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IOP Publishing Inverse Problems

Inverse Problems **34** (2018) 075002 (13pp) https://doi.org/10.1088/1361-6420/aabe56

Interface with weakly singular points always scatter

Long Li[1,2], Guanghui Hu[3,4] and Jiansheng Yang[1,2]

[1] LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, People's Republic of China
[2] Beijing Advanced Innovation Center for Imaging Technology, Capital Normal University, Beijing 100048, People's Republic of China
[3] Beijing Computational Science Research Center, Beijing 100193, People's Republic of China

E-mail: 1601110046@pku.edu.cn, hu@csrc.ac.cn and jsyang@pku.edu.cn

Received 1 January 2018, revised 24 March 2018
Accepted for publication 16 April 2018
Published 8 May 2018

Abstract

Assume that a bounded scatterer is embedded into an infinite homogeneous isotropic background medium in two dimensions. The refractive index function is supposed to be piecewise constant. If the scattering interface contains a weakly singular point, we prove that the scattered field cannot vanish identically. This implies the absence of non-scattering energies for piecewise analytic interfaces with one singular point. Local uniqueness is obtained for shape identification problems in inverse medium scattering with a single far-field pattern.

Keywords: uniqueness, inverse medium scattering, non-scattering energy, weakly singular corners, Helmholtz equation

1. Introduction

Assume a time-harmonic incoming wave u^{in} is incident onto a bounded penetrable scatterer $D \subset \mathbb{R}^2$ embedded in a homogeneous isotropic background medium. We assume that the boundary ∂D is Lipschitz continuous and piecewise analytic, and that the complement $D^e := \mathbb{R}^2 \backslash \overline{D}$ of D is connected. The wave propagation of the total field $u = u^{\text{in}} + u^{\text{sc}}$ is then modeled by the Helmholtz equation

$$\Delta u + k^2 q\, u = 0 \quad \text{in} \quad \mathbb{R}^2. \tag{1.1}$$

In this paper the refractive index (potential) function q is supposed to be a piecewise constant function, given by

[4] Author to whom any correspondence should be addressed.

$$q(x) = \begin{cases} 1, & \text{if} \quad x \in D^e, \\ q_0 \neq 1, & \text{if} \quad x \in \overline{D}. \end{cases}$$

Across the interface ∂D, we assume the continuity of the total field and its normal derivative, i.e.

$$u^+ = u^-, \quad \partial_\nu u^+ = \partial_\nu u^- \quad \text{on } \partial D. \tag{1.2}$$

Here the superscripts $(\cdot)^\pm$ stand for the limits taken from outside and inside, respectively, and $\nu \in \mathbb{S} := \{x \in \mathbb{R}^2 : |x| = 1\}$ is the unit normal on ∂D pointing into D^e. At the infinity, the perturbed scattered field u^{sc} is supposed to fulfill the Sommerfeld radiation condition

$$\lim_{|x|\to\infty} \sqrt{|x|} \left\{ \frac{\partial u^{\text{sc}}}{\partial |x|} - \mathrm{i}k u^{\text{sc}} \right\} = 0. \tag{1.3}$$

The unique solvability of the scattering problem (1.1), (1.3) and (1.2) in $H^2_{\text{loc}}(\mathbb{R}^2)$ is well known (see e.g. [6, chapter 8]). In particular, the Sommerfeld radiation condition (1.3) leads to the asymptotic expansion

$$u^{\text{sc}}(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{\sqrt{|x|}} u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|^{3/2}}\right), \quad |x| \to +\infty, \tag{1.4}$$

uniformly in all directions $\hat{x} := x/|x|$, $x \in \mathbb{R}^2$. The function $u^\infty(\hat{x})$ is an analytic function defined on $\mathbb{S}$ and is referred to as the *far-field pattern* or the *scattering amplitude*. The vector $\hat{x} \in \mathbb{S}$ is called the observation direction of the far field. The classical inverse scattering problem consists of the recovery of the boundary ∂D from the far-field patterns corresponding to one or several incident plane waves. In this paper we are concerned with the following questions:

(i) Is it possible that $u^{\text{sc}} \equiv 0$ in D^e for any incident wave ?
(ii) Does the far-field pattern of a single incoming wave uniquely determine ∂D ?

