h. \tag{2.14}$$

Here, $\Pi_h(x, y)$ denotes the Green's tensor for the Navier equation in the half space $x_2 > h$ with the homogeneous Dirichlet boundary condition on Γ_h, and $T_y\,[\Pi_h(x, y)]$ is understood as the application of T to each column of $\Pi_h(x, y)$ with respect to the argument y. The explicit expression of $\Pi_h(x, y)$ and its inverse Fourier transform on Γ_h with respect to y_1 can be found in [20].

Since each element of $M_p \exp(i\gamma_p(x_2-h)$ and $M_s \exp(i\gamma_s(x_2-h)$ is uniformly bounded in $\xi \in \mathbb{R}$, the integral (2.12) exists in the Lebesgue sense for all $x \in U_h$ when $u_h^{sc} \in L^2(\Gamma_h)^2$ so that $\hat{u}_h^{sc} \in L^2(\mathbb{R})^2$. In the weighted case of $u_h^{sc} \in H_\varrho^{1/2}(\mathbb{R})^2$ with $\varrho > -1$, we can interpret Equation (2.12) as a bounded linear functional over $H_\varrho^{1/2}(\mathbb{R})^2$. To see this, arguing analogously to the Helmholtz case, we only need to show that the function $l_x(\xi)$, defined by

$$l_x(\xi) := \frac{1}{\sqrt{2\pi}} \left(e^{i\gamma_p(\xi)\,(x_2-h)} M_p(\xi) + e^{i\gamma_s(\xi)\,(x_2-h)} M_s(\xi) \right) e^{ix_1\xi},$$

belongs to the dual space $H_{-1/2}^{-\varrho}(\mathbb{R})^2$ of $H_{1/2}^{\varrho}(\mathbb{R})^2$ for $\varrho > -1$; note that by Proposition 2.1(i) we have $\mathcal{F}u_h^{sc} \in H_{1/2}^{\varrho}(\mathbb{R})^2$. Indeed, using (2.11) there holds

$$\begin{aligned} ||l_x(\xi)||^2_{H_{-1/2}^{-\varrho}(\mathbb{R})^2} &= || \left(1+\xi^2\right)^{-1/4} l_x(\xi)\, ||^2_{H^{-\varrho}(\mathbb{R})^2} \\ &= \int_{\mathbb{R}} \left(1+\xi^2\right)^{-\varrho} |\, \mathcal{F}_{t\to\xi}[\left(1+t^2\right)^{-1/4} l_x(t)](\xi)|^2\, d\xi \\ &= \int_{\mathbb{R}} \left(1+\xi^2\right)^{-\varrho} |b_{1/2}(\xi) * \hat{l}_x(\xi)|^2\, d\xi, \end{aligned} \tag{2.15}$$

where $b_\varrho(\xi) := F_{t\to\xi} \left(1+t^2\right)^{-\varrho/2} \in L^1(\mathbb{R})$ for $\varrho > 0$ (see e.g. [25, Lemma 6.4]), with $*$ denoting convolution. Moreover, elementary calculations show that (cf. (2.12) and (2.14))

$$\hat{l}_x(y_1) = \mathcal{F}_{\xi\to y_1}[l_x(\xi)] = -i\,[T_y\, \Pi_h(x,y)]|_{\,y\in\Gamma_h}, \quad y = (y_1, y_2) \in \mathbb{R}^2,$$

and that (see [20, Theorem 2.2])

$$||\Pi_h(x,y)|| \le \frac{\mathcal{H}(x_2-h, y_2-h)}{|x_1-y_1|^{3/2}}, \quad |x_1-y_1| \ge \epsilon > 0, \quad x, y \in U_h,$$

for some function $\mathcal{H} \in C(\mathbb{R}^2)$. Together with the interior estimate for solutions to the Navier equation (see e.g. Arens [20, Appendix]), the previous estimate implies that, for a fixed $x \in U_h$, the inequality

$$||\, T_y\, \Pi_h(x,y)\, || \le C\ (1+|x_1-y_1|)^{-3/2}$$

holds uniformly in all $y \in \Gamma_h$, with the positive constant C depending only on x_2 and h. Therefore, it follows from (2.15) that

$$||\, l_x(\xi)||^2_{H_{-1/2}^{-\varrho}(\mathbb{R})^2} \le C\, ||b_{1/2}(\xi)||^2_{L^1(\mathbb{R})} \int_{\mathbb{R}} \left(1+\xi^2\right)^{-(\varrho+3/2)} d\xi,$$

which is bounded provided $\varrho > -1$. This explains why we can understand (2.12) by extending the mapping $u^{sc}(x)|_{\Gamma_h} \to u^{sc}(x)$, given by (2.12), to a bounded linear functional over $H_\varrho^{1/2}(\mathbb{R})^2$ for $\varrho > -1$.

Now we formulate the Dirichlet and impedance boundary value problems (DBVP) and (IBVP) as follows.

(DBVP): Given the incoming plane wave u^{in}, find the total field $u = u^{in} + u^{sc} \in H^1_{loc}(D)^2$ such that

$$u|_{S_h} \in H_\varrho^1(S_h)^2, \quad \forall\, h > f^+, \quad \text{for some } \varrho \in (-1, -1/2),$$

u satisfies the Navier Equation (2.5) in a distributional sense and the Dirichlet condition (2.6), and that the radiation condition (2.12) holds for all $h > f^+$.

(IBVP): Given the incoming plane wave u^{in}, find the total field $u \in H^1_{loc}(D)^2$ such that

$$u|_{S_h} \in H^1_\varrho(S_h)^2, \quad \forall\, h > f^+, \quad \text{for some } \varrho \in (-1, -1/2),$$

u satisfies the Navier Equation (2.5) in a distributional sense and the impedance boundary condition (2.7), and that the radiation condition (2.12) holds for all $h > f^+$.

2.4. *Dirichlet-to-Neumann map and variational formulations*

The purpose of this subsection is to propose equivalent variational formulations of (DBVP) and (IBVP) in the weighted Sobolev spaces $H^s_\varrho(S_h)^2$ for every $\varrho \in (-1, -1/2)$ and $h > f^+$. Note that we require $-1 < \varrho < -1/2$, because the radiation condition (2.12) is well defined for any $\varrho > -1$ and the elastic plane wave (2.4) belongs to the space $H^1_\varrho(S_h)^2$ for any $\varrho < -1/2$.

Recall the first Betti formula

$$-\int_{S_h} (\Delta^* + \omega^2) w \cdot \overline{v}\, dx = \int_{S_h} \left(\mathcal{E}_{\tilde{\mu},\tilde{\lambda}}(w, \overline{v}) - \omega^2 w \cdot \overline{v}\right) dx - \int_{\partial S_h} \overline{v} \cdot T_{\tilde{\mu},\tilde{\lambda}} w\, ds \qquad (2.16)$$

for $w, v \in H^2(S_h)^2$, where the bar indicates the complex conjugate, $\tilde{\mu}$ and $\tilde{\lambda}$ are real numbers satisfying $\tilde{\mu} + \tilde{\lambda} = \mu + \lambda$, and

$$\begin{aligned} \mathcal{E}_{\tilde{\mu},\tilde{\lambda}}(w, v) &:= (\lambda + 2\mu)\,(\partial_1 w_1\, \partial_1 v_1 + \partial_2 w_2\, \partial_2 v_2) + \mu\,(\partial_2 w_1\, \partial_2 v_1 + \partial_1 w_2\, \partial_1 v_2) \\ &\quad + \tilde{\lambda}\,(\partial_1 w_1\, \partial_2 v_2 + \partial_2 w_2\, \partial_1 v_1) + \tilde{\mu}\,(\partial_2 w_1\, \partial_1 v_2 + \partial_1 w_2\, \partial_2 v_1), \qquad (2.17) \\ T_{\tilde{\mu},\tilde{\lambda}} w &:= (\mu + \tilde{\mu})\,\partial_{\mathbf{n}} w + \tilde{\lambda}\,\mathbf{n}\,\mathrm{div}\, w + \tilde{\mu} \begin{pmatrix} n_2\,(\partial_1 w_2 - \partial_2 w_1) \\ n_1\,(\partial_2 w_1 - \partial_1 w_2) \end{pmatrix}. \end{aligned}$$

In the Dirichlet case, we have a freedom of selecting the parameters $\tilde{\mu}$ and $\tilde{\lambda}$. In our previous paper [22], the parameters $\tilde{\mu}, \tilde{\lambda}$ were taken as $\tilde{\mu} = 0, \tilde{\lambda} = \lambda + \mu$, leading to a minimal loss of coercivity for the corresponding Dirichlet-to-Neumann map on Γ_h; see [22, Remark 4]. Throughout this paper, we set $\tilde{\mu} = \mu, \tilde{\lambda} = \lambda$ so that the operator $T_{\tilde{\mu},\tilde{\lambda}} = T_{\mu,\lambda}$ coincides with the stress operator defined in (2.8). Moreover, with this choice the bilinear form $\mathcal{E}(\cdot,\cdot) = \mathcal{E}_{\mu,\lambda}(\cdot,\cdot)$ can be written as:

$$\mathcal{E}(w, \overline{w}) = \lambda\,|\mathrm{div}\, w|^2 + 2\mu \sum_{i,j=1}^{2} |\epsilon_{i,j}(w)|^2, \quad \epsilon_{i,j}(w) := (\partial_j w_i + \partial_i w_j)/2.$$

Under our assumptions on the Lamé constants, $\mu > 0, \lambda + \mu > 0$, we have the estimate (see e.g. [33, Chap. 5.4])

$$\int_{S_h} \mathcal{E}(w, \overline{w})\, dx \geq C \sum_{i,j=1}^{2} ||\epsilon_{i,j}(w)||^2_{L^2(S_h)}, \quad \forall\, w \in H^1(S_h)^2. \qquad (2.18)$$

and the classical Korn's inequality,

$$\int_{S_h}\left(\sum_{i,j=1}^{2}|\epsilon_{i,j}(w)|^2+|w|^2\right)dx \geq C\,||w||^2_{H^1(S_h)^2}, \quad \forall\, w\in H^1(S_h)^2, \tag{2.19}$$

where $C = C(S_h) > 0$ is independent of w. Korn's inequality for a half space above a Lipschitz graph was proved, e.g. by Nitsche [34], via constructing appropriate extension operators. This approach can be easily adapted to proving (2.19) over the strip S_h of finite height, and we also refer to [35, Section 2.2].

In the following, we introduce the Dirichlet-to-Neumann map $\mathcal{T}$ on the artificial boundary Γ_h, allowing us to treat the scattering problems in the truncated strip S_h in place of the domain D. Define v as the right-hand side of (2.12) with $u_h^{sc}\in C_0^\infty(\mathbb{R})$. Then, elementary calculations show

$$Tv|_{\Gamma_h} = \mathcal{T}(u_h^{sc}),$$

where the Dirichlet-to-Neumann (DtN) map $\mathcal{T} = \mathcal{T}^{\mu,\lambda}$ is given by the pseudodifferential operator

$$\mathcal{T}w := \mathcal{F}^{-1}M(\xi)\mathcal{F}\,w, \quad w\in H_\varrho^{1/2}(\mathbb{R})^2, \tag{2.20}$$

with

$$M = M^{\mu,\lambda} = \frac{i}{\xi^2+\gamma_p\gamma_s}\begin{pmatrix}\omega^2\gamma_p & -\xi\omega^2+2\xi\mu(\xi^2+\gamma_p\gamma_s)\\ \xi\omega^2-2\xi\mu(\xi^2+\gamma_p\gamma_s) & \omega^2\gamma_s\end{pmatrix}. \tag{2.21}$$

The following commutator estimate for the DtN map is crucial for establishing the main solvability results in weighted spaces. Its proof will be carried out later in Section 5, based on the commutator estimate of [25, Theorem 3.1] concerning non-smooth scalar symbols with a square root singularity.

