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Hölder stability estimate of Robin coefficient in corrosion detection with a single boundary measurement

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Abstract

This paper is concerned with the inverse problem of detecting a boundary corrosion coefficient which describes some corrosion index from a single pair of Cauchy data measured on an accessible boundary of an electrostatic conductor in two dimensions. The corroded portion is supposed to be either a line segment or a part of some circle, while the corrosion coefficient is restricted to be an analytic or a piecewise constant function. We prove two Hölder stability estimates in recovering the unknown boundary coefficient. Our arguments rely on the Schwarz reflection principle with the Robin boundary condition and a novel interior estimate derived from the elliptic Carleman estimate.

Keywords: inverse problem, corrosion detection, Robin boundary condition, reflection principle, Hölder stability, elliptic Carleman estimate

1. Introduction

This paper is concerned with an inverse problem in the non-destructive testing of the corrosion contaminating an inaccessible boundary of an electrostatic conductor (see e.g. [20] and [22]). Let $\Omega \subset \mathbb{R}^2$ be a bounded connected domain with the piecewise smooth boundary $\partial\Omega$, which represents the region occupied by the conductor. Assume that $\partial\Omega = \overline{\Gamma \cup \gamma \cup \gamma'}$ and $\gamma \cup \gamma'$ is C^2-smooth, where Γ and γ are two open, non-empty, disjoint open subsets of $\partial\Omega$ and $\gamma' = \partial\Omega \backslash (\overline{\Gamma \cup \gamma}) \neq \varnothing$. We assume that Ω is a curvilinear corner domain with two corner points between Γ and γ' (see section 2 for the precise

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definition). The boundary Γ is supposed to be the inaccessible part of the conductor which is now affected by corrosion, whereas $\gamma \cup \gamma'$ is the portion of $\partial\Omega$ that can be reached. In this paper, we consider the two-dimensional corrosion model with a linearized Robin boundary condition on Γ (see [20]):

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \partial_\nu u + q\, u = 0 & \text{on} \quad \Gamma, \\ u = g & \text{on} \quad \gamma, \\ u = 0 & \text{on} \quad \gamma'. \end{cases} \tag{1}$$

In (1), u denotes the electrostatic potential of the conductor, g is the boundary voltage and the normal derivative $\partial_\nu u$ represents the current flux. The unit normal ν on $\partial\Omega$ is supposed to point outside. The Robin boundary condition arises from, for example, the asymptotic analysis performed in [8] when corrosion of a thin coating with rapid oscillations tends to roughen a surface. It could also be derived by linearizing the more accurate nonlinear boundary condition for the voltage and current on the corroded surface. The non-negative Robin coefficient q represents the reciprocal of the surface impedance, and will be referred to as the corrosion coefficient characterizing the damage on Γ. We refer to [27, 31] for other mathematical models in corrosion detection and to [2, 3] for stability estimates of a nonlinear boundary term on Γ.

In the present study, the geometry of the corrosion boundary Γ is assumed to be known in advance, but knowledge of the index coefficient q, which gives essential information on the corrosion on Γ, is unknown. The corrosion detection problem then consists of the determination of q from the Cauchy data $(u, \partial_\nu u)\big|_\gamma$. The uniqueness in recovering a continuous non-negative Robin coefficient simply follows from the uniqueness of the Cauchy problem for the Laplace equation in Ω (see [20, lemma 3.1]). For non-negative bounded Robin coefficients, it was recently verified in [5] that a single Cauchy-pair $\left(u\big|_\gamma, \partial_\nu u\big|_\gamma\right) \in H^{1/2}(\gamma) \times L^2(\gamma)$ uniquely determines $q \in L^\infty(\Gamma)$ in two dimensions, which however does not hold in higher dimensions. In general, a single measurement is not sufficient to determine simultaneously the shape Γ and the coefficient q ([9]). In [4], two Cauchy data pairs are proven sufficient to recover Γ and q, if the two voltages (or current densities) are linearly independent and one of them is positive on $\partial\Omega\backslash\overline{\Gamma}$. It is worth mentioning that such a result also applies to the Cauchy data taken on an arbitrary sub-boundary γ of $\partial\Omega\backslash\overline{\Gamma}$, as considered in this paper.

This paper is concerned with stability estimates of q from single Cauchy data taken on a sub-boundary of the accessible part. Our approach is closest to the idea of [23] for the stability estimate of an analytic surface impedance in inverse scattering. The arguments in [23] rely heavily on the analytic extension of the wave fields into an ϵ-neighborhood ($\epsilon > 0$) of the scattering interface. This was achieved by applying the extension theorem of [26, theorem 5.7.1], which essentially requires suitable upper bounds of the derivatives of the impedance function. In our studies, there are no *a priori* assumptions on the derivatives of q. The price we should pay is an extra condition on the geometrical shape of Γ: the corroded portion is assumed to be either a (finite) line segment or a portion of some circle. Under such an assumption, we derive a Hölder stability estimate of the L^2-norm of an analytic Robin coefficient on any sub-interval of Γ for the Laplace equation. When Γ is flat, we consider general linear elliptic equations of second order with constant coefficients and justify the same kind estimate for a piecewise constant function over $L^\infty(\Gamma)$. These stability estimates are stronger than earlier logarithmic stability results established in [1], [12, section 3] and [10] in more general situations and those from [12, section 4] and [14, section 3] where $q \in C^3[0, 1]$

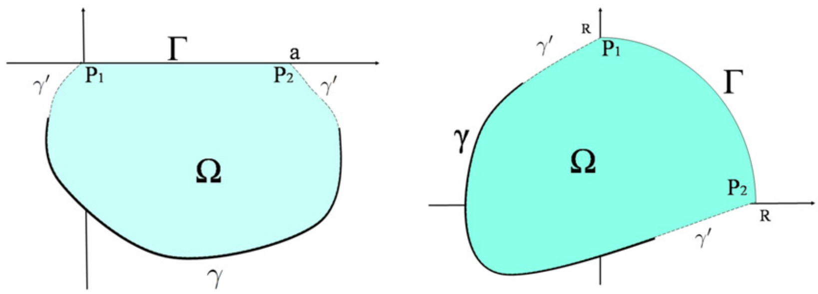

Figure 1. Configurations of Ω: $\partial\Omega = \Gamma \cup \gamma \cup \gamma'$ with $\gamma' \neq \varnothing$. Left: case (i). Right: case (ii) with $a = \pi/2$.

and Ω is a rectangular domain. Lipschitz stability estimates were obtained in [7] and [30] in general Lipschitz domains with L^∞ or piecewise constant impedance coefficients in the case where unknowns are described by a fixed finite number of parameters. Other monotone and local Lipschitz stability estimates were investigated in [11] and [14, section 2] under certain assumptions of the input current flux.

Our arguments are relatively simple compared to the existing works, and differ drastically from [13] detecting an unknown boundary coefficient in a hyperplane and the approaches in [7] and [30] as well. The key ingredients in our analysis consist of the Schwarz reflection principle under the Robin boundary condition on a flat or a circular curve in $\mathbb{R}^2$ [6, 15] and a novel interior estimate derived from the elliptic Carleman estimate. For piecewise constant Robin coefficients, it is not necessary to have *a priori* information on the number of the (finite) partition of Γ. Although our results are restricted to the case of a flat or circular boundary only, they may have practical applications, e.g. in detecting the corrosion damage on the interior boundary of a pipe where the underlying conductor is a spherical shell. With slight modifications, the inverse problem with an input current flux can be treated analogously by means of the electrostatic measurement on the accessible boundary.

The Schwarz reflection principle for harmonic functions has been well known since the pioneering paper [28]. It provides a global harmonic extension formula across a portion of flat and spherical surfaces in $\mathbb{R}^n$ subject to the Dirichlet or the Neumann boundary condition. In contrast to the odd (even) reflection law corresponding to the Dirichlet (Neumann) boundary condition, the reflection principle under the Robin boundary condition is no longer the point-to-point type. We refer to [15] for the extension formulae for linear elliptic equations with constant coefficients satisfying the Dirichlet, Neumann or Robin boundary condition with a constant Robin coefficient on a flat surface in $\mathbb{R}^n$. Recently a more general reflection principle was established in [6] for harmonic functions subject to the Robin boundary condition with analytic coefficients in $\mathbb{R}^2$. This extends the results of [15] to non-singular real-analytic curves including circles.

We organize the paper as follows. In section 2, we present assumptions on the corrosion coefficient q, the inaccessible boundary Γ and the input data g. Our stability estimates will be stated and verified in sections 3 and 4 for analytic and piecewise constant Robin coefficients, respectively. Section 3.1 is devoted to some preliminary lemmas including the Schwarz reflection principle and a novel interior estimate for linear elliptic equations of second order. A possible extension of our work to non-singular analytic curves will be discussed in section 5. For clarity, the proof of some preliminary lemmas from section 3.1 is postponed to the appendix in section A.

2. Mathematical setting

Throughout the paper, we denote by $B_r(P) \subset \mathbb{R}^2$ the disk centered at the point P with radius $r > 0$, i.e. $\{x \in \mathbb{R}^2 : |x - P| < r\}$, and we set $B_r = B_r(O)$. The inaccessible boundary Γ is supposed be either a line segment or a portion of some circle. Without loss of generality, we assume one of the following two cases holds (see figure 1):

Case (i): $\Gamma = \{(x_1, 0) : 0 < x_1 < a\}$ for some $a > 0$ and $\Omega \cap \{x : x_2 < 0\} \neq \varnothing$;
Case (ii): $\Gamma = \{R(\cos\varphi, \sin\varphi) : 0 < \varphi < a\}$ for some $R > 0$ and $a \in (0, 2\pi)$, and $\Omega \cap \{x: |x| < a\} \neq \varnothing$.