The obstacle D scatters an incident wave non-trivially, if u^{sc} does not vanish identically in D^e. A negative answer to the first question means that acoustic cloaking cannot be achieved using isotropic materials, while a positive answer implies that k^2 is a non-scattering wavenumber (energy). The study of non-scattering energies dates back to [13] in the case of a convex (planar) corner domain, where notion of *scattering support* for an inhomogeneous medium was explored. In one of the authors' previous work [8], it was shown that variable potential functions with the following corners on ∂D:

- curvilinear polygonal corners in $\mathbb{R}^2$;
- curvilinear polyhedral corners in $\mathbb{R}^3$;
- circular conic corners in $\mathbb{R}^3$;

scatters every incident wave non-trivially. Earlier publications were devoted to the absence of non-scattering energies under more restrictive assumptions on the smoothness of the potential or the angle of the corner. Here we mention the following works in the acoustic case:

- C^∞-potentials with rectangular corners in $\mathbb{R}^n$ $(n \geqslant 2)$ [1];
- Hölder continuous potentials with convex corners in $\mathbb{R}^2$, and with circular conic corners in $\mathbb{R}^3$ whose opening angle is outside of a countable subset of $(0, \pi)$ [14];
- analytical potentials with arbitrary polygonal corners or polyhedral wedge corners [7];
- Hölder continuous potentials with rectangular corners in $\mathbb{R}^3$ [9].

The argument of the pioneering work [1] was based on the use of complex geometric optics (CGO) solutions, which was later extended to [14] and [9] for treating less regular potentials and convex corners. The approach of [7] relies on the expansion of solutions to the Helmholtz equation with real-analytic potentials. For general potentials and corners, the absence of non-scattering energies can be verified via singularity analysis of the inhomogeneous Laplace equation in a cone [8]. We remark that the first question is closely related to the second one, that is, the approach for proving absence of non-scattering wavenumbers can be used to derive uniqueness to shape identification problems in inverse medium scattering. However, the two questions are not equivalent to each other. The second question seems more difficult than the first one. It was first proved in [7] that the shape of a convex penetrable obstacle of polygonal or polyhedral type with an unknown analytical potential can be uniquely determined by a single far-field pattern. The CGO-solution methods of [1, 14] also lead to uniqueness in shape identification but are confined so far to convex polygons in $\mathbb{R}^2$ and rectangular boxes in $\mathbb{R}^3$ with Hölder continuous potentials (see [9]). In [8, corollay 2.1], the uniqueness result of [7] was extended to more general potential functions using the data of a single far-field pattern.

2. Main results

The main purpose of this paper is to exclude (positive) real non-scattering energies when ∂D contains a weakly singular corner, around which the boundary is allowed to be C^1-smooth but piecewise analytic. The corners mentioned in the previous section are all strongly singular in the following sense.

Definition 2.1. A point $O \in \partial D \subset \mathbb{R}^2$ is called strongly singular if the boundary around O can be locally parameterized by a continuous and piecewise analytic function whose derivative is discontinuous at O.

Evidently, every planar corner point of a polygon with flat slides is strongly singular, because the boundary can be locally parameterized by a piecewise linear function, whose first derivative is piecewise constant. A curvilinear corner of D (see e.g. definition 2.1 of [8] for a precise description) is also strongly singular by definition 2.1. Below we state the definition of weakly singular corners to be explored within the scope of this paper.