THEOREM 2.2 *Consider the commutator*

$$\mathcal{C} := \mathcal{T} - (a^2+x_1^2)^{\varrho/2}\mathcal{T}(a^2+x_1^2)^{-\varrho/2} \tag{2.22}$$

with the parameter $a > 1$. Then, for $|\varrho| < 1$ and $a > \max\{1, 1/k_s\}$, there exists a positive constant $C = C(\varrho,\omega,\lambda,\mu)$ such that the norm of $\mathcal{C}$ on $L^2(\mathbb{R})^2$ is bounded by $a^{-1/2}C$.

The following lemma describes the continuity properties of $\mathcal{T}$.

LEMMA 2.3

(i) *For any $s\in\mathbb{R}$, the operator $\mathcal{T} = \mathcal{T}(\omega) : H^s(\mathbb{R})^2 \to H^{s-1}(\mathbb{R})^2$ is bounded, and it is also continuous with respect to ω in the operator norm.*

(ii) *For $|\varrho| < 1,\ 0\leq s\leq 1, \quad \mathcal{T} : H_\varrho^s(\mathbb{R})^2 \to H_\varrho^{s-1}(\mathbb{R})^2$ is bounded.*

Proof (i) The boundedness of $\mathcal{T}$ is a direct consequence of the estimates $\gamma_p(\xi),\gamma_s(\xi)\sim i|\xi|$ as $|\xi|\to\infty$ and $|M(\xi)z|^2 \leq c\left(1+\xi^2\right)|z|^2$ for some constant $c > 0$ uniformly in $z\in\mathbb{R}^2$, $\xi\in\mathbb{R}$. The continuity of $\mathcal{T}$ with respect to ω follows from the uniform convergence

$$\|M(\xi;\omega_1)-M(\xi;\omega_2)\|/\left(1+\xi^2\right)\to 0, \quad \text{as } \omega_1\to\omega_2,$$

in $\xi \in \mathbb{R}$. The proof of the second assertion for $\varrho \neq 0$ can be carried out in the same way as that for the Helmholtz equation (see [25, Lemma 3.3(ii)]) by applying the commutator estimate of Theorem 2.2. $\square$

We set V_ϱ as the energy space for our variational problems, i.e. $V_\varrho = V^2_{h,\varrho}$ in the Dirichlet case and $V_\varrho = H^1_\varrho(S_h)^2$ in the impedance case; see Section 2.2 for the definition of $V_{h,\varrho}$. Introduce the scalar product

$$(u, v) := \int_{S_h} u \cdot \overline{v}\, dx,$$

and define the continuous sesquilinear forms $B_j : V_\varrho \times V_{-\varrho} \to \mathbb{C}$ $(j = 1, 2)$ by

$$\begin{aligned} B_1(u, v) &:= \int_{S_h} \left(\mathcal{E}(u, \overline{v}) - \omega^2 u \cdot \overline{v}\right) dx - \int_{\Gamma_h} \gamma_- \overline{v} \cdot \mathcal{T} \gamma_- u\, ds\,, \\ B_2(u, v) &:= B_1(u, v) - i\eta\, (u, v)\,. \end{aligned} \tag{2.23}$$

Now, the variational formulation of (DBVP) resp. (IBVP) can be stated as follows: find $u \in V_\varrho$ with some $-1 < \varrho < -1/2$ such that

$$B_1(u, v) \text{ (resp. } B_2(u, v)) = \int_{\Gamma_h} g \cdot \overline{v}\, ds, \qquad g := Tu^{in}|_{\Gamma_h} - \mathcal{T}(u^{in}|_{\Gamma_h}) \in H^{-1/2}_\varrho(\mathbb{R}), \tag{2.24}$$

for all $v \in V_{-\varrho}$. To determine the function g on the right-hand side of (2.24), we introduce the notation

$$\begin{aligned} &\alpha_p := k_p \sin\theta, \quad \beta_p := k_p \cos\theta, \quad t_p := \sqrt{k_s^2 - \alpha_p^2}, \quad \rho_p := \alpha_p^2 + \beta_p\, t_p\,, \\ &\alpha_s := k_s \sin\theta, \quad \beta_s := \sqrt{k_p^2 - \alpha_s^2}, \quad t_s := k_s \cos\theta, \quad \rho_s := \alpha_s^2 + \beta_s\, t_s\,. \end{aligned}$$

By the definitions of $\gamma_p(\xi)$ and $\gamma_s(\xi)$, we have

$$\gamma_p(\alpha_p) = \beta_p, \quad \gamma_s(\alpha_p) = t_p, \quad \gamma_s(\alpha_s) = t_s, \quad \gamma_p(\alpha_s) = \beta_s.$$

Using the relation $\mathcal{F} \exp(i\alpha x_1) = \sqrt{2\pi}\,\delta(\xi - \alpha)$ (the δ-function concentrated at $\xi_1 = \alpha$) and elementary calculations, we find

$$\begin{aligned} \mathcal{T}(u_p^{in}|_{\Gamma_h}) &= \frac{i}{k_p\, \rho_p} \begin{pmatrix} 2\omega^2 \alpha_p\, \beta_p - 2\mu\alpha_p\, \beta_p\, \rho_p \\ -2\mu\alpha_p^2\, \rho_p + \omega^2(\alpha_p^2 - \beta_p\, t_p) \end{pmatrix} \exp(i\alpha_p x_1 - \beta_p h), \\ \mathcal{T}(u_s^{in}|_{\Gamma_h}) &= \frac{i}{k_s\, \rho_s} \begin{pmatrix} 2\mu\alpha_s^2\, \rho_s - \omega^2(\alpha_s^2 - \beta_s\, t_s) \\ 2\omega^2 \alpha_s\, t_s - 2\mu\alpha_s\, \beta_s\, \rho_s \end{pmatrix} \exp(i\alpha_s x_1 - t_s h). \end{aligned} \tag{2.25}$$

On the other hand, by the definition of T given in (2.8), we get

$$\begin{aligned} Tu_p^{in}|_{\Gamma_h} &= \frac{i}{k_p} \begin{pmatrix} -2\mu\alpha_p\, \beta_p \\ \omega^2 - 2\mu\alpha_p^2 \end{pmatrix} \exp(i\alpha_p x_1 - \beta_p h), \\ Tu_s^{in}|_{\Gamma_h} &= \frac{i}{k_s} \begin{pmatrix} 2\mu\alpha_s^2 - \omega^2 \\ -2\mu\alpha_s\, t_s \end{pmatrix} \exp(i\alpha_s x_1 - t_s h). \end{aligned} \tag{2.26}$$

Combining (2.25) and (2.26) yields

$$\left.\begin{aligned} T u_p^{in} - \mathcal{T} u_p^{in} &= \frac{i2\omega^2\beta_p}{k_p\,\rho_p}\,(-\alpha_p\,, t_p)^\top \exp(i\alpha_p x_1 - i\beta_p h) =: g_p(x_1),\\ T u_s^{in} - \mathcal{T} u_s^{in} &= \frac{i2\omega^2 t_s}{k_s\,\rho_s}\,(-\beta_s\,, -\alpha_s)^\top \exp(i\alpha_s x_1 - i t_s h) =: g_s(x_1), \end{aligned}\right\} \text{ on } \Gamma_h.$$

One may check that g_p and g_s take the same forms as those arising from diffraction grating problems for incident plane pressure and shear waves (see [28,29]). We now conclude that the function g on the right-hand side of (2.24) can be represented as $g = c_1 g_p + c_2 g_s$, where the coefficients c_j are the weights attached to the incident plane pressure and shear waves; see (2.4).

Remark 2.4

(i) The right-hand side of (2.24) for the Dirichlet boundary value problem does not depend on the choice of the parameters $\tilde{\mu}, \tilde{\lambda}$. In the general case of $\tilde{\mu} + \tilde{\lambda} = \mu + \lambda$, the symbol matrix $M^{\tilde{\mu},\tilde{\lambda}}$ involved in the Dirichlet-to-Neumann map $\mathcal{T}^{\tilde{\mu},\tilde{\lambda}}$ can be written as (cf. (2.21))

$$M^{\tilde{\mu},\tilde{\lambda}} := \frac{i}{\xi^2 + \gamma_p\gamma_s}\begin{pmatrix} \omega^2\gamma_p & -\xi\omega^2 + \xi(a+\mu)(\xi^2 + \gamma_p\gamma_s)\\ \xi\omega^2 - \xi(\lambda + 2\mu - b)(\xi^2 + \gamma_p\gamma_s) & \omega^2\gamma_s \end{pmatrix}.$$

To get the corresponding variational formulation in the general case, one may only replace $\mathcal{E}$ and $\mathcal{T}$ on the left-hand side of (2.24) by $\mathcal{E}_{\tilde{\mu},\tilde{\lambda}}$ and $\mathcal{T}^{\tilde{\mu},\tilde{\lambda}}$, respectively. It can be readily checked that $(T^{\tilde{\mu},\tilde{\lambda}} - \mathcal{T}^{\tilde{\mu},\tilde{\lambda}})u^{in} = (T^{\mu,\lambda} - \mathcal{T}^{\mu,\lambda})u^{in}$ on Γ_h for all $\tilde{\mu}, \tilde{\lambda} \in \mathbb{R}$ such that $\tilde{\mu} + \tilde{\lambda} = \mu + \lambda$.

(ii) Suppose $u_1 \in V_{\varrho_1}, u_2 \in V_{\varrho_2}$ are the unique solutions to (2.24) corresponding to distinct numbers ϱ_1, ϱ_2 such that $-1 < \varrho_2 < \varrho_1 < -1/2$. Then, we have $u_1 \equiv u_2$, because $V_{-\varrho_2} \subset V_{-\varrho_1}$ and thus u_1 also satisfies (2.24) with $\varrho = \varrho_2$. This implies that the solution to (2.24) belongs to the space $\cap_{-1<\varrho<-1/2} V_\varrho$, provided the variational Equation (2.24) is uniquely solvable for each $\varrho \in (-1, -1/2)$.

The equivalence of (DBVP) resp. (IBVP) and the variational formulations in (2.24) can be established using the following lemma, which extends the results of [22, Lemma 1] for $\varrho = 0$ to the weighted case.