Throughout the paper we suppose that γ' is not empty and has two disconnected components. Denote by $P_1, P_2 \in \mathbb{R}^2$ the boundary points of Ω connecting γ' and Γ. For $0 < \epsilon < a/2$ sufficiently small, introduce the subdomain Ω_ϵ of Ω by

$$\Omega_\epsilon := \left\{ x \in \Omega : \left| x - P_j \right| > \epsilon,\ j = 1, 2 \right\}. \tag{2}$$

It is supposed that $\gamma \subset \partial\Omega_\epsilon$ (see figure 1). Note that these are technical assumptions for studying the corner singularity of u around P_j. If γ' has only one component or $\gamma' = \varnothing$, then we need an extra assumption on the angle formed by Γ and $\partial\Omega \backslash \overline{\Gamma}$; see remark 3.3.

The boundary $\gamma \cup \gamma'$ is supposed be C^2-smooth, whereas the boundary around the transition points P_j is to be piecewise smooth. More precisely, for $P = P_j$ we assume there exists a neighborhood V of P and a diffeomorphism $\Psi : V \to B_1$ of C^∞ such that (see e.g. [24, chapter 1.3.7])

$$\nabla\Psi(P) = I, \quad \Psi(P) = O, \quad \Psi(V \cap \Omega) = \{(r, \varphi) : r < 1, 0 < \varphi < \omega(P)\} \tag{3}$$

for some angle $\omega(P) \in (0, 2\pi)$. The point P is called a curvilinear corner of Ω if $\omega(P) \neq \pi$. In particular, $\omega(P) = \pi$ if $\partial\Omega$ is smooth at P. Note that the required regularity of $\partial\Omega$ excludes the possible inner peaks at P_j.

In order to formulate problem (1), we need to introduce appropriate Sobolev spaces. For $s \in \mathbb{R}$, define

$$H^s(\gamma) := \left\{ u\,|_\gamma : u \in H^s(\partial\Omega),\ \mathrm{Supp}\ u \subset \overline{\gamma} \right\}$$

endowed with the norm

$$||u||_{H^s(\gamma)} := \inf_{U \in H^s(\partial\Omega), U|_\gamma = u} \left\{ ||U||_{H^s(\partial\Omega)} \right\}.$$

It is well known that for $g \in H^{1/2}(\gamma)$, $q > 0$ i.e. on Γ, there exists a unique solution $u \in X := \{u \in H^1(\Omega) : u = 0 \text{ on } \gamma'\}$ to the mixed boundary value problem (1). Throughout the paper, we assume that the input data g are given from the admissible set

$$\mathcal{G}_M := \left\{ g \in H^{3/2}(\gamma) : 0 < ||g||_{H^{3/2}(\gamma)} < M, \quad g \geqslant 0 \quad \text{on} \quad \gamma \right\} \quad \text{for some} \quad M > 0,$$

and that q is non-identically vanishing on Γ subject to the condition

$$0 \leqslant q(x) \leqslant N, \qquad N > 0.$$

It is worth noting that we do not have the higher regularity $u \in H^2(\Omega)$ for general $g \in H^{3/2}(\gamma)$, because the solution may be 'singular' around the transition points between Γ

and γ', even if q and $\partial\Omega$ are both smooth. We refer to lemma 3.2 (see also [25, corollary 3.1] or [32]) for descriptions of the singular part of the solution.

3. Estimate of analytic coefficients

In this subsection we consider the stability estimate of an analytic Robin coefficient defined on line segments or circles.

Let the parameter $a > 0$ be specified as in case (i) or (ii). For $0 < \epsilon < a/2$ we define a subdomain Γ_ϵ of Γ as

$$\begin{aligned} &\text{in case (i)}: \quad \Gamma_\epsilon := \left\{ \left(x_1, 0\right) : \epsilon < x_1 < a - \epsilon \right\}; \\ &\text{in case (ii)}: \quad \Gamma_\epsilon := \{ R(\cos\varphi, \sin\varphi) : \epsilon < \varphi < a - \epsilon \}. \end{aligned} \tag{4}$$

Since q is analytic and $g \in \mathcal{G}_M$, it follows from the boundary and interior regularity estimates for elliptic boundary value problems that

$$||u||_{H^2(\Omega_\epsilon)} \leqslant C \quad \text{for some} \quad C = C(M, N, \Omega, \epsilon) > 0, \tag{5}$$

where Ω_ϵ is defined by (2). Applying the trace lemma yields $(\partial_\nu u)\big|_\gamma \in H^{1/2}(\gamma)$ and $u\,|_{\Gamma_\epsilon} \in H^{3/2}(\Gamma_\epsilon)$. The main result of this section is stated as follows.

Theorem 3.1. *Assume q_j are analytic functions on Γ and $g_j \in \mathcal{G}_M$ for $j = 1, 2$ and some $M > 0$. Denote by $u_j \in H^1(\Omega)$ the unique solution to (1) with $g = g_j$. There holds the stability estimate*

$$\left|\left| q_1 - q_2 \right|\right|_{L^2(\Gamma_\epsilon)} \leqslant C \left(\left|\left| g_1 - g_2 \right|\right|_{H^1(\gamma)} + \left|\left| \partial_\nu (u_1 - u_2) \right|\right|_{L^2(\gamma)} \right)^\kappa$$

for some $C > 0$, $\kappa \in (0, 1)$ depending on N, M, *ϵ and $\partial\Omega$.*

In the theorem, the estimation of $q_1 - q_2$ is limited to a proper subset Γ_ϵ of Γ, in order to apply an elliptic interior estimate but not to obtain the Hölder estimate.

Before carrying out the proof of theorem 3.1, we state some preliminary lemmas in the subsequent subsection. From now on, unless otherwise stated we always use C to denote a generic positive constant depending on the *a priori* data specified in theorem 3.1 which may vary from line to line.

3.1. Preliminary lemmas

We first prove by using a compact argument that the solution to (1) has a uniform positive lower bound on $\overline{\Gamma}_\epsilon$, if the data g belongs to the admissible set $\mathcal{G}_M$. Then we state two reflection principles under the Robin boundary condition and an interior estimate for elliptic equations, which are the essential ingredients for proving theorems 3.1. In the following lemma, we denote by $\omega_j \in (0, 2\pi)$ the angle of the boundary $\partial\Omega$ at the transition point P_j between Γ and γ'. We have $\omega_j = \pi$ if $\partial\Omega$ is smooth at P_j, and $\omega_j \neq \pi$ if P_j is a curvilinear corner point.

Lemma 3.2. *Let $u_g \in H^1(\Omega)$ be the unique solution to (1) for $g \in \mathcal{G}_M$, and assume $\gamma \subset \partial\Omega_\epsilon$ (cf (2)). Choose $m_j \geqslant 2$ to be the minimal integer such that $(m_j - 1)\pi/\omega_j + 1/2$ is not an integer, i.e. $m_j := \min\{m \in \mathbb{N} : m \geqslant 2,\ (m - 1)\pi/\omega_j + 1/2 \notin \mathbb{N}\}$.*

(*i*) *It holds that $u_g\,|_\Gamma \in H^{1/2+s}(\Gamma)$ for some $s > 0$, and* $\mathrm{u_g}$ *is continuous on $\overline{\Gamma}$.*

(ii) We have $u_g \in H^\sigma(\Omega)$ *where*

$$\begin{array}{lll} \sigma = 2 & \textit{if} & \max\{\omega_1, \omega_2\} < \pi/2, \min\{m_1, m_2\} \geqslant 3; \\ \sigma \textit{ is any number less than } 2 & \textit{if} & \max\{\omega_1, \omega_2\} < \pi/2, \min\{m_1, m_2\} = 2. \end{array}$$

If $\max\{\omega_1, \omega_2\} \geqslant \pi/2$, *then* σ *is allowed to be any number less than* $\min\{\pi/(2\omega_1), \pi/(2\omega_2)\} + 1 < 2$. *In particular,* $u_g \in H^{1+s}(\Omega)$ *for any* $0 \leqslant s < 1/2$ *when* $\omega_1 = \omega_2 = \pi$.

(iii) We have

$$u_g > 0 \quad \text{on} \quad \Gamma, \qquad \min_{x \in \overline{\Gamma}_\epsilon} \left\{ u_g(x) \right\} \geqslant \eta > 0,$$

for all $g \in \mathcal{G}_M$ *and some positive number* $\eta(\epsilon) > 0$.

Proof. (i) We only need to investigate the regularity of u_g around P_j. Write $\gamma_j' = B_\epsilon(P_j) \cap \gamma'$, where the number $\epsilon > 0$ is defined in (4). Obviously, $\gamma_j' \neq \varnothing$ for $j = 1, 2$ due to the assumption $\gamma \subset \partial\Omega_\epsilon$, and u_g fulfills the homogeneous Dirichlet boundary condition $u_g |_{\gamma'} = 0 \in H^m(\gamma')$ for any $m > 0$. In view of the geometrical assumption on the transition points P_j, we first apply the local coordinate transformation $y = \Psi(x)$ (see (3) for the definition) and then employ the perturbation argument of [24, chapter 1.3.7] in a neighborhood of the origin in the transformed domain. Consequently, it suffices to consider the plane corner singularity for the Laplace equation with mixed Dirichlet and Robin boundary conditions in the sector $\{(r, \theta) : r < 1, 0 < \theta < \omega_j\}$. Since such arguments are standard, we omit the details for brevity.