Definition 2.2. A point $O = (0, 0) \in \partial D$ is called weakly singular if the subboundary $B_\epsilon(O) \cap \partial D$ for some $\epsilon > 0$ can be parameterized by the polynomial function $x_2 = f(x_1)$, $x_1 \in (-1, 1)$, where

$$f(x_1) = \begin{cases} c_1 x_1^{\alpha_1}, & 1 > x_1 \geqslant 0, \\ c_2 x_1^{\alpha_2}, & -1 < x_1 \leqslant 0, \end{cases} \tag{2.1}$$

and the coefficients $c_j \in \mathbb{R}$ and $\alpha_j \in \mathbb{N}^+$ are assumed to fulfill the relations

$$(c_1, \alpha_1) \neq (c_2, \alpha_2),\ c_1^2 + c_2^2 \neq 0,\ \ \alpha_j \geqslant 2.$$

The order of the singularity at O is defined as

$$\beta := \begin{cases} \min\{\alpha_1, \alpha_2\} & \text{if} \quad c_1 \neq 0, c_2 \neq 0, \\ \alpha_1 & \text{if} \quad c_2 = 0, \\ \alpha_2 & \text{if} \quad c_1 = 0. \end{cases}$$

The boundary around a weakly singular point of order β is $C^{\beta-1}$-smooth but piecewise C^{β}-smooth, that is, the βth derivative is discontinuous at O. A singular point of order one must be strongly singular in the sense of definition 2.1.

Definition 2.3. A point $O \in \partial D$ is called singular if it is either strongly singular in the sense of definition 2.1 or weakly singular in the sense of definition 2.2.

The singular points defined by definition 2.3 form only a subset of non-analytic points of the boundary. In fact, the polynomial functions described in (2.1) can be regarded as the leading terms of the Taylor expansion of an analytic function at $x_1 = 0^{\pm}$. If q is a piecewise constant function in $\mathbb{R}^2$, we shall prove that

Theorem 2.4. *The obstacle $D \subset \mathbb{R}^2$ scatters every incoming wave, if ∂D contains a singular point.*

Only local properties of the Helmholtz equation are involved in the proof of theorem 2.4. Consequently, we get a local uniqueness result to the inverse scattering for shape identification:

Theorem 2.5. *Let D_j ($j = 1, 2$) be two penetrable obstacles in $\mathbb{R}^2$ with the piecewise constant potential functions q_j, respectively. If ∂D_2 differs from ∂D_1 in the presence of a singular point lying on the boundary of the unbounded component of $\mathbb{R}^2 \backslash \overline{(D_1 \cup D_2)}$, then the far-field patterns corresponding to D_j and q_j incited by any incoming wave cannot coincide.*

Theorem 2.5 can be used to distinguish two penetrable scatterers with a piecewise constant potential. Equivalently, theorem 2.5 can be reformulated as follows:

Corollary 2.6. *Let $D_j \subset \mathbb{R}^2$ ($j = 1, 2$) be two penetrable obstacles in $\mathbb{R}^2$ with the piecewise constant potential functions q_j, respectively. Assume that ∂D_j are piecewise analytic and all non-analytical points of the boundary are singular corners defined by definition 2.3. If the far-field patterns corresponding to (D_j, q_j) incited by a single incoming wave are identical, then the boundary of the unbounded component of $\mathbb{R}^2 \backslash \overline{(D_1 \cup D_2)}$ must be analytic.*

If ∂D possesses a strongly singular corner, the proofs of theorems 2.4 and 2.5 are implicitly contained in [8, proposition 12]. In the case that D is a polygon, the proofs simply follow from [7] where variable analytical potential functions were treated. The main contribution of this paper is to deal with scattering interfaces with weakly singular points in $\mathbb{R}^2$ (that is, the order of singularity is $\beta \geqslant 2$). The above results imply that a Lipschitz domain with a singular point on the boundary cannot scatter every incoming wave trivially in two dimensions. Theorem 2.4 follows straightforwardly from lemma 3.1 below for weakly singular points. Theorem 2.5 and corollary 2.6 can be verified in the same manner as the proof of theorem 2.4. We omit the proofs for simplicity. Note that if the far-field data is available for all incident directions but at fixed energy, uniqueness was verified based on the idea of Isakov; see [11, 12]. We also refer to [2, 10] and [6, chapter 10] for unique determination of potential functions from the data of infinitely many plane waves or the Dirichlet-to-Neumann map.