Lemma 2.5 *Let $|\varrho| < 1$.*

(i) *If (2.24) holds with $u_h^{sc} \in H_\varrho^{1/2}(\Gamma_h)^2$, then $u^{sc} \in H_\varrho^1(U_h \backslash \bar{U}_H)^2$ for every $H > h$.*

(ii) *Furthermore, we have $(\Delta^* + \omega^2)u^{sc} = 0$ in U_h, $\gamma_+ u^{sc} = u_h^{sc}$, and*

$$\int_{\Gamma_h} \bar{v}\, \cdot \mathcal{T}\gamma_+ u^{sc}\, dx + \omega^2 \int_{U_h} u \cdot \bar{v}\, d\mathrm{x} - \int_{U_h} \mathcal{E}(u, v)\, dx = 0, \quad \forall v \in C_0^\infty(D)^2\,.$$

As in the case of the Helmholtz equation [17] for $\varrho = 0$, assertion (ii) is a consequence of (i). We will prove Lemma 2.5(i) in Section 5 applying our commutator estimates. Using the arguments from [17,22], we deduce from Lemma 2.5, Remark 2.4(ii), and the well-posedness of (2.24) (see Theorem 3.1 below) the following lemma.

LEMMA 2.6 *If u is a solution of* (DBVP) (resp. (IBVP)), *then $u|_{S_h}$ satisfies the variational problem* (2.24). *Conversely, let w be the unique solution of* (2.24). *If we set $u = w$ in S_h and define $u = u^{in} + u^{sc}$ in U_h, where u^{sc} is given by right-hand side of* (2.12) *with $u_h^{sc} = \gamma_-(w - u^{in})$, then u is the unique solution of* (DBVP) (resp. (IBVP)).

3. Existence and uniqueness results in weighted spaces

From Lemma 2.3(ii), it is seen that the sesquilinear forms B_j ($j = 1, 2$) are well defined and continuous on $V_\varrho \times V_{-\varrho}$ for $|\varrho| < 1$. Denote by $\mathcal{B}_\varrho^{(j)} : V_\varrho \to V_{-\varrho}^*$ the continuous linear operator generated by B_j, where $V_{-\varrho}^*$ is the dual of $V_{-\varrho}$ with respect to the scalar product $(\cdot, \cdot)$ in $L^2(S_h)^2$. This enables us to rewrite the variational formulations (2.24) as the operator equations

$$\mathcal{B}_\varrho^{(j)}(u) = \mathcal{G} \quad \text{in} \quad V_{-\varrho}^*, \quad j = 1, 2, \quad \mathcal{G}(v) := \int_{\Gamma_h} g \cdot \overline{v} \, ds, \quad \forall\, v \in V_{-\varrho}\,, \tag{3.1}$$

for $\varrho \in (-1, -1/2)$. In this section, we investigate the unique solvability of problems (3.1) and thus of the boundary value problems (IBVP) and (DBVP). We shall follow the approach of Chandler-Wilde and Elschner [25] by using the results in the non-weighted case ($\varrho = 0$) and a perturbation argument based on commutator estimates. The main theorem of this paper is stated as follows.

THEOREM 3.1 *Under the assumptions* (2.2) *and* (2.3), *the operators $\mathcal{B}_\varrho^{(j)} : V_\varrho \to V_{-\varrho}^*$, $j = 1, 2$, are invertible for $|\varrho| < 1$. In particular, the boundary value problems (DBVP) and (IBVP) both admit a unique solution that belongs to $\cap_{-1<\varrho<-1/2} V_\varrho$.*

The proof of Theorem 3.1 will be carried out below in Sections 3.1 and 3.2.

3.1. *Proof for the Dirichlet boundary value problem*

We first recall the invertibility of $\mathcal{B}_\varrho^{(1)}$ in the non-weighted case when $\varrho = 0$. It was proved in [22] that $\mathcal{B}_0^{(1)}$ is invertible for any frequency of the incident wave, and for some constant $c_0 = c_0(\omega, \lambda, \mu, h, L) > 0$, there holds

$$||(\mathcal{B}_0^{(1)})^{-1}||_{V_0 \to V_0^*} \leq c_0. \tag{3.2}$$

This generalizes the results of Chandler-Wilde and Monk [17] to the case of elastic scattering. The proof of (3.2) is based on Rellich-type identities for both the Helmholtz and Navier equations and a perturbation argument for semi-Fredholm operators. However, in contrast to the Helmholtz case, the Dirichlet-to-Neumann map for the Navier equation does not have a definite real part, leading to essential difficulties in establishing explicit bounds on solutions as in [17].

To investigate the case when $\varrho \neq 0$, we introduce equivalent norms

$$\|u\|_{L^2_\varrho(S_h)^2} = \|(a^2 + x^2)^{\varrho/2} u\|_{L^2(S_h)^2}$$

with parameter $a > 0$ sufficiently large and modify the norm (2.9) in V_ϱ correspondingly. As in [25], we reformulate the variational form (2.23)–(2.24) as a perturbation of the problem in the non-weighted case. For $u \in V_\varrho,\ v \in V_{-\varrho}$, set

$$\varphi = (a^2 + x_1^2)^{\varrho/2} u \in V_0 , \quad \psi = (a^2 + x_1^2)^{-\varrho/2} v \in V_0 .$$

Then from (2.23), we obtain

$$B_1(u, v) = B_1(\varphi, \psi) + K(\varphi, \psi) , \tag{3.3}$$

where $K = K_1 + K_2$ with

$$\begin{aligned}
K_1(\varphi, \psi) &:= \int_{S_h} \Big[\mathcal{E}((a^2 + x_1^2)^{-\varrho/2}\varphi,\ (a^2 + x_1^2)^{\varrho/2}\psi) - \mathcal{E}(\varphi, \psi) \Big]\, dx, \\
K_2(\varphi, \psi) &:= \int_{\Gamma_h} \Big[\overline{\psi} \cdot \mathcal{T}\varphi - (a^2 + x_1^2)^{\varrho/2}\overline{\psi} \cdot \mathcal{T}(a^2 + x_1^2)^{-\varrho/2}\varphi \Big]\, ds \\
&= \int_{\Gamma_h} \overline{\psi} \cdot \mathcal{C}\varphi\, ds.
\end{aligned}$$

Recall that the operator $\mathcal{C}$ is the commutator defined in (2.22). By the definition of $\mathcal{E}(\cdot, \cdot)$ (see (2.17)), the sesquilinear form K_1 can be evaluated as

$$\begin{aligned}
|K_1(\varphi, \psi)| \le c(\lambda, \mu)\big\{ &||\varphi||_{L^2}\, ||\psi||_{L^2}\, |\mathcal{E}(\textstyle\sum_{j=1,2}(a^2 + x_1^2)^{-\varrho/2} e_j, \sum_{j=1,2}(a^2 + x_1^2)^{\varrho/2} e_j)| \\
&+ ||\varphi||_{L^2}\, (a^2 + x_1^2)^{\varrho/2}\, |\mathcal{E}(\textstyle\sum_{j=1,2}(a^2 + x_1^2)^{-\varrho/2} e_j, \psi)| \\
&+ ||\psi||_{L^2}\, (a^2 + x_1^2)^{-\varrho/2}\, |\mathcal{E}(\varphi, \textstyle\sum_{j=1,2}(a^2 + x_1^2)^{\varrho/2} e_j)| \big\}.
\end{aligned}$$

Here, $e_1 = (1, 0)^\top, e_2 = (1, 0)^\top$ denote the unit vectors in $\mathbb{R}^2$ and the norm $||\cdot||_{L^2(S_h)^2}$ is written as $||\cdot||_{L^2}$ for simplicity. Moreover, using the estimates

$$\begin{aligned}
&\sup_{S_h} \left|\nabla(a^2 + x_1^2)^{\varrho/2}\right| (a^2 + x_1^2)^{-\varrho/2} \le |\varrho|/2a, \\
&\sup_{S_h} \left|\nabla(a^2 + x_1^2)^{\varrho/2} \cdot \nabla(a^2 + x_1^2)^{\varrho/2}\right| \le (|\varrho|/2a)^2,
\end{aligned}$$

we obtain

$$\begin{aligned}
|K_1(\varphi, \psi)| \ &\le c(\lambda, \mu) \left\{ \left(\frac{|\varrho|}{2a}\right)^2 \|\varphi\|_{L^2}\|\psi\|_{L^2} + \left(\frac{|\varrho|}{2a}\right) \left(\|\nabla\varphi\|_{L^2}\|\psi\|_{L^2} + \|\varphi\|_{L^2}\|\nabla\psi\|_{L^2}\right) \right\} \\
&\le c(\lambda, \mu) \left\{ \frac{|\varrho|}{2a} \max\left(1,\ \frac{|\varrho|}{2a}\right) \|\varphi\|_{V_0}\|\psi\|_{V_0} \right\}.
\end{aligned} \tag{3.4}$$

Applying Theorem 2.2 to K_2, we get

$$|K_2(\varphi, \psi)| \le c(\omega, \lambda, \mu, \varrho)\, a^{-1/2}\, \|\varphi\|_{L^2(\Gamma_h)^2}\, \|\psi\|_{L^2(\Gamma_h)^2}. \tag{3.5}$$

The estimates (3.4) and (3.5) then imply that the norm of the operator $\mathcal{K}_0 : V_0 \to V_0^*$ generated by the form K tends to zero as $a \to \infty$. By (3.3) we have

$$\mathcal{B}_\varrho^{(1)} = (a^2 + x_1^2)^{-\varrho/2}(\mathcal{B}_0^{(1)} + \mathcal{K}_0)(a^2 + x_1^2)^{\varrho/2}.$$

Now it can be concluded that $\mathcal{B}_\varrho^{(1)} : V_\varrho \to V_{-\varrho}^*$ is invertible provided a is sufficiently large, with the norm of its inverse bounded by some positive constant $c = c(\omega, \lambda, \mu, \varrho, L, h)$. Hence, the variational formulation (2.24) always admits a unique solution for each $\varrho \in (-1, -1/2)$; note that $u^{in}|_{S_h} \in V_\varrho$ for such ϱ. By Remark 2.4 (ii) and Lemma 2.6, the solution to (2.24) is indeed the unique solution to (DBVP) belonging to the space $\cap_{-1<\varrho<-1/2} V_\varrho$. □

3.2. *Proof for the impedance boundary value problem*

The mathematical analysis in Section 3.1 applies to the impedance boundary value problem, provided the invertibility of $\mathcal{B}_0^{(2)}$ holds in the non-weighted space. The following lemma shows that the operator $(\mathcal{B}_0^{(2)})^{-1}$ exists and is bounded if we can establish an a priori bound for the solution $w \in V_0$ of the equation

$$\mathcal{B}_0^{(2)} w = \tilde{g}, \quad \tilde{g} \in V_0. \tag{3.6}$$

LEMMA 3.2 *Assume there exists some constant* $c = c(\omega, \lambda, \mu, \eta, h, L) > 0$ *such that*

$$||w||_{H^1(S_h)^2} \leq c\,||\tilde{g}||_{H^1(S_h)^2} \tag{3.7}$$

for all $w, \tilde{g} \in H^1(S_h)^2$ *satisfying the Equation* (3.6). *Then the operator* $\mathcal{B}_0^{(2)} : H^1(S_h)^2 \to (H^1(S_h)^2)^*$ *is invertible, with the norm of its inverse bounded by some constant depending on* $\omega, \lambda, \mu, \eta, h$ *and* L.