Since the mapping Ψ is of C^∞-smooth and q is analytic, the transformed Robin coefficient $\tilde{q}(y) := q(\Psi^{-1}(y))$ is still a non-negative C^∞-smooth function. This enables us to apply Grivard's regularity theorem (see [17] or [25, 32]) to a neighborhood of the origin. Therefore, it holds that

$$u_g - \sum_{0 < \lambda_{j,k} < m_j - 1} a_{j,k} \, v_{j,k} \in H^{m_j}\left(V_j\right), \quad j = 1, 2, \tag{6}$$

where V_j is a small neighborhood of P_j in Ω, the number m_j is specified as in lemma 3.2, $a_{j,k} \in \mathbb{C}$ and $\lambda_{j,k} = (k - 1/2)\pi/\omega_j$ for $k = 1, 2, \cdots$. The singular functions $v_{j,k}$ are given by the following rule:

$$v_{j,k}(r, \theta) := \begin{cases} r^{\lambda_{j,k}} \cos\left(\lambda_{j,k}\theta\right) & \text{if} \quad \lambda_{j,k} \notin \mathbb{N}, \\ r^{\lambda_{j,k}} \left(\ln r \cos\left(\lambda_{j,k}\theta\right) - \theta \sin\left(\lambda_{j,k}\theta\right) \right) & \text{if} \quad \lambda_{j,k} \in \mathbb{N}. \end{cases} \tag{7}$$

Here without fear of confusion, by the same notations (r, θ) we denote the local polar coordinates around P_j, $j = 1, 2$. Moreover, it can be derived from (7) that $v_{j,k} \in H^s(V_j)$ for all $0 < s < \lambda_{j,k} + 1$. Observing that $\lambda_{j,k} + 1 < m_j$ and $\lambda_{j,k} + 1 \leqslant \lambda_{j,1} + 1 = \pi/(2\omega_j) + 1$ for $k = 1, 2, \cdots$, we obtain

$$u_g \in H^s\left(V_j\right) \quad \text{for all} \quad 0 < s < \sigma_j := \min\left\{ \pi/\left(2\omega_j\right) + 1, m_j \right\}. \tag{8}$$

Hence, by the trace lemma, $u_g \in H^{1/2+s}(\Gamma \cup \gamma_1' \cup \gamma_2')$ for all $0 < s < \min\{\sigma_1, \sigma_2\} - 1$. Applying the Sobolev embedding theorems yields $u_g \in C^{0,s}(\Gamma)$ for some $s > 0$. The first assertion is thus proven.

(ii) If $\omega_j < \pi/2$ and $m_j \geqslant 3$, it is seen from (8) that $u_g \in H^{2+s}(V_j)$ for some $s > 0$. Combining this with the fact that $g \in \mathcal{G}_M$, we see $u \in H^2(\Omega)$. If $\omega_j \geqslant \pi/2$ or $m_j = 2$ for

$j = 1, 2$, the regularity of u in Ω is dominated by that in V_j depending on the size of the angle at P_j, as shown in the second assertion of the lemma.

(iii) Obviously, $u(P_j) \geqslant 0$ for $j = 1, 2$, since $u_g|_{\partial\Omega}$ is continuous in a neighborhood of P_j and $u_g \geqslant 0$ on $\gamma \cup \gamma'$. We first prove that $u_g > 0$ on Γ for arbitrarily fixed $g \in \mathcal{G}_M$. Assume on the contrary that $u_g(x^*) \leqslant 0$ for some $x^* \in \Gamma$. If $u_g(x^*) < 0$, then by $u(P_j) \geqslant 0$ for $j = 1, 2$, we see that $x^* \neq P_1, P_2$, and we can assume that $u_g(x_*) = \min_{\overline{\Gamma}} u_g$. Next if $u_g(x^*) = 0$, then $u(P_j) \geqslant 0$ for $j = 1, 2$ implies $u_g(P_j) \geqslant u_g(x^*)$ and we can assume also $u_g(x_*) = \min_{\overline{\Gamma}} u_g = 0$. Therefore in both cases, $u_g(x^*) = \min_{\overline{\Gamma}} u_g$. Recalling the minimum principle for harmonic functions and again using the fact that $u_g \geqslant 0$ on $\gamma \cup \gamma'$, we see $u_g(x^*) = \min\{u_g(x) : x \in \overline{\Omega}\}$, that is, u_g attains the global minimum at x^* on $\overline{\Omega}$. Moreover, it holds that $u_g(x) > u_g(x^*)$ for $x \in \Omega$. Since x^* is an interior point of Γ and Γ is smooth in a neighborhood of x^*, by choosing a small subdomain $U_{x^*} \subset \Omega$ such that $\partial U_{x^*} \subset \Gamma \cup \Omega$ and ∂U_{x^*} is smooth, we can apply Hopf's lemma to obtain

$$\partial_\nu \left[u_g\left(x^*\right) \right] < 0, \quad x^* \in \Gamma.$$

However, the impedance boundary condition of u_g leads to

$$\partial_\nu \left[u_g\left(x^*\right) \right] = -q\left(x^*\right) u_g\left(x^*\right) \geqslant 0.$$

This contradiction implies that $u_g(x) > 0$ for $x \in \Gamma$ and thus $\min_{x \in \overline{\Gamma}_\epsilon} \{u_g(x)\} \geqslant \eta_0 > 0$ for some $\eta_0 = \eta_0(\epsilon) > 0$. It remains to find a uniform lower bound for all $g \in \mathcal{G}_M$. Suppose on the contrary that

$$\inf \left\{ \min_{x \in \overline{\Gamma}_\epsilon} \left\{ u_g(x) \right\} : g \in \mathcal{G}_M \right\} = \tau \leqslant 0.$$

Then we can find a sequence $\{g_j\}_{j=1}^\infty \subset \mathcal{G}_M$ such that $\min_{x \in \overline{\Gamma}_\epsilon} \{u_{g_j}(x)\} \to \tau$ as $j \to \infty$. By the compact embedding of $H^{3/2}(\gamma)$ into $H^1(\gamma)$, there always exists a subsequence, which we still denote by g_j, such that $g_j \to g_0$ in $H^1(\gamma)$ for some $g_0 \in H^1(\gamma)$. It is seen from the proof of the first assertion that

$$u_{g_j} \in H^{s+1/2}(\Gamma) \subset C^{0,s}\left(\overline{\Gamma}\right), \quad j = 0, 1, 2, \cdots, \quad \text{for some} \quad s > 0.$$

Together with the trace lemma and the well-posedness of elliptic problems (1) and (20), we get the convergence

$$\sup_{x \in \overline{\Gamma}} |u_{g_j}(x) - u_{g_0}(x)| \leqslant C \left|\left| u_{g_j} - u_{g_0} \right|\right|_{H^{s+1/2}(\Gamma)} \leqslant C \left|\left| g_j - g_0 \right|\right|_{H^1(\gamma)} \to 0$$

as $j \to \infty$. In particular,

$$\min_{x \in \overline{\Gamma}_\epsilon} \left\{ u_{g_0}(x) \right\} = \lim_{j \to \infty} \min_{x \in \overline{\Gamma}_\epsilon} \left\{ u_{g_j}(x) \right\} = \tau \leqslant 0.$$

This contradicts the fact that $\min_{x \in \overline{\Gamma}_\epsilon} \{u_{g_0}(x)\} \geqslant \tilde{\eta}_0(\epsilon) > 0$ which can be proved by repeating the argument at the beginning of our proof. Lemma 3.2 is thus proven. □

Remark 3.3. If $\gamma' = \varnothing$, then we have the boundary data $u\Big|_\gamma = g \in H^{3/2}(\gamma)$ on the accessible part around the point P_j in place of the homogeneous Dirichlet boundary value. In this case, the positivity of u_g over $\overline{\Gamma}_\epsilon$ can be verified by repeating the arguments in the proof of lemma 3.2, provided the following assumption holds (see [17] or [25, corollary 3.1]):

$$\omega_j/\pi + 1/2 \notin \mathbb{N}, \quad j = 1, 2.$$

In this paper we do not consider the inverse problem in the case of $\gamma' = \varnothing$.

Below we present the Schwarz reflection principle for some elliptic equations subject to the Robin boundary condition. First, we select a neighboring area $\Sigma^- \subset \Omega$ of Γ so that the reflection of Σ^- with respect to Γ lies in $\mathbb{R}^2\backslash\overline{\Omega}$. Let ϵ be given as in (4). In case (i), choose $0 < \epsilon_0 < \epsilon$, $M_0 > 0$ to be such that

$$\begin{cases} \Sigma^- := \left\{ (x_1, x_2) : \epsilon_0 < x_1 < a - \epsilon_0, - M_0 < x_2 < 0 \right\} \subset \Omega, \\ \Sigma^+ := \left\{ (x_1, x_2) : \epsilon_0 < x_1 < a - \epsilon_0, 0 < x_2 < M_0 \right\} \subset \mathbb{R}^2\backslash\overline{\Omega}. \end{cases} \tag{9}$$

Clearly, the domain Σ^+ is the mirror image of Σ^- with respect to the x_1-axis.

In case (ii), the parameters ϵ_0 and M_0 are chosen to satisfy

$$\begin{cases} \Sigma^- := \left\{ r(\cos\varphi, \sin\varphi) : \epsilon_0 < \varphi < a - \epsilon_0, R - M_0 < r < R \right\} \subset \Omega, \\ \Sigma^+ := \left\{ r(\cos\varphi, \sin\varphi) : \epsilon_0 < \varphi < a - \epsilon_0, R < r < R + M_0^* \right\} \subset \mathbb{R}^2\backslash\overline{\Omega}, \end{cases} \tag{10}$$

with $M_0^* := R^2/M_0$. Then Σ^+ is the inversion of Σ^- with respect to a circle in the latter case. When Ω is convex, we may take

$$\epsilon_0 = 0, \quad M_0 = \inf_{x\in\Gamma}\{\ell > 0 : x - \ell\nu(x) \in \gamma \bigcup \gamma'\},$$

where $\nu(x)$ is the unit normal direction at $x \in \partial\Omega$ pointing into $\mathbb{R}^2\backslash\overline{\Omega}$.