3. Weakly singular points always scatter

This section is devoted to the proof of theorem 2.4 when ∂D contains a weakly singular point. Without loss of generality, we suppose that $O = (0, 0)$ is the singular point lying on ∂D. Assuming that u^{sc} vanishes in D^e, we shall derive a contradiction. Suppose that the boundary ∂D in a neighborhood of O can be expressed as $\Gamma = \{(x_1, f(x_1)) : x_1 \in (-1, 1)\}$, where the function f is given by (2.1). Since $u^{sc} = 0$ in D^e, the Cauchy data of u on Γ coincide with those

of u^{in}, which are analytic. Observing that q is a constant on $\overline{D}$ and Γ is piecewise analytic, by Cauchy–Kovalevskaya theorem, one may extend u analytically from $\overline{D} \cap B_1$ to a small neighborhood of O in the exterior domain $D^e \cap B_1$. For notational convenience, we suppose the extended domain contains B_1. Further, the extended function, which we still denote by u, satisfies the Helmholtz equation

$$\Delta u + k^2 q_0 u = 0 \quad \text{in} \quad B_1.$$

Hence, we deduce the transmission problem for the Helmholtz equations

$$\begin{cases} \Delta u_j + q_j u_j = 0, \quad j = 1, 2, & \text{in} \quad B_1, \\ u_1 = u_2, \quad \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} & \text{on} \quad \Gamma, \end{cases} \tag{3.1}$$

where

$$u_1 = u^{\text{in}}, \quad u_2 = u, \qquad q_1 = k^2, \quad q_2 = k^2 q_0.$$

To prove theorem 2.4 in the weakly singular case, it is essential to prove that

Lemma 3.1. *Suppose that $u_j \in H^2(B_1)$ $(j = 1, 2)$ are solutions to (3.1). If $q_1 \neq q_2$, then $u_1 = u_2 \equiv 0$ in B_1.*

By lemma 3.1 and the unique continuation, u^{in} vanishes identically in $\mathbb{R}^2$ which is impossible. Hence, a piecewise constant potential with a weakly singular point on the boundary of the support always scatter. It seems non-trivial to prove the above lemma by extending the analysis for treating curvilinear corners [8, proposition 12] to the case of weakly singular points. The analytical approach of using polar coordinates (see [7]) also turns out to be complicated. Below we shall present a novel approach by using the expansion of solutions to the Helmholtz equation in the Cartesian coordinate system.

Since u_j satisfies the Helmholtz equation and q_j is constant, the solution u_j is analytic in B_1. Hence, u_j can be expanded into the convergent Taylor expansion

$$u_j = \sum_{n \geqslant 0} \sum_{m \geqslant 0} a_{n,m}^{(j)} x_1^n x_2^m \quad \text{in} \quad B_1,$$

where the coefficients $a_{n,m}^{(j)} \in \mathbb{C}$ satisfy the relation

$$(n+1)(n+2) a_{n+2,m}^{(j)} + (m+1)(m+2) a_{n,m+2}^{(j)} + q_j a_{n,m}^{(j)} = 0. \tag{3.2}$$

Set $u = u_1 - u_2$. Then u admits the Taylor expansion

$$u = \sum_{n \geqslant 0} \sum_{m \geqslant 0} a_{n,m} x_1^n x_2^m \quad \text{in} \quad B_1, \qquad a_{n,m} := a_{n,m}^{(1)} - a_{n,m}^{(2)},$$

and satisfies the equation $\Delta u + q_1 u = (q_2 - q_1) u_2$ in B_1. The implies that the coefficients $a_{n,m}$ fulfills the recursive relation

$$(n+1)(n+2) a_{n+2,m} + (m+1)(m+2) a_{n,m+2} + q_1 a_{n,m} = (q_2 - q_1) a_{n,m}^{(2)}. \tag{3.3}$$