We sketch the proof of Lemma 3.2 based on the argument of [22] for elastic scattering from rigid rough surfaces due to an inhomogeneous source term. The proof extends the result of [17] in acoustic scattering to the case of the Navier equation under the impedance boundary condition.

Proof of Lemma 3.2 Using Korn's inequality (2.19), from (3.7), one can derive the a priori estimate

$$||w||_{H^1(S_h)^2} \leq c\,||\mathcal{B}_0^{(2)} w||_{(H^1(S_h)^2)^*} \quad \text{for all} \quad w \in H^1(S_h)^2, \tag{3.8}$$

at arbitrary frequency $\omega \in \mathbb{R}^+$. Indeed, (3.8) can be verified by arguing analogously to [22, Lemma 4] where the same a priori estimate for $\mathcal{B}_0^{(1)}$ was justified. The estimate (3.8) implies that $\mathcal{B}_0^{(2)} : H^1(S_h)^2 \to (H^1(S_h)^2)^*$ is a semi-Fredholm operator. Such an estimate combined with the invertibility of $\mathcal{B}_0^{(2)} : H^1(S_h)^2 \to (H^1(S_h)^2)^*$ for small frequencies leads to the existence and boundedness of $(\mathcal{B}_0^{(2)})^{-1}$ at any frequency; we refer the reader to [22, Sections 4 and 5] for the details using perturbation arguments for semi-Fredholm operators. Note that, under the assumption $\eta > 0$ for the impedance coefficient, the invertibility of $\mathcal{B}_0^{(2)}$ for small frequencies can be established in the same way as in [22, Section 4]. □

Now we turn to establishing the crucial a priori estimate (3.7) in the case $\varrho = 0$. Due to the positive impedance coefficient η on Γ, the mathematical argument below appears simpler compared to the Dirichlet case. It also provides a shorter proof of the well-posedness of acoustic scattering from impedance rough surfaces in the non-weighted Sobolev space (see [13, Chapter 3.4]) at arbitrary wavenumber. Our approach rests heavily on the well-posedness of the Dirichlet boundary value problem in the case $\varrho = 0$. To prove (3.7), we need the following lemma describing the positivity of the matrix $\operatorname{Re} M := (M + M^*)/2$ for large $|\xi|$.

LEMMA 3.3 *Let the matrix* M *be given as in* (2.21). *There exists a sufficiently large number* $\Lambda > 0$ *such that the matrix* $\operatorname{Re} M$ *is positive definite for all* $|\xi| > \Lambda$.

In the case of $\tilde{\mu} = 0$ and $\tilde{\lambda} = \lambda + \mu$, Lemma 3.3 was proved in [22, Section 4] by choosing $\Lambda = k_s$. Since the approach there applies to our present case of $\tilde{\mu} = \mu$, $\tilde{\lambda} = \lambda$, we omit the proof for the sake of brevity. A corresponding result for diffraction gratings can be found in [28, Lemma 2].

Assume $w \in H^1(S_h)^2$ is a solution to (3.6). In order to evaluate the non-definite part occurring in the DtN map, we follow [22] and extend w to S_H via (2.12) for some $H > h$. Without loss of generality, we assume $H = h + 1$. Note that this extension is a solution of the inhomogeneous Navier equation $(\Delta^* + \omega^2)w = \tilde{g}$ in S_H, with $\tilde{g} \equiv 0$ in $S_H \backslash S_h$, and it also satisfies the impedance boundary condition on Γ and the UPRC in U_H. Hence, for all $v \in H^1(S_H)^2$,

$$\int_{S_H} \left(\mathcal{E}(w, \overline{v}) - \omega^2 w \cdot \overline{v} \right) dx - i\eta \int_\Gamma w \cdot \overline{v}\, ds - \int_{\Gamma_H} \gamma_- \overline{v} \cdot \mathcal{T} \gamma_- w\, ds = \int_{S_H} \tilde{g} \cdot v\, dx. \quad (3.9)$$

Taking the imaginary part of (3.9) with $v = w$ and making use of the identity (see e.g. [22])

$$\mathrm{Im} \int_{\Gamma_H} \gamma_- \overline{w} \cdot \mathcal{T} \gamma_- w\, ds = 2\omega^2 \left(\int_{|\xi|<k_p} \gamma_p^2(\xi)\, |P(\xi)|^2\, d\xi + \int_{|\xi|<k_s} \gamma_s^2(\xi)\, |S(\xi)|^2\, d\xi \right) > 0 \quad (3.10)$$

with $P(\xi) := (-i/k_p^2)\mathcal{F}(\mathrm{div}\, u|_{\Gamma_H})$, $S(\xi) := (i/k_s^2)\mathcal{F}(\mathrm{curl}\, u|_{\Gamma_H})$, we find

$$||w||^2_{L^2(\Gamma)^2} \le c(\omega)\, \eta^{-1}\, ||\tilde{g}||_{L^2(S_H)^2}\, ||w||_{L^2(S_H)^2}. \quad (3.11)$$

To estimate the L^2 norm of w on the strip S_H, we study the auxiliary boundary value problem of finding $u \in V_0$ such that

$$(\Delta^* + \omega^2)\, u = \overline{w} \quad \text{in} \quad S_H\,, \quad u = 0 \quad \text{on} \quad \Gamma, \quad Tu = \mathcal{T}(\gamma_- u) \quad \text{on} \quad \Gamma_H. \quad (3.12)$$

The consideration of the above problem is motivated by [22,26] where the a priori estimate for solutions of the Helmholtz equation is verified in unbounded periodic and non-periodic structures. It follows from [22, Lemma 8] that problem (3.12) is well-posed, with the unique solution u satisfying the bound

$$||u||_{H^1(S_H)^2} \le c\; ||w||_{H^1(S_H)^2}, \quad c = c(\omega, \lambda, \mu, H, L) > 0. \quad (3.13)$$

Moreover, using (3.13), the L^2-norms of div u and curl u on the scattering surface can be estimated by

$$||\mathrm{div}\, u||_{L^2(\Gamma)^2} + ||\mathrm{curl}\, u||_{L^2(\Gamma)^2} \le c||w||^{1/2}_{L^2(S_H)^2} ||\partial_2 u||^{1/2}_{L^2(S_H)^2} \le c||w||^{1/2}_{L^2(S_H)^2} ||w||^{1/2}_{H^1(S_H)^2}, \quad (3.14)$$

where the first inequality follows from [22, Lemma 6] through Rellich identities for the Helmholtz equation under the assumption (2.3). Since $u = 0$ on Γ, it is easy to check that

$$n_2\, |\partial_{\mathbf{n}} u|^2 = n_2\, |\nabla u|^2 = n_2(|\mathrm{curl}\, u|^2 + |\mathrm{div}\, u|^2) \quad \text{on} \quad \Gamma.$$

Therefore, by (3.14), the L^2-norm of $\partial_{\mathbf{n}} u$ on Γ and thus that of Tu can be also bounded by the left-hand side of (3.14), i.e.

$$||Tu||_{L^2(\Gamma)^2} \le c\, ||w||^{1/2}_{L^2(S_H)^2}\, ||w||^{1/2}_{H^1(S_H)^2}\,. \quad (3.15)$$

Using integration by parts and again the fact that $u = 0$ on Γ, we obtain

$$\begin{aligned}\|w\|^2_{L^2(S_H)^2} &= \int_{S_H} w\cdot\overline{w}\,dx = \int_{S_H} w\cdot(\Delta^* u + \omega^2 u)\,dx \\ &= \int_{S_H} (\Delta^* w + \omega^2 w)\cdot u\,dx + \int_{S_H} (\Delta^* u\cdot w - \Delta^* w\cdot u)\,dx \\ &= \int_{S_H} \tilde{g}\cdot u\,dx + \int_{\Gamma\cup\Gamma_H} (Tu\cdot w - Tw\cdot u)\;ds \\ &= \int_{S_H} \tilde{g}\cdot u\,dx + \int_{\Gamma} Tu\cdot w\,ds + \int_{\Gamma_H} (\mathcal{T}u\cdot w - \mathcal{T}w\cdot u)\;ds.\end{aligned}$$

It follows from the symmetry $M(-\xi) = M(\xi)^\top$ and the Plancherel identity that

$$\begin{aligned}\int_{\Gamma_H} \mathcal{T}u\cdot w\,ds &= \int_{\mathbb{R}} M(\xi)\,\hat{u}_H(\xi)\cdot\hat{w}_H(-\xi)\,d\xi \\ &= \int_{\mathbb{R}} \hat{u}_H(-\xi)\cdot M(\xi)\,\hat{w}_H(\xi)\,d\xi = \int_{\Gamma_H} \mathcal{T}w\cdot u\,ds,\end{aligned}$$

where $w_H = w|_{\Gamma_H}$. Hence, using (3.11), (3.13) and (3.15),

$$\begin{aligned}\|w\|^2_{L^2(S_H)^2} &= \int_{S_H} \tilde{g}\cdot u\,dx + \int_{\Gamma} Tu\cdot w\,ds \\ &\le c\,\left(||\tilde{g}||_{L^2(S_H)^2}\,||u||_{L^2(S_H)^2} + ||Tu||_{L^2(\Gamma)^2}\,||w||_{L^2(\Gamma)^2}\right) \\ &\le c\,\left(||\tilde{g}||_{L^2(S_H)^2}\,||w||_{H^1(S_H)^2} + ||w||_{L^2(S_H)^2}\,||\tilde{g}||^{1/2}_{L^2(S_H)^2}\,||w||^{1/2}_{H^1(S_H)^2}\right),\end{aligned}$$

for some constant $c = c(\omega, H, L, \eta, \mu) > 0$. Together with Young's inequality and the relation $\tilde{g} = 0$ in $S_H\backslash S_h$, this leads to the following estimate of the L^2-norm of w on S_H,

$$\|w\|^2_{L^2(S_H)^2} \le c\,||\tilde{g}||_{L^2(S_h)^2}\,||w||_{H^1(S_H)^2}. \tag{3.16}$$

Taking the real part of (3.9) with $v = w$ and using (2.20), we get

$$\begin{aligned}&\int_{S_H} \mathcal{E}(w,\overline{w})dx - \mathrm{Re}\int_{|\xi|>\Lambda} M(\xi)\,\hat{w}_H(\xi)\cdot\overline{\hat{w}}_H(\xi)\,d\xi \\ &\quad = -\mathrm{Re}\int_{S_H} \tilde{g}\cdot\overline{w}\,dx + \mathrm{Re}\int_{|\xi|\le\Lambda} M(\xi)\,\hat{w}_H(\xi)\cdot\overline{\hat{w}}_H(\xi)\,d\xi + \omega^2\int_{S_H} |w|^2 dx\,,\end{aligned} \tag{3.17}$$

where $\Lambda > 0$ is taken as in Lemma 3.3 so that the second term on the left-hand side of (3.17) is positive. The second term on the right-hand side of (3.17), which is non-definite, can be estimated by (see [22, formula (5.40)])

$$\mathrm{Re}\int_{|\xi|\le\Lambda} M(\xi)\hat{w}_H(\xi)\cdot\overline{\hat{w}}_H(\xi)\,d\xi \le c\,||\tilde{g}||_{H^1(S_h)^2}\,(||\tilde{g}||_{H^1(S_h)^2}+||\partial_2 w||_{L^2(S_H)^2})\,, \tag{3.18}$$

for some constant $c = c(\omega, \lambda, \mu, h, L, \Lambda) > 0$. Adding up (3.17) and (3.16) and using the inequalities (2.18), (2.19) and (3.18), we arrive at

$$\begin{aligned}||w||^2_{H^1(S_h)^2} &\le c\,(||\tilde{g}||^2_{H^1(S_h)^2} + ||\tilde{g}||_{H^1(S_h)^2}\,||w||_{H^1(S_h)^2} + ||w||^2_{L^2(S_H)^2}) \\ &\le c\,(||\tilde{g}||^2_{H^1(S_h)^2} + ||\tilde{g}||_{H^1(S_h)^2}\,||w||_{H^1(S_h)^2})\end{aligned} \tag{3.19}$$

where the last step follows again from (3.16). Finally, recalling that $H = h+1$ and applying Young's inequality, we obtain

$$||w||_{H^1(S_h)^2} \leq ||w||_{H^1(S_H)^2} \leq c\,||\tilde{g}||_{H^1(S_h)^2}, \quad c = c(\omega, \lambda, \mu, h, L, \eta) > 0.$$

This proves the estimate (3.7).