Lemma 3.4. *Let q be an analytic function on Γ. Then the solution* u *to (1) can be analytically extended into the domain Σ^+. In particular, when $q(x) = q_0$ is a constant, the extension formula is given explicitly by*

$$\text{case (i)} : u(x_1, x_2) = u(x_1, -x_2) + 2q_0 \int_0^{-x_2} \mathrm{e}^{-(x_2+t)q_0} u(x_1, t)\mathrm{d}t; \tag{11}$$

$$\text{case (ii)} : u(r, \varphi) = u\left(r^*, \varphi\right) + 2q_0 \int_R^{r^*} \frac{(t\ r)^{-q_0}}{t} u(t, \varphi)\,\mathrm{d}t, \quad r^* := R^2/r, \tag{12}$$

for $x = (x_1, x_2) = r(\cos\varphi, \sin\varphi) \in \Sigma^+$.

We refer to [6] for a proof performed for general non-singular real-analytic curves in $\mathbb{R}^2$ based on the idea of constructing *reflected fundamental solutions* (see [16]). The explicit extension formulae can also be directly justified following the argument in section 6.2 of this paper. Note that when $q_0 = 0$, (11) and (12) reduce to the classical (even) reflection principle for harmonic functions subject to the Neumann boundary condition. For the readers' convenience, we will present a straightforward justification of (11) for the Helmholtz equation in the appendix. Note that the extension formula (12), which is valid for the harmonic function in the circular case, does not apply to the Helmholtz equation; see [19].

Our arguments in proving the stability mainly depend on an interior conditional estimate for elliptic equations of second order with lateral Cauchy data. The proof is done by applying a Carleman estimate (e.g. [18, 21]) for elliptic equations. For self-contained discussions, we shall prove the Carleman estimate in the appendix. Our proof is more explicit than [18] and [21] for general partial differential equations where the level sets are not given concretely.

We introduce some notation before stating the elliptic interior estimate. An essential ingredient in the proof is the solution estimates in a level set $\Lambda(x, \lambda, \nu) + \delta\nu$ defined below. For $x \in \mathbb{R}^2$, $\lambda > 0$ and a unit vector ν (i.e. $|\nu| = 1$), denote by $\Lambda(x, \lambda, \nu)$ a paraboloidal domain with the vertex located at x and the axis parallel to ν which is congruent to $x_2 < -\lambda x_1^2$. For $\delta > 0$, set

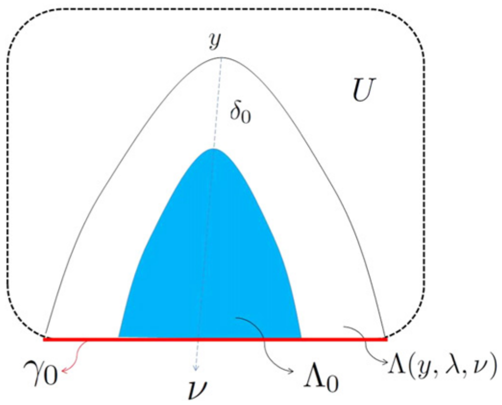

Figure 2. Configurations of $\Lambda_0 := (\Lambda(y, \lambda, \nu) + \delta_0 \nu) \cap U$ with $y \in U$ and $\gamma_0 := \partial U \cap \Lambda(y, \lambda, \nu)$.

$$\Lambda(x, \lambda, \nu) + \delta\nu := \{x : x - \delta\nu \in \Lambda(x, \lambda, \nu)\} = \bigcup_{x \in \Lambda(x,\lambda,\nu)} \{x + \delta\nu\},$$

that is, the translation of $\Lambda(x, \lambda, \nu)$ along the direction ν. The direction of the unit vector ν is supposed to be chosen such that $\Lambda(x, \lambda, \nu) + \delta\nu \subset \Lambda(x, \lambda, \nu)$ for $\delta > 0$.

Let $U \subset \mathbb{R}^2$ be a bounded connected domain. In this paper, the paraboloidal domain $\Lambda(x, \lambda, \nu)$ with $x \in U$ always means the connected component of $\Lambda(x, \lambda, \nu) \cap U$ containing x. Analogously, the notation $\Lambda(x, \lambda, \nu) \cap \partial U$ always means the intersection of the boundary of this connected domain with ∂U. This rule also applies to the paraboloidal domain $\Lambda(x, \lambda, \nu) + \delta\nu$. Note that $\Lambda(x, \lambda, \nu) \cap U$ may have several disconnected components if U is not convex.

Lemma 3.5 **(interior estimate).** *Let $U \subset \mathbb{R}^2$ be a bounded connected domain with the boundary ∂U of C^2-smooth. Let $y \in U$, $\gamma_0 = \partial U \cap \Lambda(y, \lambda, \nu)$ and $\ell = \min\{t : y + t\nu \in \partial U, t > 0\}$. For $0 < \delta_0 < \ell$, set $\Lambda_0 := (\Lambda(y, \lambda, \nu) + \delta_0 \nu) \cap U$ (see figure 2). Suppose that $u \in H^2(U)$ is a solution to the elliptic equation with variable coefficients*

$$\mathcal{L}u := \sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b_i(x) \frac{\partial u}{\partial x_i} + c(x)\, u = 0 \text{in} \quad U, \tag{13}$$

where $a_{ij} = a_{ji} \in C^1(\overline{U})$, $b_i, c \in L^\infty(U)$. Then there exist constants $C > 0$ and $\kappa \in (0, 1)$, which depend on l, δ_0, λ, a_{ij}, b_i and c, such that

$$||u||_{H^1(\Lambda_0)} \leqslant \Big(||u||_{H^1(\gamma_0)} + ||\partial_\nu u||_{L^2(\gamma_0)} \Big) + C \Big(||u||_{H^1(\gamma_0)} + ||\partial_\nu u||_{L^2(\gamma_0)} \Big)^\kappa ||u||^{1-\kappa}_{H^1(U)}. \tag{14}$$

Here C *and* κ *do not depend on* γ_0.

The proof will be carried out in appendix 6.1, based on an elliptic Carleman estimate. Lemma 3.5 yields a stability estimate for u provided that $||u||_{H^1(U)}$ is bounded which is called a conditional stability estimate.

Remark. In the lemma, since $\lambda > 0$ can be chosen arbitrarily large, even if γ_0 is an arbitrarily small sub-boundary, for any $y \in U$ we can construct a family of paraboloidal

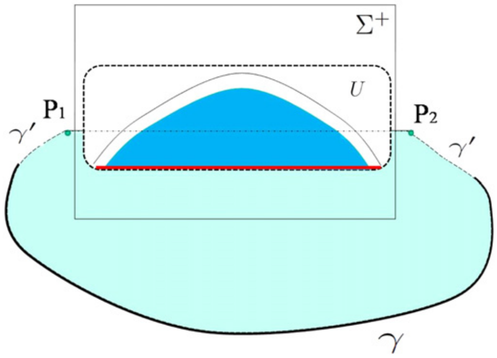

Figure 3. The paraboloidal domains constructed in the proof of theorem 3.1.

domains in U containing y and γ_0 and repeat the argument in lemma 3.5 to prove the uniqueness in the continuation by Cauchy data on an arbitrarily small sub-boundary γ_0, which is a classical result. More precisely, let γ_0 be a non-empty relatively open connected sub-boundary of ∂U. If $\mathcal{L}u = 0$ in U and $u = \partial_\nu u = 0$ on γ_0, then $u \equiv 0$ in U.

3.2. Proof of theorem 3.1

Relying on lemmas 3.2–3.5, we are now ready to prove theorem 3.1.

Proof of theorem 3.1. Since q is an analytic function and $\epsilon > 0$, we have $u \in H^2(\Gamma_\epsilon)$. Set $w = u_1 - u_2$ in Ω. From the Robin boundary conditions

$$\partial_\nu u_j + q_j\, u_j = 0 \quad \text{on} \quad \Gamma,\ \ j = 1, 2,$$

it follows that

$$\Delta w = 0 \quad \text{in} \quad \Omega, \qquad \partial_\nu w + q_1\, w = (q_2 - q_1) u_2 \quad \text{on} \quad \Gamma. \tag{15}$$

Let $\eta > 0$ be the number specified in lemma 3.2, and let $\Gamma_\epsilon \subset \Gamma$ be defined as in (4). Applying lemma 3.2, we derive from the boundary condition for w in (15) that

$$\begin{aligned}\eta^2\, \|q_2 - q_1\|^2_{L^2(\Gamma_\epsilon)} &\leqslant \int_{\Gamma_\epsilon} |q_2 - q_1|^2\, |u_2|^2 ds \\ &= \int_{\Gamma_\epsilon} |\partial_\nu w + q_1\, w|^2\, ds \\ &\leqslant \max\{1, N^2\} \left(\|\partial_\nu w\|^2_{L^2(\Gamma_\epsilon)} + \|w\|^2_{L^2(\Gamma_\epsilon)} \right) \\ &\leqslant \max\{1, N^2\} \left(\|\partial_\nu w\|^2_{L^2(\Gamma_\epsilon)} + \|w\|^2_{H^1(\Gamma_\epsilon)} \right),\end{aligned} \tag{16}$$

where N is the *a priori* upper bound of q_j.