Combining (3.2) and (3.3), we deduce that

$$\begin{aligned} 0 = {} & (m+4)(m+3)(m+2)(m+1) a_{n,m+4} + (n+4)(n+3)(n+2)(n+1) a_{n+4,m} \\ & + 2(n+2)(n+1)(m+2)(m+1) a_{n+2,m+2} \\ & + (q_1 + q_2)(n+2)(n+1) a_{n+2,m} + (q_1 + q_2)(m+2)(m+1) a_{n,m+2} \\ & + q_2 q_1 a_{n,m}. \end{aligned} \tag{3.4}$$

We shall prove $a_{n,m}=0$ for all $n,m\in\mathbb{N}$ through (3.4) and the transmission conditions

$$u=\partial_\nu u=0 \quad \text{on} \quad \Gamma.$$

This together with (3.3) would give rise to $a^{(2)}_{n,m}=a^{(1)}_{n,m}=0$.

Denote by $\Gamma_1:=\{(x_1,f(x_1):x_1\in[0,1)\}$ and $\Gamma_2=\{(x_1,f(x_1):x_1\in(-1,0])\}$, with the normal directions given by

$$\nu^{(1)}(x_1)=(\alpha_1c_1x_1^{\alpha_1-1},-1)^\top,\quad x_1>0;\qquad \nu^{(2)}(x_1)=(\alpha_2c_2x_1^{\alpha_2-1},-1)^\top,\quad x_1<0,$$

respectively. Observe that

$$\frac{\partial u}{\partial x_1}=\sum_{n\geqslant0}\sum_{m\geqslant0}a_{n+1,m}(n+1)x_1^nx_2^m,\quad \frac{\partial u}{\partial x_2}=\sum_{n\geqslant0}\sum_{m\geqslant0}a_{n,m+1}(m+1)x_1^nx_2^m.$$

It follows from $\partial_\nu u=0$ on Γ_j $(j=1,2)$ that

$$\alpha_1\sum_{n\geqslant0}\sum_{m\geqslant0}a_{n+1,m}(n+1)x_1^{n+\alpha_1m+\alpha_1-1}c_1^{m+1}-\sum_{n\geqslant0}\sum_{m\geqslant0}a_{n,m+1}(m+1)x_1^{n+\alpha_1m}c_1^m=0, \tag{3.5}$$

$$\alpha_2\sum_{n\geqslant0}\sum_{m\geqslant0}a_{n+1,m}(n+1)x_1^{n+\alpha_2m+\alpha_2-1}c_2^{m+1}-\sum_{n\geqslant0}\sum_{m\geqslant0}a_{n,m+1}(m+1)x_1^{n+\alpha_2m}c_2^m=0. \tag{3.6}$$

Without loss of generality, we suppose that $\alpha_1\leqslant\alpha_2$. In order to prove lemma 3.1, we will consider two cases:

Case 1: $\alpha_1=\alpha_2\geqslant2$; Case 2: $2\leqslant\alpha_1<\alpha_2$.

The proofs in cases 1 and 2 will be carried out in the subsequent two subsections, separately.

3.1. Proof of lemma 3.1 when $\alpha_1=\alpha_2\geqslant2$

For notational convenience we set $\alpha:=\alpha_1=\alpha_2\geqslant2$. Equating coefficients of ${x_1}^l$ $(l\in\mathbb{N},l\geqslant\alpha-1)$ in (3.5) and (3.6) and changing properly the summation indices, we obtain

$$\sum_{n+\alpha m=l-\alpha+2,n\geqslant1,m\geqslant0}\alpha na_{n,m}c_1^{m+1}-\sum_{n+\alpha m=l+\alpha,n\geqslant0,m\geqslant1}ma_{n,m}c_1^{m-1}=0, \tag{3.7}$$

$$\sum_{n+\alpha m=l-\alpha+2,n\geqslant1,m\geqslant0}\alpha na_{n,m}c_2^{m+1}-\sum_{n+\alpha m=l+\alpha,n\geqslant0,m\geqslant1}ma_{n,m}c_2^{m-1}=0. \tag{3.8}$$