Having established the a priori estimate for solutions to (3.6), we can verify Theorem 3.1 for the impedance boundary value problem in the same way as that for (DBVP). We omit the details. The proof of Theorem 3.1 is thus complete. □

Remark 3.4 In proving Theorem 3.1 we have used the identity (3.10) and the inequality (3.18), which were justified in [22] for the Dirichlet problem when the parameters $\tilde{\mu}, \tilde{\lambda}$ are taken as $\tilde{\mu} = 0, \tilde{\lambda} = \lambda + \mu$. However, (3.10) and (3.18) remain valid in the general case of $\tilde{\mu}, \tilde{\lambda} \in \mathbb{R}$ such that $\tilde{\mu} + \tilde{\lambda} = \mu + \lambda$.

4. Applications

4.1. *Elastic scattering by diffraction gratings*

As an application of Theorem 3.1, we prove the quasiperiodicity of solutions to (DBVP) and (IBVP) for diffraction gratings (periodic structures) whenever the incident wave is quasiperiodic. For simplicity, we assume the scattering surface Γ is 2π-periodic in x_1, that is, the Lipschitz function f given in (2.2) satisfies $f(x_1 + 2\pi) = f(x_1)$ for all $x_1 \in \mathbb{R}$. Recall that u is called quasiperiodic in D with phase shift α (or α-quasiperiodic) if the function $u(x)\exp(i\alpha x_1)$ is 2π-periodic in x_1, or equivalently

$$u(x_1 + 2\pi, x_2) = \exp(i2\pi\alpha)u(x_1, x_2), \quad x \in D.$$

Obviously, the incident pressure and shear waves u_p^{in}, u_s^{in} are α-quasiperiodic with $\alpha = k_p \sin\theta$, $\alpha = k_s \sin\theta$, respectively.

COROLLARY 4.1 *Suppose the grating profile function f is 2π-periodic in x_1 and the incident wave $\tilde{u}^{in}$ is α-quasiperiodic in D. Then, the unique solution to (DBVP) or (IBVP) is also α-quasiperiodic. Moreover, the scattered field $u^{sc} = u - \tilde{u}^{in}$ satisfies the following outgoing Rayleigh expansion*

$$u^{sc}(x) = \sum_{n\in\mathbb{Z}} \left\{ A_{p,n} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} e^{i\alpha_n x_1 + i\beta_n x_2} + A_{s,n} \begin{pmatrix} \gamma_n \\ -\alpha_n \end{pmatrix} e^{i\alpha_n x_1 + i\gamma_n x_2} \right\} \tag{4.1}$$

for $x_2 > f^+$, where $A_{p,n},\ A_{s,n} \in \mathbb{C}$ are the Rayleigh coefficients, $\alpha_n := \alpha + n$ and

$$\beta_n := \begin{cases} \sqrt{k_p^2 - \alpha_n^2} & \text{if} \quad |\alpha_n| \leq k_p\,, \\ i\,\sqrt{\alpha_n^2 - k_p^2} & \text{if} \quad |\alpha_n| > k_p, \end{cases} \qquad \gamma_n = \begin{cases} \sqrt{k_s^2 - \alpha_n^2} & \text{if} \quad |\alpha_n| \leq k_s\,, \\ i\,\sqrt{\alpha_n^2 - k_s^2} & \text{if} \quad |\alpha_n| > k_s. \end{cases}$$

Proof Assume u is the unique solution to (DBVP) or (IBVP). Then, one can check that the function $w(x) = \exp(-i2\pi\alpha)u(x_1 + 2\pi, x_2)$ is also a solution, using the periodicity of Γ and the quasiperiodicity of the incident wave. By the uniqueness shown in Theorem 3.1, this implies the identity

$$\exp(-i2\pi\alpha)u(x_1 + 2\pi, x_2) = u(x) \quad \text{in } D,$$

that is, u is quasiperiodic with the same phase shift as the incident wave. The equivalence of the UPRC (2.14) for quasiperiodic solutions to the Rayleigh expansion (4.1) can be found in [22, Remark 1]. □

Corollary 4.1 shows that a solution of the form $u = \tilde{u}^{in} + u^{sc}$ satisfying the Dirichlet or impedance boundary condition on Γ, where u^{sc} is α-quasiperiodic and admits the Rayleigh expansion (4.1), is the unique solution to (DBVP) or (IBVP) for diffraction gratings.

Remark 4.2 In the case of general elastic plane waves of the form (2.4), the unique solution of (DBVP) or (IBVP) for diffraction gratings belongs to the sum of a $k_p \sin \theta$- and a $k_s \sin\theta$-quasiperiodic Sobolev space by linear superposition. The diffraction of other non-quasiperiodic incident waves, e.g. a point source wave generated by the free space (non-quasiperiodic) Green's tensor to the Navier equation, can be treated as a special case of the scattering by rough surfaces (see Corollary 4.3 below).

4.2. *Scattering of elastic point source waves*

As an immediate consequence of the solvability results in weighted Sobolev spaces, we obtain well-posedness of the scattering of elastic point source waves (spherical waves) from rough surfaces. For $y = (y_1, y_2) \in \mathbb{R}^2$ with $y_2 > f^+$ and some polarization vector $\mathbf{a} \in \mathbb{C}^2$, the incident elastic point source wave $G^{in}_{\mathbf{a}}(x, y)$ is defined as $G^{in}_{\mathbf{a}}(x, y) = G(x, y)\mathbf{a}$, $x \neq y$, where $G(x, y)$ is the free-space elastic Green's tensor given by (see e.g. [36])

$$G(x, y) = \frac{i}{4\mu} H_0^{(1)}(k_s|x-y|)\, I + \frac{i}{4\omega^2} \operatorname{grad}_x \operatorname{grad}_x^\top \Big[H_0^{(1)}(k_s|x-y|) - H_0^{(1)}(k_p|x-y|) \Big].$$

Here $H_0^{(1)}(t)$ denotes the first kind Hankel function of order zero. Each column of $G(x, y)$ satisfies the Kupradze radiation condition as $|x| \to \infty$. The asymptotic behavior of the Hankel function for large arguments implies that

$$G^{in}_{\mathbf{a}}(x, y),\ \nabla_x G^{in}_{\mathbf{a}}(x, y) \sim \mathcal{O}(|x|^{-1/2}) \quad \text{as } |x| \to \infty.$$

Therefore, the incident wave satisfies $G^{in}_{\mathbf{a}}(x, y) \in H^1_\varrho(S_h)^2$ for every $\varrho < 0$ and $f^+ < h < y_2$. Note that $G^{in}_{\mathbf{a}}(x, y) \notin H^1_\varrho(S_h)^2$ for $h > y_2$, since it has a logarithmic singularity at the point source $x = y$. By Lemma 2.6 and the proof of Theorem 3.1, we have

COROLLARY 4.3 *Given an incident elastic point source wave $G^{in}_{\mathbf{a}}(x, y)$ with $y_2 > f^+$, there exists a unique solution $u = G^{in}_{\mathbf{a}}(\cdot, y) + u^{sc}$ to the boundary value problem (DBVP) or (IBVP), where u^{sc} lies in the intersection of weighted Sobolev spaces $\bigcap_{-1<\varrho<0} V_\varrho(S_h)$ for any $h > f^+$.*

5. Commutator estimates

This section is devoted to the proof of Theorem 2.2 and Lemma 2.5 (i). Introduce the parameter $a > 0$ and consider the pseudodifferential operator $\mathcal{T}_a$ on $\mathbb{R}$, with symbol $M_a(\xi)$:

$$\mathcal{T}_a\, v(t) = \mathcal{F}^{-1}\, M_a(\xi)\, \mathcal{F} v\,(\xi), \qquad M_a(\xi) := M(\xi/a),$$

where the matrix $M = M^{\mu,\lambda}$ is given in (2.21). Set $\rho^{(a)}(|\xi|) = |\xi|^2 + \gamma_p^{(a)}(|\xi|)\,\gamma_s^{(a)}(|\xi|)$, with

$$\gamma_p^{(a)}(|\xi|) := a\gamma_p(\xi/a) = \sqrt{k_p^2 a^2 - |\xi|^2}, \quad \gamma_s^{(a)}(|\xi|) := a\gamma_s(\xi/a) = \sqrt{k_s^2 a^2 - |\xi|^2}.$$

Then, the matrix $M_a(\xi)$ can be rewritten as:

$$M_a(\xi) = ia \begin{pmatrix} \omega^2\gamma_p^{(a)}(\xi)/\rho^{(a)}(\xi) & -\omega^2\xi/\rho^{(a)}(\xi) + 2\xi\mu/a^2 \\ \omega^2\xi/\rho^{(a)}(\xi) - 2\xi\mu/a^2 & \omega^2\gamma_s^{(a)}(\xi)/\rho^{(a)}(\xi) \end{pmatrix}.$$

Consider the commutator

$$\mathcal{C}_a := \mathcal{T}_a - \left(1 + x_1^2\right)^{\varrho/2} \mathcal{T}_a \left(1 + x_1^2\right)^{-\varrho/2}, \quad a > 0. \tag{5.1}$$

To reduce the norm estimate of Theorem 2.2 for the commutator $\mathcal{C}$ to a corresponding estimate for $\mathcal{C}_a$, we will make use of the following lemma, which follows immediately from a standard scaling argument (see [25, p.2573]).

Lemma 5.1 *For $a > 0$, the norm of the commutator $\mathcal{C}$ on $L^2(\mathbb{R}^2)^2$ is bounded by $Ca^{-1/2}$ if and only if this is true for the commutator $\mathcal{C}_a$.*

To estimate the norm of $\mathcal{C}_a$ on $L^2(\mathbb{R}^2)^2$, we need to study the commutator corresponding to each entry of M_a on $L^2(\mathbb{R})$. Introduce the symbols

$$m_a^{(0)} = \xi\mu/a,\ m_a^{(1)} = a\,\gamma_p^{(a)}(\xi)/\rho^{(a)}(\xi),\ m_a^{(2)} = a\,\gamma_s^{(a)}(\xi)/\rho^{(a)}(\xi),\ m_a^{(3)} = a\,\xi/\rho^{(a)}(\xi), \tag{5.2}$$

and define analogous commutators $\mathcal{C}_a^{(j)}$ $(j = 0, 1, 2, 3)$ of $\mathcal{C}_a$ with M_a replaced by $m_a^{(j)}$. Obviously, the symbol of the pseudodifferential operator $\mathcal{C}_a^{(0)}$ is smooth, whereas those of $\mathcal{C}_a^{(j)}$ $(j = 1, 2, 3)$ are only continuous functions. In the following two lemmas, we collect some commutator estimates for pseudodifferential operators with smooth and non-smooth scalar symbols established by Chandler-Wilde and Elschner [25].