We need to extend w from Ω to a neighborhood of Γ_ϵ. Let the domains $\Sigma^\pm$ be given by (9) and (10), respectively, with some $\epsilon_0 < \epsilon$. Then there holds the relation $\Gamma_\epsilon \subset \Gamma_{\epsilon_0} = \overline{\Sigma^+} \cap \overline{\Sigma^-}$. By lemma 3.4, u_j ($j = 1, 2$) and w can be analytically extended from Ω to Σ^+. Set $\widetilde{\Omega} \coloneqq \Omega \cup \Sigma^+ \cup \Gamma_{\epsilon_0}$ and choose a paraboloidal domain

$$\Lambda(y,\ \lambda,\ \nu) \subset \widetilde{\Omega}, \quad \text{with some} \quad y \in \Sigma^+,\ \lambda > 0,\ \nu \in \mathbb{R}^2,\ |\nu| = 1,$$

such that

$$\Gamma_\epsilon\ \subset\ \Lambda(y,\ \lambda,\ \nu) \bigcap \partial\left(\Sigma^+\right)\ \subset \Gamma_{\epsilon_0}.$$

Obviously, the choice of $\Lambda(y,\ \lambda,\ \nu)$ is not unique, depending on the length of Γ, the height of Σ^+ as well as the number ϵ. Since $\epsilon_0 < \epsilon$, we can always find a smooth domain U and choose a $\delta_0 = \delta_0(\epsilon,\ \epsilon_0,\ \delta) > 0$ such that (see figure 3)

$$\left(\Lambda(y,\ \lambda,\ \nu) \bigcap \Sigma^+\right) \subset\ U\ \subset \widetilde{\Omega}, \qquad \Gamma_\epsilon\ \subset\ \left(\Lambda(y,\ \lambda,\ \nu) + \delta_0\nu\right) \bigcap \Gamma. \tag{17}$$

Now, as done in lemma 3.5 we set

$$\Lambda_0 \coloneqq \left(\Lambda(y,\ \lambda,\ \nu) + \delta_0\nu\right) \bigcap\ U, \quad \gamma_0 \coloneqq \partial U \bigcap\ \Lambda(y,\ \lambda,\ \nu) \subset \Omega.$$

Applying lemma 3.5 to w in U, we arrive at

$$\begin{aligned} &||w||_{H^1(\Lambda_0)} \\ &\leqslant \left(||w||_{H^1(\gamma_0)} + \left|\left|\partial_\nu w\right|\right|_{L^2(\gamma_0)}\right) + C\left(||w||_{H^1(\gamma_0)} + \left|\left|\partial_\nu w\right|\right|_{L^2(\gamma_0)}\right)^\kappa ||w||^{1-\kappa}_{H^1(U)}, \end{aligned} \tag{18}$$

with some $\kappa \in (0,\ 1)$. Since $U \subset \widetilde{\Omega}$, using the extension formula and applying the elliptic interior estimate leads to

$$\left|\left|u_j\right|\right|_{H^2(U)} \leqslant C\ \left|\left|u_j\right|\right|_{H^2(U\bigcap\Omega)} \leqslant C\ \left|\left|u_j\right|\right|_{H^1(\Omega)} \leqslant C,$$

from which the relation $||w||_{H^2(U)} \leqslant C$ follows. Hence, it is seen from (18) that

$$||w||_{H^1(\Lambda_0)} \leqslant C\left(||w||_{H^1(\gamma_0)} + \left|\left|\partial_\nu w\right|\right|_{L^2(\gamma_0)}\right)^{\kappa_0}$$

for some $\kappa_0 \in (0,\ 1]$. Making use of the Sobolev interpolation inequality, we see

$$\begin{aligned} ||w||_{H^{3/2}(\Lambda_0)} &\leqslant C\ ||w||^{1/2}_{H^1(\Lambda_0)}||w||^{1/2}_{H^2(\Lambda_0)} \\ &\leqslant C\left(||w||_{H^1(\gamma_0)} + \left|\left|\partial_\nu w\right|\right|_{L^2(\gamma_0)}\right)^\kappa ||w||^{1/2}_{H^2(U)} \\ &\leqslant C\left(||w||_{H^1(\gamma_0)} + \left|\left|\partial_\nu w\right|\right|_{L^2(\gamma_0)}\right)^\kappa \end{aligned}$$

with $\kappa \coloneqq \kappa_0/2 \in (0,\ 1)$. Since $\Gamma_\epsilon \subset \Gamma_0$ (see (17)), we obtain from the trace lemma that

$$||w||_{H^1(\Gamma_\epsilon)} + \left|\left|\partial_\nu w\right|\right|_{L^2(\Gamma_\epsilon)} \leqslant C\ ||w||_{H^{3/2}(\Lambda_0)} \leqslant C\left(||w||_{H^1(\gamma_0)} + \left|\left|\partial_\nu w\right|\right|_{L^2(\gamma_0)}\right)^\kappa.$$

Together with (16), this proves the estimate

$$\left|\left|q_2 - q_1\right|\right|_{L^2(\Gamma_\epsilon)} \leqslant C\left(||w||_{H^1(\gamma_0)} + \left|\left|\partial_\nu w\right|\right|_{L^2(\gamma_0)}\right)^\kappa, \tag{19}$$

where the constant C depends on M, N, ϵ and $\partial\Omega$. Note that γ_0 is a C^2-smooth curve inside Ω.

To finish the proof, we need to estimate the left-hand side of (19) in terms of the given Cauchy data on γ by modifying the previous arguments as follows. We can first take a piecewise linear curve in Ω connecting a point $y \in \gamma_0$ and an interior point of γ and construct a family of paraboloidal domains in Ω with axes included in the linear curve. Then, we repeat to apply lemma 3.5 in such a family. After a finite number of steps, we can obtain

$$||w||_{H^{3/2}(\gamma_0)} \leqslant C\left(||w||_{H^1(\gamma)} + \left|\left|\partial_\nu w\right|\right|_{L^2(\gamma)}\right)^\kappa.$$

This together with (19) finishes the proof of theorem 3.1. □

4. Estimate of piecewise constant coefficients

Throughout this section Γ is supposed to be the line segment specified in case (i) for some $a > 0$. We consider a more general linear elliptic boundary value problem of second order with constant coefficients

$$\begin{cases} \sum_{i,j=1}^{2} A_{ij} \dfrac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{2} B_i \dfrac{\partial u}{\partial x_i} + c\, u = 0 & \text{in} \quad \Omega, \\ \partial_\nu u + q\, u = 0 & \text{on} \quad \Gamma, \\ u = g & \text{on} \quad \gamma, \\ u = 0 & \text{on} \quad \gamma', \end{cases} \tag{20}$$

where $A_{ij} = A_{ji}$ for $i, j = 1, 2$ and $c \leqslant 0$. In this case, we suppose that the corrosion coefficient $q(x)$ is a piecewise constant function, belonging to the admissible class

$$\mathcal{Q}_\delta := \left\{ q(x) : \begin{array}{l} \text{there exists a finite partition} \bigcup_{j=1}^{N_0} \Gamma_j \text{ of } \Gamma \text{ such that} \\ q(x) = q_j > 0 \text{ on } \Gamma_j \text{ and that } \mathrm{mes}\left(\Gamma_j\right) > \delta, j = 1, 2, \cdots N_0 \end{array} \right\} \tag{21}$$

for some $\delta > 0$. Here we note that $N_0 \in \mathbb{N}$ depends on $q \in \mathcal{Q}_\delta$ and mes (Γ_j) denotes the Lebesgue measure of Γ_j. For two different Robin coefficients $q_j \in \mathcal{Q}_\delta$ $(j = 1, 2)$, we can obtain a stability estimate of the L^∞-norm of the difference $q_1 - q_2$ for the elliptic equation (20). In the following theorem, the number of partitions (i.e. N_0) of Γ is assumed to be unknown.

Theorem 4.1. *Suppose that Γ is the line segment specified in case (i). Let $g_j \in \mathcal{G}_M$ and $q_j \in \mathcal{Q}_\delta$ $(j = 1, 2)$ for some $M > 0$, $\delta > 0$. Denote by $u_j \in H^1(\Omega)$ the unique solution to (20) with $g = g_j$. Then*

$$||q_1 - q_2||_{L^\infty(\Gamma)} \leqslant C \left(||g_1 - g_2||_{H^1(\gamma)} + ||\partial_\nu (u_1 - u_2)||_{L^2(\gamma)} \right)^\kappa \tag{22}$$

for some $C > 0$, $\kappa \in (0, 1)$ depending only on M, N, δ, A_{ij}, B_i, c and Ω.

Theorem 4.1 can be extended to the case where Γ is a non-singular real-analytic curve, provided the configuration of Ω fulfills a certain geometrical condition; see section 5 for a brief discussion. When Γ is circular, our proof carries over to the Laplace equation only by employing the extension formula (12). In the following we state the reflection principle for the linear elliptic equation of second order with constant coefficients.

Lemma 4.2. *Let $q = q_0$ be a positive constant. Then the solution* u *to (20) can be analytically extended into the domain Σ^+. Moreover, the extended solution still satisfies the elliptic equation in (20).*

Lemma 4.2 was first verified in [15] under the Dirichlet, Neumann or Robin boundary conditions in $\mathbb{R}^n$ $(n \geqslant 2)$; see also section 6.2 in the appendix. For piecewise constant Robin coefficients, we cannot extend the solution u from Σ^- to the whole Σ^+ as an analytic function. Based on the extension formula described in lemma 4.2, we shall repeat the arguments in the proof of theorem 3.1 on each sub-interval of Γ. Below we state the regularity of the solution in Ω and the positive lower bound on Γ_ϵ.

Lemma 4.3. *Let $u_g \in H^1(\Omega)$ be the unique solution to (20) for some $g \in \mathcal{G}_M$ and $q \in \mathcal{Q}_\delta$. Assume $\gamma \subset \partial\Omega_\epsilon$ (cf (2)) for some $0 < \epsilon < a/2$. Then*

i. u_g is continuous on $\overline{\Gamma}$. Further, $\min_{x \in \overline{\Gamma}_\epsilon} \{u_g(x)\} \geqslant \eta(\epsilon) > 0$ uniformly in $g \in \mathcal{G}_M$.

ii. $u_g|_\Gamma \in H^{1/2+s}(\Gamma)$, $u_g \in H^{1+s}(\Omega)$ for some $s > 0$.

Proof. When q is a piecewise constant function, u_g is piecewise $C^{1,\alpha}$-smooth on Γ and $u_g \in C^{0,\alpha}(\Gamma)$ for some $0 < \alpha < 1$. The proof was contained in the proof of [30, theorem 3.2]) or [29, lemma 3.3] for the Laplace equation, and carries over to the elliptic equation in (20) without difficulties. Therefore, it holds that $(\partial_\nu u)|_\Gamma = -(qu)|_\Gamma \in H^s(\Gamma)$ for some $0 < s < 1/2$, implying that $u_g \in H^{1+s}(\Gamma_\epsilon)$ for any fixed $a/2 > \epsilon > 0$. The regularity of u_g in a neighborhood of the end points of Γ and the uniformly positive low bound of $u_g|_{\overline{\Gamma}_\epsilon}$ can be verified in the same way as the proof of lemma 3.2. Hence, $u_g|_\Gamma \in H^{1/2+s}(\Gamma)$ and $u_g \in H^{1+s}(\Omega)$ for some $s > 0$. □

Proof of theorem 4.1. Denote by $u_j \in H^{1+s}(\Omega)$ $(s > 0)$ the unique solution to (20) with $g = g_j \in \mathcal{G}_M$, $q = q_j \in \mathcal{Q}_\delta$ for some $M, \delta > 0$. Assume that

$$q_1(x) = \lambda_j \quad \text{on} \quad \Gamma_j^{(1)} := [a_j, a_{j+1}), \quad j = 1, 2, \cdots, N_1, \quad a_1 = 0, \ a_{N_1+1} = a,$$
$$q_2(x) = \mu_j \quad \text{on} \quad \Gamma_j^{(2)} := \left[b_j, b_{j+1}\right), \quad j = 1, 2, \cdots, N_2, \quad b_1 = 0, \ b_{N_2+1} = a,$$

for some $N_1, N_2 > 0$, where $\{\Gamma_j^{(1)}\}_{j=1}^{N_1}$ and $\{\Gamma_j^{(2)}\}_{j=1}^{N_1}$ are two partitions of $\Gamma = \{(x_1, 0) : 0 \leqslant x_1 \leqslant a\}$ satisfying $\mathrm{mes}(\Gamma_j^{(i)}) > \delta$ for $i = 1, 2$ and every j. Let $\epsilon_0 < \delta/2$ and $M_0 > 0$ be given in (9). Without loss of generality, we suppose that $b_2 \leqslant a_2$ and set $I_1 = (\epsilon_0, b_2)$ (otherwise we set $I_1 = (\epsilon_0, a_2)$). For clarity we divide our proof into three steps.

Step 1. Applying lemma 4.2 to u_j with $\Gamma = I_1$, we can extend u_j from Ω to the domain

$$\Sigma_1^+ := \left\{(x_1, x_2) : x_1 \in I_1, \ 0 < x_2 < M_0\right\}.$$

Select a paraboloidal domain $\Lambda(y, \lambda, \nu) \subset \Omega \cup \Sigma_1^+ \cup I_1$ with $y \in \Sigma_1^+$ and a $\delta_0 > 0$ such that

$$mes\left\{\{\Lambda(y, \lambda, \nu) + \delta_0\nu\} \bigcap \Gamma\right\} > (b_2 - \epsilon_0)/2.$$

It is clear that $w := u_1 - u_2 \in H^2(\Omega \cup \Sigma_1^+ \cup I_1)$. Repeating the argument in the proof of theorem 3.1 and making use of the fact that $q_1(x) = \lambda_1$ and $q_2(x) = \mu_1$ on I_1, we arrive at (cf (19))

$$|\lambda_1 - \mu_1|\left(\frac{b_2 - \epsilon_0}{2}\right) \leqslant C\left(||w||_{H^1(\gamma)} + ||\partial_\nu w||_{L^2(\gamma)}\right)^\kappa$$

for some $\kappa \in (0, 1)$. Since $b_2 > \delta$ and $\epsilon_0 < \delta/2$, we see

$$\max_{0 \leqslant x_1 \leqslant b_2} |q_1(x_1) - q_2(x_1)| = |\lambda_1 - \mu_1| \leqslant C\left(||w||_{H^1(\gamma)} + ||\partial_\nu w||_{L^2(\gamma)}\right)^\kappa. \tag{23}$$

Step 2. If $b_3 \leqslant a_2$, then we set $I_2 = (b_2, b_3)$. Since $q_1(x_1) = \lambda_1$ and $q_2(x_1) = \mu_2$ on I_2, by lemma 4.2 we can extend u_j from Ω to

$$\Sigma_2^+ := \left\{(x_1, x_2) : x_1 \in I_2, \ 0 < x_2 < M_0\right\}.$$

By arguing analogously to step 1, we apply the interior estimate of lemma 3.5 to a family of paraboloidal domains contained in $\Omega \cup \Sigma_2^+ \cup I_2$, leading to

$$\max_{b_1 \leqslant x_1 \leqslant b_3} \left| q_1(x_1) - q_2(x_1) \right| \leqslant C \left(||w||_{H^1(\gamma)} + \left|\left| \partial_\nu w \right|\right|_{L^2(\gamma)} \right)^\kappa. \tag{24}$$

Repeating this process, we can obtain an estimate analogous to (24) with b_3 replaced by some b_j satisfying $b_3 \leqslant b_j \leqslant a_2 < b_{j+1}$. To proceed with the proof, we define

$$I_j = \left(b_j, a_2 \right) \quad \text{if} \quad a_2 - b_j > \delta/2, \qquad \text{or} \quad I_j = \left(a_2, b_{j+1} \right) \quad \text{if} \quad b_{j+1} - a_2 > \delta/2.$$

This is possible due to the assumption $b_{j+1} - b_j > \delta$. In either of the two cases, we can obtain an estimate of $\left| \lambda_1 - \mu_j \right|$ bounded by the right-hand side of (24) following the proof of (23). Consequently,

$$\max_{a_1 \leqslant x_1 \leqslant a_2} \left| q_1(x_1) - q_2(x_1) \right| \leqslant C \left(||w||_{H^1(\gamma)} + \left|\left| \partial_\nu w \right|\right|_{L^2(\gamma)} \right)^\kappa. \tag{25}$$

In the case that $b_3 > a_2$, the equality (25) can be verified in the same manner.

Step 3. Considering the interval (a_2, a_3) and repeating the arguments in step 2 yield an estimate of $\left|\left| q_1(x_1) - q_2(x_1) \right|\right|_{L^\infty([b_1, b_{j+1}])}$ with some $b_{j+1} > a_2$ bounded by the right-hand side of (25). After a finite number of steps, we can obtain the estimate (22). Theorem 4.1 is thus proven. □

5. Concluding remarks

In this paper we have derived two Hölder stability estimates in recovering an unknown Robin coefficient from a single pair of Cauchy data taken on the accessible part of the boundary. We assume *a priori* information that the inaccessible boundary Γ is flat or circular, which corresponds to special cases of real-analytic curves. Under such a condition, the solutions to the Laplace or elliptic equation with constant coefficients can be analytically extended into a neighboring (rectangular or annular) area of any sub-interval of Γ in $\mathbb{R}^2 \backslash \overline{\Omega}$ with a constant thickness. This is due to the global extension formula for flat and circular boundaries subject to the Robin boundary condition [6, 15]. If Γ is a non-singular real-analytic curve, then a local extension is still possible only if the Robin coefficient is analytic; see [6]. However, the extended area is determined by both the Schwarz function associated with Γ and the geometrical shape of Ω. Theorems 3.1 and 4.1 can be generalized to non-singular analytic curves, provided the thickness of the extended area keeps a positive lower bound from zero (this can be guaranteed, e.g. by the *a priori* bound on the derivatives of the Robin coefficient [23]). To extend our results to three dimensions, one only needs to analyze the regularity for the Laplace equation with mixed Dirichlet–Robin boundary conditions in a polyhedral or smooth domain, which is more challenging than the 2D case because of the co-existence of corner and edge singularities. Note that the extension formulae and the Carleman estimate all remain valid in higher dimensions.

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Appendix

6.1. Proof of lemma 3.5

To prove lemma 3.5 we need the following Carleman estimate for elliptic equations. Its proof can be found in [18] or [21]. Recall from lemma 3.5 that $U \subset \mathbb{R}^2$ is a bounded connected domain with the C^2-smooth boundary $\partial\Omega$.

Lemma A.1 **(Carleman estimate).** *Let the elliptic operator $\mathcal{L}$ be given by (13). Define the function*

$$\varphi(x) = \mathrm{e}^{\lambda_0\, d(x)}, \qquad \lambda_0 > 0,$$

where $d \in C^2(\overline{U})$ satisfying $|\nabla d| \neq 0$ on $\overline{U}$. Suppose $D \subset U$ is a domain with smooth boundary such that $\overline{D} \subset U$. Then there exist $C > 0$, $s_0 > 0$ such that

$$\int_D \left\{ s\,|\nabla u|^2 + s^3\,|u|^2 \right\} \mathrm{e}^{2s\varphi}\,\mathrm{d}x \leqslant C \int_D |\mathcal{L}u|^2 \mathrm{e}^{2s\varphi}\,\mathrm{d}x + Ce^{Cs} \int_{\partial D} \left\{ |\nabla u|^2 + |u|^2 \right\} \mathrm{d}s \tag{26}$$

for all $s > s_0$ and all $u \in H^2(D)$.