Lemma 5.2 *Consider the scalar symbol $m_a(\xi) \in C^1(\mathbb{R})$ with parameter $a > 0$ and define the commutators*

$$\tilde{\mathcal{C}}_a := \mathcal{M}_a - \left(1 + x_1^2\right)^{\varrho/2} \mathcal{M}_a \left(1 + x_1^2\right)^{-\varrho/2}, \quad \mathcal{M}_a := \mathcal{F}^{-1}\, m_a\, \mathcal{F} \tag{5.3}$$

for $|\varrho| \le 1$.

(i) *Assume there exist positive constants C_0 and C_1 such that*

$$|m_a(\xi)| \le C_0 \left(1 + \xi^2\right)^{1/2}, \quad |m'_a(\xi)| \le C_1\, a^{-1/2} \quad \textit{on} \quad \mathbb{R}. \tag{5.4}$$

Then $\tilde{\mathcal{C}}_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are bounded operators with norm less than $a^{-1/2}\, C(\varrho)$ for some constant $C(\varrho) > 0$ depending only on ϱ.

(ii) *Assume there exist positive constants C_0 and C_1 such that, for $a = 1$,*

$$|m_1(\xi)| \le C_0\,, \quad |m'_1(\xi)| \le C_1 \left(1 + \xi^2\right)^{-1/2} \quad \textit{on} \quad \mathbb{R}. \tag{5.5}$$

Then the pseudodifferential operator $\mathcal{M}_1 : L^2_\varrho(\mathbb{R}) \to L^2_\varrho(\mathbb{R})$ and the commutator $\tilde{\mathcal{C}}_1 : L^2(\mathbb{R}) \to H^1(\mathbb{R})$ both can be bounded by some constant $C(\varrho) > 0$.

The results of Lemma 5.2, which are shown in [25, Remark 6.6(ii),(iii)], can be verified by using standard estimates for pseudodifferential operators; see also the proof of [25, Theorem 6.2(i)]. More general results on pseudodifferential operators with smooth symbols in weighted Sobolev spaces can be found in [30,31]. We also refer the reader to the monograph [37] by Eskin concerning the theory of smooth pseudodifferential operators, including their applications to boundary value problems for elliptic equations in a half space. The following lemma from [25, Section 6] presents norm estimates for pseudodifferential operators with non-smooth (continuous) symbols.

LEMMA 5.3 *Assume $ka > 1$ and $|\varrho| < 1$.*

(i) *The commutator $\tilde{\mathcal{C}}_a$ defined in (5.3) with $m_a(\xi) = a^{-1}\sqrt{k^2a^2 - \xi^2}$ has norm less than $C(\varrho)\sqrt{k/a}$ on $L^2(\mathbb{R})$.*

(ii) *Suppose $a = 1$ and $m_1(\xi) = \exp(i\,(x_2-h)\sqrt{k^2 - \xi^2})$, where $x_2 \in (h, H)$ for some $H > h$. Then the operators $\mathcal{M}_1 : L^2_\varrho(\mathbb{R}) \to L^2_\varrho(\mathbb{R})$ and $\tilde{\mathcal{C}}_1 : L^2(\mathbb{R}) \to H^1(\mathbb{R})$ are bounded by some constant $C(\varrho, \omega, \lambda, \mu, H - h) > 0$ uniformly in $x_2 \in (h, H)$.*

The main idea in the proof of Lemma 5.3(i) in [25] is the use of cut-off functions vanishing in a neighborhood of the singularities $\xi = \pm ka$, splitting the square-root symbol into a sum of a compactly supported non-smooth symbol and a C^∞-smooth symbol. We do believe that such an approach applies to our commutator estimates in the elastic case as well, with only an additional complexity arising from the four singularities $\xi = \pm k_pa, \pm k_sa$ of the symbol matrix M_a. However, in the following, we prefer to verify Theorem 2.2 (via Lemma 5.1) and Lemma 2.5(i) in an alternative way by reducing the proofs to the estimates of Lemmas 5.2 and 5.3 using an appropriate decomposition of the symbols in (5.2).

5.1. *Proof of Theorem 2.2*

Applying Lemma 5.2(i) to the commutator $\mathcal{C}_a^{(0)}$, it follows that

$$||\mathcal{C}_a^{(0)}||_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \le a^{-1/2}C(\varrho) \quad \text{for} \quad a > 1.$$

To verify the same estimate for $\mathcal{C}_a^{(j)}$, $j = 1, 2, 3$, we introduce the auxiliary symbols

$$\begin{aligned}\tilde{m}_a^{(j)}(\xi) &:= a^{-1}\sqrt{k_p^2a^2 - \xi^2}\,C_p^{(j)} + a^{-1}\sqrt{k_s^2a^2 - \xi^2}\,C_s^{(j)} \\ &= a^{-1}C_p^{(j)}\,\gamma_p^{(a)}(\xi) + a^{-1}C_s^{(j)}\,\gamma_s^{(a)}(\xi),\end{aligned}$$

with $C_p^{(j)}, C_s^{(j)} \in \mathbb{R}$ $(j = 1, 2, 3)$ to be determined later. Obviously,

$$(\tilde{m}_a^{(j)})'(\xi) = a^{-1}C_p^{(j)}\,(\gamma_p^{(a)})'(\xi) + a^{-1}C_s^{(j)}\,(\gamma_s^{(a)})'(\xi), \quad j = 1, 2, 3. \tag{5.6}$$

where $(\gamma_p^{(a)})'(\xi) = -\xi/\sqrt{k_p^2a^2 - \xi^2}$ is singular at $\xi = \pm k_p\,a$, while $(\gamma_s^{(a)})'(\xi) = -\xi/\sqrt{k_s^2a^2 - \xi^2}$ is singular at $\xi = \pm k_s\,a$. These singular points coincide with those for $m_a^{(j)}$.

For $j = 1, 2$, we select $C_p^{(j)}, C_s^{(j)}$ such that $m_a^{(j)}(\xi) - \tilde{m}_a^{(j)}(\xi)$ are continuously differentiable functions in $\xi \in \mathbb{R}$. In the case $j = 1$, a simple calculation shows

$$\begin{aligned}(m_a^{(1)})'(\xi) &= a(\gamma_p^{(a)})'(\xi)/\rho^{(a)}(\xi) - a\gamma_p^{(a)}(\rho^{(a)})'(\xi)/[\rho^{(a)}(\xi)]^2 \\ &= (\gamma_p^{(a)})'(\xi)\,\frac{a}{\rho^{(a)}(\xi)}\left[1 - \frac{\gamma_p^{(a)}(\xi)\gamma_s^{(a)}(\xi)}{\rho^{(a)}(\xi)}\right] - (\gamma_s^{(a)})'(\xi)\,\frac{a[\gamma_p^{(a)}(\xi)]^2}{[\rho^{(a)}(\xi)]^2} \\ &\quad - 2a\xi\,\gamma_p^{(a)}(\xi)/[\rho^{(a)}(\xi)]^2.\end{aligned} \tag{5.7}$$

This suggests that $m_a^{(1)}(\xi) - \tilde{m}_a^{(1)}(\xi) \in C^1(\mathbb{R})$ if we take (cf. (5.6) and (5.7))

$$\begin{aligned}C_p^{(1)} &= \lim_{|\xi|\to k_p a} \frac{a^2}{\rho^{(a)}(\xi)}\left[1 - \frac{\gamma_p^{(a)}(\xi)\gamma_s^{(a)}(\xi)}{\rho^{(a)}(\xi)}\right] = \frac{1}{k_p^2}\,, \\ C_s^{(1)} &= -\lim_{|\xi|\to k_s a} \frac{a^2[\gamma_p^{(a)}(\xi)]^2}{[\rho^{(a)}(\xi)]^2} = \frac{k_s^2 - k_p^2}{k_s^4}\,.\end{aligned}$$

Moreover, with such a choice the estimates in (5.4) apply to the difference

$$\begin{aligned}m_a(\xi) := m_a^{(1)}(\xi) - \tilde{m}_a^{(1)}(\xi) &= a^{-1}\gamma_p^{(a)}(\xi)[a^2/\rho^{(a)}(\xi) - 1/k_p^2] - a^{-1}\gamma_s^{(a)}(\xi)(k_s^2 - k_p^2)/k_s^4 \\ &= \gamma_p(\zeta)[1/\rho(\zeta) - 1/k_p^2] - \gamma_s(\zeta)(k_s^2 - k_p^2)/k_s^4 \\ &=: m(\zeta),\end{aligned}$$

where $\zeta = \xi/a$, $\rho(\zeta) := \zeta^2 + \gamma_p(\zeta)\gamma_s(\zeta)$ and $m(\zeta) \in C^1(\mathbb{R})$. In fact, the first estimate in (5.4) simply follows from the uniform boundedness

$$m_a(\xi) = m(\zeta) \le C_0\left(1 + \zeta^2\right)^{1/2} \le C_0\left(1 + \xi^2\right)^{1/2}, \quad \forall\,\xi \in \mathbb{R}.$$

To prove the second inequality in (5.4), we observe that, for $|\zeta| \ne k_p, k_s$,

$$m'(\zeta) = \gamma_p'(\zeta)[1/\rho(\zeta) - 1/k_p^2] - \gamma_s'(\zeta)(k_s^2 - k_p^2)/k_s^4 + \gamma_p(\zeta)\rho'(\zeta)/\rho^2(\zeta).$$

By virtue of the asymptotic behavior

$$\rho'(\zeta) \sim -\frac{(k_p^2 - k_s^2)^2\,|\zeta|^3}{(|\zeta|^2 - k_p^2)^{3/2}(|\zeta|^2 - k_s^2)^{3/2}} \quad \text{as} \quad |\zeta| \to \infty, \tag{5.8}$$

and the uniform boundedness

$$k_p^2 \le |\rho(\zeta)| \le k_s^2, \quad \forall\,\zeta \in \mathbb{R},$$

we get

$$m_a'(\xi) = m'(\zeta)/a \le C_1/a \le C_1 a^{-1/2}, \quad a > 1.$$

By Lemma 5.2(i), the commutator (5.3) corresponding to the symbol $m_a := m_a^{(1)}(\xi) - \tilde{m}_a^{(1)}(\xi)$ has norm less than $C(\varrho)a^{-1/2}$ over $L^2(\mathbb{R})$. On the other hand, applying Lemma 5.3(i) we arrive at the same bound for the commutator associated with the symbol $\tilde{m}_a^{(1)}(\xi)$ when $k_s a > 1$; note that the constants $C_p^{(1)}$ and $C_s^{(1)}$ are independent of a. Therefore,

$$||\mathcal{C}_a^{(1)}||_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \le C(\varrho)a^{-1/2}, \quad \text{for all} \quad a > \max\{1, 1/k_s\}.$$

Analogously, taking $C_p^{(2)} = (k_p^2 - k_s^2)/k_p^4$ and $C_s^{(2)} = 1/k_s^2$ yields the same bound for $\mathcal{C}_a^{(2)}$.