Proof of lemma 3.5. Without loss of generality, by translation and rotation we can formulate the paraboloidal domain in lemma 3.5 as

$$\Lambda(y, \lambda, \nu) = \left\{ (x_1, x_2) : x_2 < -\lambda x_1^2 + \ell \right\}, \qquad \lambda, \ell > 0$$

with $\nu = (0, -1)$, $y = (0, \ell)$ and we assume that the origin O is on γ_0 and U is located above γ_0 near O; see figure 2. In order to apply lemma A.1, we set

$$d(x) = -x_2 - \lambda x_1^2 + \ell, \qquad D_\delta = \{x \in U : d(x) > \delta\}, \qquad \delta \geqslant 0.$$

We note that $D_{\delta_2} \subset D_{\delta_1}$ if $\delta_1 < \delta_2$ and

$$D_\delta = (\Lambda(y, \lambda, \nu) + \delta\nu) \bigcap U.$$

We have Cauchy data on $\gamma_0 = \partial D_0 \cap \partial U$ but no data on $\partial D_0 \backslash \partial U$. Thus we introduce a cut-off function $\chi \in C^\infty(\mathbb{R}^2)$ such that (see e.g. [21])

$$0 \leqslant \chi \leqslant 1, \qquad \chi(x) = \begin{cases} 1 & \text{in} \quad D_{2\delta_0/3}, \\ 0 & \text{in} \quad D_0 \backslash D_{\delta_0/3}, \end{cases}$$

where $0 < \delta_0 < \ell$ is specified in lemma 3.5. Set $v = \chi u$. Then $v = 0$ on $\partial D_0 \backslash \partial U$ and

$$\mathcal{L}v = \sum_{i,j=1}^{2} a_{ij} \left\{ \frac{\partial\chi}{\partial x_i}\frac{\partial u}{\partial x_j} + \frac{\partial\chi}{\partial x_j}\frac{\partial u}{\partial x_i} + \frac{\partial^2\chi}{\partial x_i \partial x_j} u \right\} + \sum_{i=1}^{2} b_i \frac{\partial\chi}{\partial x_i} u, \qquad x \in U. \tag{27}$$

Applying the Carleman estimate (26) to v in D_0 and taking $s_0 > 0$ sufficiently large yield

$$\begin{aligned}
&\int_{D_0} \left\{ s\,|\nabla v|^2 + s^3\,|v|^2 \right\} \mathrm{e}^{2s\varphi}\,\mathrm{d}x \\
&\quad \leqslant C \int_{D_0} \left| \sum_{i,j=1}^{2} a_{ij} \left\{ \frac{\partial \chi}{\partial x_i}\frac{\partial u}{\partial x_j} + \frac{\partial \chi}{\partial x_j}\frac{\partial u}{\partial x_i} + \frac{\partial^2 \chi}{\partial x_i \partial x_j} u \right\} \right|^2 \mathrm{e}^{2s\varphi}\,\mathrm{d}x \\
&\qquad + Ce^{Cs} \int_{\gamma_0} \left\{ |\nabla v|^2 + |v|^2 \right\} \mathrm{d}s
\end{aligned} \tag{28}$$

for all $s > s_0$. In the Carleman estimate, the weighted L^2-norms of the lower order terms on the right-hand side of (27) have been absorbed by the left-hand side of (28). More precisely, the first integral on the right-hand side of (28) does not vanish only if $x \in D_{\delta_0/3} \backslash D_{2\delta_0/3}$, because χ is constant in $D_{2\delta_0/3}$ and $D_0 \backslash D_{\delta_0/3}$. Therefore, observing that

$$\begin{aligned}
&d(x) > \delta_0, && \varphi(x) > \mathrm{e}^{\lambda_0 \delta_0} && \text{for} \quad x \in D_{\delta_0}, \\
&d(x) \leqslant 2\delta_0/3, && \varphi(x) \leqslant \mathrm{e}^{2\lambda_0 \delta_0/3} && \text{for} \quad x \in D_{\delta_0/3} \backslash D_{2\delta_0/3},
\end{aligned}$$

the first integral on the right-hand side is bounded by

$$Ce^{2s\,\exp(2\lambda_0\delta_0/3)} \int_{D_{\delta_0/3}\backslash D_{2\delta_0/3}} \left\{ s\,|\nabla u|^2 + s^3\,|u|^2 \right\} \mathrm{d}x.$$

By the definition of χ and the fact that $D_{\delta_0} \subset D_0$,

$$\begin{aligned}
\int_{D_{\delta_0}} \left\{ s\,|\nabla u|^2 + s^3\,|u|^2 \right\} \mathrm{e}^{2s\varphi}\,\mathrm{d}x &= \int_{D_{\delta_0}} \left\{ s\,|\nabla v|^2 + s^3\,|v|^2 \right\} \mathrm{e}^{2s\varphi}\,\mathrm{d}x \\
&\leqslant \int_{D_0} \left\{ s\,|\nabla v|^2 + s^3\,|v|^2 \right\} \mathrm{e}^{2s\varphi}\,\mathrm{d}x.
\end{aligned} \tag{29}$$

Hence we derive from (28) and (29) that

$$\begin{aligned}
&e^{2s\,\exp(\lambda_0\delta_0)} \int_{D_{\delta_0}} \left\{ s\,|\nabla u|^2 + s^3\,|u|^2 \right\} \mathrm{d}x \\
&\quad \leqslant \int_{D_{\delta_0}} \left\{ s\,|\nabla u|^2 + s^3\,|u|^2 \right\} \mathrm{e}^{2s\varphi}\,\mathrm{d}x \\
&\quad \leqslant C\, e^{2s\,\exp(2\lambda_0\delta_0/3)} \int_{D_{\delta_0/3}\backslash D_{2\delta_0/3}} \left\{ s\,|\nabla u|^2 + s^3\,|u|^2 \right\} \mathrm{d}x + Ce^{Cs} \int_{\gamma_0} \left\{ |\nabla u|^2 + |u|^2 \right\} \mathrm{d}s
\end{aligned}$$

for all $s \geqslant s_0$, which implies that

$$||u||^2_{H^1(D_{\delta_0})} \leqslant Ce^{-2sr_0}\, ||u||^2_{H^1(U)} + Ce^{Cs} \left(||u||^2_{H^1(\gamma_0)} + \left|\left| \partial_\nu u \right|\right|^2_{L^2(\gamma_0)} \right), \quad s \geqslant s_0,$$

for some $0 < r_0 < \exp(\lambda_0\delta_0) - \exp(2\lambda_0\delta_0/3)$. Replacing C by Ce^{Cs_0}, we reach

$$||u||^2_{H^1(D_{\delta_0})} \leqslant C\, e^{-2sr_0}\, ||u||^2_{H^1(U)} + Ce^{Cs} \left(||u||^2_{H^1(\gamma_0)} + \left|\left| \partial_\nu u \right|\right|^2_{L^2(\gamma_0)} \right), \quad s \geqslant 0. \tag{30}$$

We consider two cases separately.

Case (a) Let $||u||^2_{H^1(U)} > ||u||^2_{H^1(\gamma_0)} + ||\partial_\nu u||^2_{L^2(\gamma_0)}$. Then we can choose $s > 0$ such that

$$e^{-2sr_0} \ ||u||^2_{H^1(U)} = e^{Cs}\Big(||u||^2_{H^1(\gamma_0)} + \big|\big|\partial_\nu u\big|\big|^2_{L^2(\gamma_0)}\Big)$$

which can make the right-hand side of (30) close to the minimum in s. That is, setting

$$s = \frac{1}{C + 2r_0} \log \frac{||u||^2_{H^1(U)}}{||u||^2_{H^1(\gamma_0)} + \big|\big|\partial_\nu u\big|\big|^2_{L^2(\gamma_0)}},$$

we see from (30) that

$$\begin{aligned} ||u||^2_{H^1(D_{\delta_0})} &\leqslant 2Ce^{Cs}\Big(||u||^2_{H^1(\gamma_0)} + \big|\big|\partial_\nu u\big|\big|^2_{L^2(\gamma_0)}\Big) \\ &= 2C \ ||u||^{2C/(C+2r_0)}_{H^1(U)}\Big(||u||^2_{H^1(\gamma_0)} + \big|\big|\partial_\nu u\big|\big|^2_{L^2(\gamma_0)}\Big)^{2r_0/(C+2r_0)} \\ &= 2C \ ||u||^{2(1-\kappa)}_{H^1(U)}\Big(||u||^2_{H^1(\gamma_0)} + \big|\big|\partial_\nu u\big|\big|^2_{L^2(\gamma_0)}\Big)^{\kappa} \end{aligned} \tag{31}$$

with $\kappa = 2r_0/(C + 2r_0) \in (0, 1)$.

Case (b). Let

$$||u||^2_{H^1(U)} \leqslant ||u||^2_{H^1(\gamma_0)} + \big|\big|\partial_\nu u\big|\big|^2_{L^2(\gamma_0)}.$$

Then

$$||u||_{H^1(D_{\delta_0})} \leqslant ||u||_{H^1(U)} \leqslant ||u||_{H^1(\gamma_0)} + \big|\big|\partial_\nu u\big|\big|_{L^2(\gamma_0)}. \tag{32}$$

Combining (31) and (32), we finally obtain

$$\begin{aligned} ||u||_{H^1(D_{\delta_0})} \leqslant 2C\,||u||^{1-\kappa}_{H^1(U)}\Big(||u||_{H^1(\gamma_0)} + \big|\big|\partial_\nu u\big|\big|_{L^2(\gamma_0)}\Big)^{\kappa} \\ + \Big(||u||_{H^1(\gamma_0)} + \big|\big|\partial_\nu u\big|\big|_{L^2(\gamma_0)}\Big), \end{aligned}$$

which proves lemma 3.5. □

6.2. Proof of lemma 4.2

In this subsection, we first provide a direct proof of the extension formula (11) for the Helmholtz equation subject to the Robin boundary condition, and then remark on how to extend it to elliptic equations with constant coefficients. Let

$$\begin{aligned} \Sigma^- &= \big\{ (x_1, x_2) : 0 < x_1 < b,\ x_2 < 0\big\}, \\ \Sigma^+ &= \big\{ (x_1, x_2) : 0 < x_1 < b,\ x_2 > 0\big\}, \\ \Gamma &= \big\{ (x_1, x_2) : 0 < x_1 < b,\ x_2 = 0\big\}. \end{aligned}$$

Lemma A.2. *Assume $\kappa, q_0 \in \mathbb{C}$ and*

$$(\Delta + \kappa)u = 0 \quad \text{in} \quad \Sigma^-, \qquad \frac{\partial u}{\partial x_2} + q_0 u = 0 \quad on \quad \Gamma.$$

Define $v := u$ in Σ^- and

$$v(x_1, x_2) := u(x_1, -x_2) + 2q_0 \int_0^{-x_2} \mathrm{e}^{-(x_2+t)q_0} u(x_1, t)\mathrm{d}t \qquad \text{in} \quad \Sigma^+.$$

Then $(\Delta + \kappa)v = 0$ in $\Sigma^+ \cup \Gamma \cup \Sigma^-$.

Proof. Straightforward computations show that

$$v^+ = u^-, \quad \frac{\partial v^+}{\partial x_2} = -\frac{\partial u^-}{\partial x_2} - 2q_0 u^- \quad \text{on} \quad \Gamma,$$

where the superscripts $(\,\cdot\,)^\pm$ denote the limits taken from above and below, respectively. This implies that

$$\frac{\partial v^+}{\partial x_2} + q_0 v^+ = -\frac{\partial u^-}{\partial x_2} - q_0 u^- = 0 \quad \text{on} \quad \Gamma.$$

Hence $v^+ = v^-$ and $\partial_\nu v^+ = \partial_\nu v^-$ on Γ. To prove the lemma, we only need to verify that v satisfies the Helmholtz equation in Σ^+. Since this is true for the function $x \to u(x_1, -x_2)$ and q_0 is a constant, it is sufficient to check that $(\Delta + \kappa)U = 0$ in Σ^+, where

$$U(x) = \int_0^{-x_2} \mathrm{e}^{-(x_2+t)q_0} u(x_1, t)\mathrm{d}t, \qquad x \in \Sigma^+.$$

By elementary calculations, we see

$$\begin{aligned}
\partial_2^2 U(x_1, x_2) &= q_0^2\, U(x_1, x_2) + q_0\, u(x_1, -x_2) + \partial_2 u(x_1, -x_2),\\
\partial_1^2 U(x_1, x_2) &= \int_0^{-x_2} \mathrm{e}^{-(x_2+t)q_0} \partial_1^2 u(x_1, t)\mathrm{d}t\\
&= -\int_0^{-x_2} \mathrm{e}^{-(x_2+t)q_0} \Big(\partial_t^2 + \kappa\Big) u(x_1, t)\mathrm{d}t\\
&= -\kappa\, U(x) - \int_0^{-x_2} \mathrm{e}^{-(x_2+t)q_0}\, \partial_t^2 u(x_1, t)\mathrm{d}t.
\end{aligned}$$

Integrating by parts twice and using the Robin boundary condition, we have

$$\int_0^{-x_2} \mathrm{e}^{-(x_2+t)q_0}\, \partial_t^2 u(x_1, t)\mathrm{d}t = \partial_2 u(x_1, -x_2) + q_0 u(x_1, -x_2) + q_0^2 U(x_1, x_2).$$

Combining the previous three equations, we obtain $(\Delta + \kappa)U = 0$ which completes the proof. □

Proof of lemma A.2. The linear elliptic equation with constant coefficients in (20) can be reduced to the Helmholtz equation via an appropriate orthogonal transformation of variables and a certain function substitution. Moreover, under such transformations the Robin boundary condition with a constant coefficient is transformed into the same kind. Hence, the first part of lemma A.2 is a consequence of lemma A.2 (see [15] for the details). Note that the extended function in Σ^+ still satisfies the original elliptic equation, since the transformations involved are invertible. This together with lemma A.2 completes the proof of lemma A.2. □

References

[1] Alessandrini G, del Piero L and Rondi L 2003 Stable determination of corrosion by a single electrostatic boundary measurement *Inverse Problems* **19** 973–84
[2] Alessandrini G and Sincich E 2006 Detecting nonlinear corrosion by electrostatic measurements *Appl. Anal.* **85** 107–28
[3] Alessandrini G and Sincich E 2007 Solving elliptic Cauchy problems and the identification of nonlinear corrosion *J. Comput. Appl. Math.* **198** 307–20
[4] Bacchelli V 2009 Uniqueness for the determination of unknown boundary and impedance with the homogeneous Robin condition *Inverse Problems* **25** 015004
[5] Baratchart L, Bourgeois L and Leblond J Uniqueness results for inverse Robin problems with bounded coefficient arXiv:1412.3283
[6] Belinskiy B P and Savina T V 2008 The Schwarz reflection principle for harmonic functions in $\mathbb{R}^2$ subject to the Robin condition *J. Math. Anal. Appl.* **348** 685–91
[7] Bourgeois L 2013 A remark on Lipschitz stability for inverse problems *Comptes Rendus Mathematique* **351** 187–90
[8] Buttazzo G and Khon R V 1987 Reinforcement by a thin layer with oscillating thickness *Appl. Math. Optim.* **16** 247–61
[9] Cakoni F and Kress R 2007 Integral equations for inverse problems in corrosion detection from partial Cauchy data *Inverse Problems and Imaging* **1** 229–45
[10] Chaabane S, Fellah I, Jaoua M and Leblond J 2004 Logarithmic stability estimates for a Robin coefficient in two-dimensional Laplace inverse problems *Inverse Problems* **20** 47–59
[11] Chaabane S and Jaoua M 1999 Identification of Robin coefficients by the means of boundary measurements *Inverse Problems* **15** 1425–38
[12] Cheng J, Choulli M and Lin J 2008 Stable determination of a boundary coefficient in an elliptic equation *Math. Models Methods Appl. Sci.* **107** 107–23
[13] Choulli M 2002 Stability estimates for an inverse elliptic problem *J. Inverse Ill-Posed Probl.* **10** 601–10
[14] Choulli M 2004 An inverse problem in corrosion detection: Stability estimates *J. Inverse Ill-Posed Probl.* **12** 349–67
[15] Diaz J B and Ludford G S S 1955 Reflection principles for linear elliptic second order partial differential equations with constant coefficients *Annali di Matematica Pura ed Applicata* **39** 87–95
[16] Garabedian P R 1960 Partial differential equations with more than two independent variables in the complex domain *J. Math. Mech.* **9** 241–71
[17] Grisvard P 1985 *Boundary Value Problems in Non-smooth Domains* (*Monographs and Studies in Mathematics*) (Boston, MA: Pitman)
[18] Hörmander L 1963 *Linear Partial Differential Operators* (New York: Springer)
[19] Hu G and Liu X 2014 Unique determination of balls and polyhedral scatterers with a single point source wave *Inverse Problems* **30** 065010
[20] Inglese G 1997 An inverse problem in corrosion detection *Inverse Problems* **13** 977–94
[21] Isakov V 2006 Inverse problems for partial differential equations *Applied Mathematical Sciences* (*127*) (New York: Springer)
[22] Kaup P G and Santosa F 1995 Nondestructive evaluation of corrosion damage using electrostatic measurements *J. Nondestructive Eval.* **14** 127–36
[23] Labreuche C 1999 Stability of the recovery of surface impedances in inverse scattering *J. Math. Anal. Appl.* **231** 161–76
[24] Maz'ya V, Nazarov S and Plamenevskij B A 2000 *Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains* ed G Heinig and C Posthoff Vol. I (Berlin: Birkhäuser) (translated from German)
[25] Mghazli Z 1992 Regularity of an elliptic problem with mixed Dirichlet–Robin boundary conditions in a polygonal domain *Calcolo* **29** 241–67
[26] Morrey C 1966 *Multiple Integrals in the Calculus of Variations* (Berlin: Springer)
[27] Santosa F, Vogelius M and Xu J M 1998 An effective nonlinear boundary condition for a corroding surface. Identification of the damage based on steady state electric data *Z. Angew. Math. Phys.* **49** 656–79

[28] Schwarz H A 1869 Über einige Abbildungsaufgaben *Journal für die reine und angewandte Mathematik* **70** 105–20 (See also Gesammelte Mathematische Abhandlungen, Berlin, 1890, 65-83).
[29] Sincich E 2005 Stability and reconstruction for the determination of boundary terms by a single measurements *PhD Thesis* SISSA-ISAS, Trieste
[30] Sincich E 2007 Lipschitz stability for the inverse Robin problem *Inverse Problems* **23** 1311–26
[31] Vogelius M and Xu J M 1998 A nonlinear elliptic boundary value problem related to corrosion modeling *Q. Appl. Math.* **56** 479–505
[32] Wendland W, Stephan E and Hsiao G 1979 On the integral equation method for the plane mixed boundary problem of the Laplacian *Math. Meth. Appl. Sci.* **1** 265–321