In the case $j = 3$, we have $m_a^{(3)}(\xi) = \zeta/\rho(\zeta)$, $\zeta = \xi/a$, and for $|\xi| \neq k_p a, k_s a$,

$$\begin{aligned}(m_a^{(3)})'(\xi) &= a[1/\rho^{(a)}(\xi) - \xi(\rho^{(a)})'(\xi)/[\rho^{(a)}(\xi)]^2] = [1/\rho(\zeta) - \zeta\rho'(\zeta)/\rho^2(\zeta)]/a \\ &= -(\gamma_p^{(a)})'(\xi)\frac{a\,\xi\,\gamma_s^{(a)}(\xi)}{[\rho^{(a)}(\xi)]^2} - (\gamma_s^{(a)})'(\xi)\frac{a\,\xi\,\gamma_p^{(a)}(\xi)}{[\rho^{(a)}(\xi)]^2} + \frac{a}{\rho^{(a)}(\xi)}\left[1 - \frac{2\xi^2}{\rho^{(a)}(\xi)}\right].\end{aligned}$$

Define a function $\chi(\xi) \in C^1(\mathbb{R})$ such that $\chi(\xi) = 1$ for $\xi > 1$, $\chi(\xi) = -1$ for $\xi < -1$. Consider the symbol $m_a^{(3)}(\xi) - \tilde{m}_a^{(3)}(\xi)\chi(\xi)$, where the coefficients of $\tilde{m}_a^{(3)}(\xi)$ in (5.6) are taken as $C_p^{(3)} = -\sqrt{k_s^2 - k_p^2}/k_p$ and $C_s^{(3)} = -\sqrt{k_p^2 - k_s^2}/k_s$, so that this symbol is continuously differentiable for all $\xi \in \mathbb{R}$. Again using (5.8), it follows that the symbol can be also estimated as in (5.4). Employing the same argument as for $\mathcal{C}_a^{(1)}$ implies that the norm of $\mathcal{C}_a^{(3)} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is bounded by $C(\varrho)a^{-1/2}$ for all $a > \{1, 1/k_s\}$.

Now, it can be concluded that the commutator $\mathcal{C}_a : L^2(\mathbb{R}^2)^2 \to L^2(\mathbb{R}^2)^2$ given by (5.1) can be bounded by $C(\varrho)a^{-1/2}$ for all $a > \{1, 1/k_s\}$, since this is true for the commutators $\mathcal{C}_a^{(j)}$ $(j = 0, 1, 2, 3)$ that correspond to the entries of M_a. Recalling Lemma 5.1, we finish the proof of Theorem 2.2. □

5.2. *Proof of Lemma 2.5(i)*

Set $v = u^{sc}$ in D and $v_h = v|_{\Gamma_h}$. It follows from (2.12) that

$$\begin{aligned}&v(x) = \mathcal{F}^{-1}_{\xi\to x_1} N_0(\xi, x_2)\,\mathcal{F}_{x_1\to\xi}\, v_h =: \mathcal{N}_0\, v_h, \quad (x_1, x_2) \in U_h,\\ &N_0(\xi, x_2) := \exp(i\gamma_p(\xi)(x_2 - h))M_p(\xi) + \exp(i\gamma_s(\xi)(x_2 - h))M_s(\xi)\,,\end{aligned}$$

with the matrixes $M_p(\xi), M_s(\xi) \in \mathbb{C}^{2\times 2}$ given in (2.13). Introduce the differential operator

$$\tilde{T}v := \begin{pmatrix}\mu\partial_2 & \mu\partial_1\\ \lambda\partial_1 & (\lambda+2\mu)\partial_2\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix},$$

which coincides with the stress operator $T = T^{\mu,\lambda}$ on Γ_b for any $b > h$. Then, for some constant $C(\omega, \lambda, \mu) > 1$, there holds the inequality

$$C(|\partial_1 v|^2 + |\tilde{T}v|^2) \geq C\,|\partial_1 v|^2 + |\tilde{T}v|^2 \geq \frac{1}{2}\big(|\partial_1 v|^2 + |\partial_2 v|^2\big) \tag{5.9}$$

on D. The differential operators ∂_1 and $\tilde{T}$ acting on v can be expressed as:

$$\begin{aligned}\partial_1 v(x_1, x_2) &= \mathcal{F}^{-1}[i\xi\, N_0(\xi, x_2)]\,\mathcal{F}\, v_h =: \mathcal{N}_1 v_h,\\ \tilde{T}v(x_1, x_2) &= \mathcal{F}^{-1}[M(\xi)\, N_0(\xi, x_2)]\,\mathcal{F}\, v_h =: \mathcal{N}_2 v_h.\end{aligned} \tag{5.10}$$

Now assume that $v_h \in C_0^\infty(\Gamma_h)$. We have to prove the estimate

$$||v||_{H^1_\varrho(U_h\backslash\overline{U}_H)^2} \leq C(\varrho, H, h)\,||v_h||_{H^{1/2}_\varrho(\Gamma_h)^2}, \qquad |\varrho| < 1,\ H > h.$$

Employing the equivalent norm (2.10) and recalling (5.9), we only need to verify that

$$\int_h^H \int_{\mathbb{R}} \left(1 + x_1^2\right)^\varrho \left(|\mathcal{N}_0 v_h|^2 + |\mathcal{N}_1 v_h|^2 + |\mathcal{N}_2 v_h|^2\right) dx_1 dx_2 \leq C\,||\left(1 + x_1^2\right)^{\varrho/2}\ v_h||^2_{H^{1/2}(\Gamma_h)^2}. \tag{5.11}$$

In the following lemma, we will first prove (5.11) in the case $\varrho = 0$ and then reduce the proof in the weighted case to norm estimates for the operator $\mathcal{N}_0$ only.

LEMMA 5.4

(i) *If* $v_h \in H^{1/2}(\Gamma_h)^2$, *then* $v \in H^1(U_h \backslash \overline{U}_H)^2$ *for every* $H > h$.

(ii) *In the general case* $|\varrho| < 1$, *the assertion of Lemma* 2.5(i) *holds if the following operators*

$$\begin{aligned} &\mathcal{N}_0 : L^2_\varrho(\mathbb{R}^2)^2 \to L^2_\varrho(\mathbb{R}^2)^2, \\ &\mathcal{N}_0 - \left(1+x_1^2\right)^{\varrho/2} \mathcal{N}_0 \left(1+x_1^2\right)^{-\varrho/2} : L^2(\mathbb{R}^2)^2 \to H^1(\mathbb{R}^2)^2 \end{aligned} \tag{5.12}$$

are uniformly bounded in $x_2 \in (h, H)$.

Proof (i) By the Plancherel identity we get

$$\begin{aligned} &\int_h^H \int_{\mathbb{R}} \left(|\mathcal{N}_0 v_h|^2 + |\mathcal{N}_1 v_h|^2 + |\mathcal{N}_2 v_h|^2\right) dx_1 dx_2 \\ &\quad \le \int_h^H \int_{\mathbb{R}} \left(||N_0(\xi, x_2)||^2 + ||i\xi N_0(\xi, x_2)||^2 + ||M(\xi) N_0(\xi, x_2)||^2\right) |\hat{v}_h(\xi)|^2 d\xi dx_2 \\ &\quad \le C \int_{\mathbb{R}} \left(1+\xi^2\right) |\hat{v}_h(\xi)|^2 \int_h^H ||N_0(\xi, x_2)||^2 dx_2 d\xi. \end{aligned} \tag{5.13}$$

Below we shall prove that

$$\int_h^H ||N_0(\xi, x_2)||^2 dx_2 \le C \left(1+\xi^2\right)^{-1/2}, \quad \forall \xi \in \mathbb{R}. \tag{5.14}$$

The relation $M_p + M_s = I$ allows us to rewrite N_0 as

$$N_0(\xi, x_2) = \left(\exp(i\gamma_p(\xi)(x_2-h)) - \exp(i\gamma_s(\xi)(x_2-h))\right) M_p(\xi) + \exp(i\gamma_s(\xi)(x_2-h)) I. \tag{5.15}$$

Applying the mean value theorem to the function $t \to \exp(t(x_2 - h))$ yields the identity

$$e^{i\gamma_p(\xi)(x_2-h)} - e^{i\gamma_s(\xi)(x_2-h)} = e^{t(\xi)(x_2-h)} (x_2 - h) \, |\gamma_p(\xi) - \gamma_s(\xi)|,$$

where the values of $t(\xi)$ lie between $i\gamma_p(\xi)$ and $i\gamma_s(\xi)$ for large $|\xi|$. Hence, by (5.15) and the definition of M_p,

$$\begin{aligned} &\int_h^H ||N_0(\xi, x_2)||^2 dx_2 \\ &\quad \le C \left(|\gamma_p(\xi) - \gamma_s(\xi)|^2 \xi^2 \int_0^{H-h} |e^{t(\xi)x_2} x_2|^2 dx_2 + \int_0^{H-h} |e^{i\gamma_s(\xi)x_2}|^2 dx_2\right). \end{aligned}$$

Making use of the asymptotic behavior

$$|\gamma_p(\xi)|, |\gamma_s(\xi)| \sim \left(1+\xi^2\right)^{1/2}, \quad |\gamma_p(\xi) - \gamma_s(\xi)| \sim 1/|\xi|, \quad \text{as } |\xi| \to \infty, \tag{5.16}$$

we obtain after some elementary calculations (see e.g. [17, Lemma 2.2])

$$|\gamma_p(\xi) - \gamma_s(\xi)|^2 \xi^2 \int_0^{H-h} |e^{t(\xi)x_2} x_2|^2 dx_2 \le C \int_0^{H-h} |e^{t(\xi)x_2} x_2|^2 dx_2 \le C \left(1+\xi^2\right)^{-1/2},$$
$$\int_0^{H-h} |e^{i\gamma_s(\xi)x_2}|^2 dx_2 \le C \left(1+\xi^2\right)^{-1/2},$$

from which the inequality (5.14) follows. Insertion of (5.14) into (5.13) yields (5.11) for $\varrho = 0$. This proves the first assertion.

(ii) We shall prove the second assertion following the lines in the proof of Lemma 3.4(i) and Lemma 3.3(ii) of [25]. Denote by $\mathcal{A}$ one of the operators ∂_1 and $\tilde{T}$. Then, there holds the identity

$$\begin{aligned} \mathcal{A}\mathcal{N}_0 - \left(1+x_1^2\right)^{\varrho/2} \mathcal{A}\mathcal{N}_0 \left(1+x_1^2\right)^{-\varrho/2} &= \mathcal{A}\left(\mathcal{N}_0 - \left(1+x_1^2\right)^{\varrho/2} \mathcal{N}_0 \left(1+x_1^2\right)^{-\varrho/2}\right) \\ &+ \left(\mathcal{A} - \left(1+x_1^2\right)^{\varrho/2} \mathcal{A} \left(1+x_1^2\right)^{-\varrho/2}\right) \left(1+x_1^2\right)^{\varrho/2} \mathcal{N}_0 \left(1+x_1^2\right)^{-\varrho/2}. \end{aligned} \tag{5.17}$$

Since the two operators in (5.12) are uniformly bounded and the operators

$$\mathcal{A} : H^1(\mathbb{R})^2 \to L^2(\mathbb{R})^2, \quad \mathcal{A} - \left(1+x_1^2\right)^{\varrho/2} \mathcal{A} \left(1+x_1^2\right)^{-\varrho/2} : L^2(\mathbb{R})^2 \to L^2(\mathbb{R})^2$$

are also bounded, we derive from (5.17) and (5.10) that the commutators

$$\mathcal{N}_j - \left(1+x_1^2\right)^{\varrho/2} \mathcal{N}_j \left(1+x_1^2\right)^{-\varrho/2}, \quad j = 0, 1, 2,$$

are uniformly bounded on $L^2(\mathbb{R})^2$ with respect to $x_2 \in (h, H)$. Further, this implies that

$$\mathcal{C}_j = \mathcal{C}_j(x_2) := \left(1+x_1^2\right)^{-\varrho/2} \mathcal{N}_j - \mathcal{N}_j \left(1+x_1^2\right)^{-\varrho/2}, : L^2(\mathbb{R})^2 \to L^2_\varrho(\mathbb{R})^2, \quad j = 0, 1, 2, \tag{5.18}$$

are uniformly bounded in x_2. By the continuous imbedding of $H^{1/2}(\mathbb{R})^2$ into $L^2(\mathbb{R})^2$, we see the boundedness of $\mathcal{C}_j : H^{1/2}(\mathbb{R})^2 \to L^2_\varrho(U_h \backslash \overline{U}_H)^2$. On the other hand, the operators

$$\left(1+x_1^2\right)^{-\varrho/2} \mathcal{N}_j : H^{1/2}(\mathbb{R})^2 \to L^2_\varrho(U_h \backslash \overline{U}_H)^2, \qquad j = 0, 1, 2, \tag{5.19}$$

are also bounded, because by assertion (i) the operators $\mathcal{N}_j : H^{1/2}(\mathbb{R})^2 \to L^2(U_h \backslash \overline{U}_H)^2$ are bounded. Now combining (5.18) and (5.19), we can conclude the boundedness of

$$\mathcal{N}_j \left(1+x_1^2\right)^{-\varrho/2} : H^{1/2}(\mathbb{R})^2 \to L^2_\varrho(U_h \backslash \overline{U}_H)^2, \qquad j = 0, 1, 2,$$

which implies the estimate (5.11). □

Remark 5.5 In [25] the uniform boundedness of the operators in (5.12) with

$$\mathcal{N}_0 = \mathcal{F}^{-1} \exp\left(i\sqrt{k^2 - \xi^2}(x_2 - h)\right) \mathcal{F}$$

(see Lemma (5.3)(ii)) plays an essential role in proving Lemma 2.5(i) for the Helmholtz equation.

We proceed with the proof of Lemma 2.5(i). By Lemma 5.4(ii), it suffices to estimate the norm of the operators in (5.12). For this purpose, we shall adopt the same approach as in the proof of Theorem 2.2 by using the second assertion of Lemma 5.2 and the result of Lemma 5.3(ii) for non-smooth symbols.

Motivated by the proof of Theorem 2.2, we introduce the auxiliary symbol

$$\begin{aligned} W(\xi, x_2) = {} & [\exp(i\gamma_p(\xi)(x_2-h))\,\Pi_p^+ + \exp(i\gamma_s(\xi)(x_2-h))\,\Pi_s^+]\,\chi(\xi) \\ & + [\exp(i\gamma_p(\xi)(x_2-h))\,\Pi_p^- + \exp(i\gamma_s(\xi)(x_2-h))\,\Pi_s^-]\,(1-\chi(\xi)), \end{aligned}$$

where $\chi(\xi) \in C^\infty(\mathbb{R})$ satisfies $\chi = 1$ for $\xi > k_p/3$ and $\chi = 0$ for $\xi < -k_p/3$. We shall select the entries of $\Pi_p^\pm, \Pi_s^\pm \in \mathbb{C}^{2\times 2}$ so that $Q := N_0 - W$ is a continuously differentiable matrix in $\xi \in \mathbb{R}$.

Elementary calculations show

$$\frac{\partial W}{\partial \xi} = i(x_2-h)[\exp(i\gamma_p(\xi)(x_2-h))\Pi_p^\pm \gamma_p'(\xi) + \exp(i\gamma_s(\xi)(x_2-h))\Pi_s^\pm \gamma_s'(\xi)] := J_1(\xi) \tag{5.20}$$

for $\xi \gtrless \pm k_p/3$, and $\partial N_0/\partial\xi = J_0(\xi) + J_2(\xi)$ where

$$\begin{aligned} J_0(\xi) &:= i(x_2-h)[\exp(i\gamma_p(\xi)(x_2-h))\,M_p(\xi)\gamma_p'(\xi) + \exp(i\gamma_s(\xi)(x_2-h))\,M_s(\xi)\gamma_s'(\xi)], \\ J_2(\xi) &:= \exp(i\gamma_p(\xi)(x_2-h))\,M_p'(\xi) + \exp(i\gamma_s(\xi)(x_2-h))\,M_s'(\xi). \end{aligned} \tag{5.21}$$

Comparing (5.20) and (5.21) and using elementary calculations, we obtain the desired expressions for $\Pi_p^\pm, \Pi_s^\pm$ depending on x_2, k_p and k_s:

$$\Pi_p^\pm(x_2) = \begin{pmatrix} 1 & \pm k_p^{-1}\sqrt{k_s^2-k_p^2} \\ 0 & 0 \end{pmatrix} + \frac{1 - e^{i\sqrt{k_s^2-k_p^2}(x_2-h)}}{i(x_2-h)\,k_p^2} \begin{pmatrix} -\sqrt{k_s^2-k_p^2} & \mp k_p^{-1}(k_s^2-k_p^2) \\ \pm k_p & \sqrt{k_s^2-k_p^2} \end{pmatrix},$$

$$\Pi_s^\pm(x_2) = \begin{pmatrix} 0 & 0 \\ \mp k_s^{-1}\sqrt{k_p^2-k_s^2} & 1 \end{pmatrix} + \frac{1 - e^{i\sqrt{k_p^2-k_s^2}(x_2-h)}}{i(x_2-h)\,k_s^2} \begin{pmatrix} \sqrt{k_p^2-k_s^2} & \mp k_s \\ \pm k_s^{-1}(k_p^2-k_s^2) & -\sqrt{k_p^2-k_s^2} \end{pmatrix}.$$

Since the matrices $\Pi_p^\pm(x_2)$, $\Pi_s^\pm(x_2)$ are uniformly bounded in $x_2 \in [h, H]$, applying Lemma 5.3(ii) to W yields the uniform boundedness of the operators

$$\begin{aligned} \mathcal{W} := \mathcal{F}^{-1}\, W\,\mathcal{F}:\ L^2_\varrho(\mathbb{R}^2)^2 \to L^2_\varrho(\mathbb{R}^2)^2\,, \\ \mathcal{W} - \left(1+x_1^2\right)^{\varrho/2} \mathcal{W} \left(1+x_1^2\right)^{-\varrho/2}:\ L^2(\mathbb{R}^2)^2 \to H^1(\mathbb{R}^2)^2\,. \end{aligned}$$

Now it is sufficient to prove the uniform boundedness of the operators in (5.12) with the C^1-smooth matrix Q in place of N_0. In the following, we shall apply Lemma 5.2(iii) and check the validity of the inequalities in (5.5) with m_1 replaced by each entry of Q for large $|\xi|$. Since $Q = N_0 - W$ and $\partial Q/\partial\xi = J_0 + J_1 + J_2$, it is enough to show that there exist a positive number $K > 0$ and some constant $C(\varrho, N, H-h) > 0$ such that

$$||N_0|| + ||W|| \le C, \quad ||J_n|| \le C(1+\xi^2)^{1/2}, \quad |\xi| > K, \quad j = 0, 1,\ n = 0, 1, 2. \tag{5.22}$$

We first prove (5.22) for J_0. Observing that $\gamma_p'(\xi) = \xi/\gamma_p(\xi)$, $\gamma_s'(\xi) = \xi/\gamma_s(\xi)$ and $M_p(\xi) + M_s(\xi) = I$, we represent J_0 as $J_0 = J_0^{(1)} + J_0^{(2)}$ with

$$J_0^{(1)}(\xi) := i(x_2 - h)\,\xi \left\{ e^{i(x_2-h)\gamma_p(\xi)}/\gamma_p(\xi) - e^{i(x_2-h)\gamma_s(\xi)}/\gamma_s(\xi) \right\} M_p(\xi),$$
$$J_0^{(2)}(\xi) := i(x_2 - h)\,\big(\xi/\gamma_s(\xi)\big)\, e^{i(x_2-h)\gamma_s(\xi)}\, I.$$

The matrix function $J_0^{(2)}$ can be bounded as

$$||J_0^{(2)}|| \le ||i(x_2 - h)\gamma_s(\xi) e^{i(x_2-h)\gamma_s(\xi)}||\; |\xi/\gamma_s^2(\xi)| \le C\,(1 + \xi^2)^{-1/2}, \tag{5.23}$$

where $C > 0$ is independent of $x_2 \in (h, H)$. Applying the mean value theorem to the function $t \to \exp((x_2 - h)t)/t$ gives the relation

$$\begin{aligned} &e^{i(x_2-h)\gamma_p(\xi)}/\gamma_p(\xi) - e^{i(x_2-h)\gamma_s(\xi)}/\gamma_s(\xi) \\ &\quad = \frac{1}{(x_2 - h)}\, e^{(x_2-h)t(\xi)}\, [-(x_2 - h)t(\xi) + ((x_2 - h)t(\xi))^2]\, \frac{|\gamma_p(\xi) - \gamma_s(\xi)|}{t^3(\xi)}, \end{aligned} \tag{5.24}$$

where again the values of $t(\xi)$ lie between $i\gamma_p(\xi)$ and $i\gamma_s(\xi)$ for large ξ. Inserting (5.24) into the expression for $J_0^{(1)}$ and applying the asymptotic behavior (5.16), we obtain $||J_0^{(1)}|| \le C(1 + \xi^2)^{-1/2}$. This together with (5.23) proves the inequality in (5.22) for J_0.

The other estimates in (5.22) for J_n $(n = 1, 2)$ can be obtained in the same manner as for J_0. The boundedness of W in (5.22) follows straightforwardly from the uniform boundedness of $\Pi_p^{\pm}(x_2)$, $\Pi_s^{\pm}(x_2)$, $\exp(i\gamma_p(\xi)(x_2 - h))$ and $\exp(i\gamma_s(\xi)(x_2 - h))$ in $x_2 \in (h, H)$, whereas the estimate for N_0 can be verified by first using the relation $M_p + M_s = I$ and then again applying the mean value theorem to the resulting expression. The proof of Lemma 2.5(i) is thus complete. □

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