On the other hand, the Dirichlet condition $u=0$ on Γ gives the relations

$$\sum_{n\geqslant0}\sum_{m\geqslant0}a_{n,m}x_1^{n+\alpha m}{c_1}^m=0,\quad \sum_{n\geqslant0}\sum_{m\geqslant0}a_{n,m}x_1^{n+\alpha m}c_2^m=0. \tag{3.9}$$

Equating coefficients of ${x_1}^l$ $(l\in\mathbb{N},l\geqslant0)$ in (3.9), we get

$$\sum_{n+\alpha m=l,n\geqslant0,m\geqslant0}a_{n,m}{c_1}^m=0,\quad \sum_{n+\alpha m=l,n\geqslant0,m\geqslant0}a_{n,m}{c_2}^m=0. \tag{3.10}$$

By (3.7), (3.8) and (3.10) we shall prove $a_{n,m}=0$ for all $n+\alpha m=j$ $(j\in\mathbb{N})$ by an induction argument on the index $j\in\mathbb{N}$. We divide the proof into four steps.

Step 1: Prove $a_{j,0}=0$ for all $j=0,1,\ldots\alpha-1$. This follows from (3.10) with $n+\alpha m=l$ for $l=0,1,\ldots\alpha-1$.

Step 2: Prove $a_{j,0} = a_{j-\alpha,1} = 0$ for all $j = \alpha, \alpha+1, \cdots, 2\alpha-1$. Setting $l = \alpha, \alpha+1, \ldots, 2\alpha-1$ in (3.10), we obtain

$$a_{l,0} + c_1 a_{l-\alpha,1} = 0, \quad a_{l,0} + c_2 a_{l-\alpha,1} = 0.$$

Since $c_1 \neq c_2$, we see $a_{l,0} = a_{l-\alpha,1} = 0$.

Step 3: Prove $a_{j,0} = a_{j-\alpha,1} = a_{j-2\alpha,2} = 0$ for all $j = 2\alpha, 2\alpha+1, \cdots, 3\alpha-1$. As done in previous two steps, setting $n + \alpha m = 2\alpha, 2\alpha+1, \ldots, 3\alpha-1$ in (3.10), we find

$$a_{j,0} + c_1 a_{j-\alpha,1} + {c_1}^2 a_{j-2\alpha,2} = 0, \quad a_{j,0} + c_2 a_{j-\alpha,1} + {c_2}^2 a_{j-2\alpha,2} = 0. \tag{3.11}$$

On the other hand, one may conclude from Steps 1 and 2 that

$$a_{n,m} = 0 \quad \text{if} \quad n + \alpha m < j.$$

This together with $\alpha \geqslant 2$ implies that

$$0 = \sum_{n+\alpha m = j-2\alpha+2, n\geqslant 1, m\geqslant 0} \alpha n a_{n,m} c_1^{m+1} = \sum_{n+\alpha m = j, n\geqslant 1, m\geqslant 0} \alpha n a_{n,m} c_2^{m+1}.$$

Hence, setting $l = j - \alpha$ in (3.7) and (3.8) gives the relations

$$a_{j-\alpha,1} + 2c_1 a_{j-2\alpha,2} = 0, \quad a_{j-\alpha,1} + 2c_2 a_{j-2\alpha,2} = 0. \tag{3.12}$$

Therefore, combining (3.11) and (3.12) yields $a_{j,0} = a_{j-\alpha,1} = a_{j-2\alpha,2} = 0$. Further, we conclude from Steps 1–3 that

$$a_{n,m} = 0 \quad \text{if} \quad n + \alpha m < 3\alpha. \tag{3.13}$$

Step 4: Prove $a_{j,0} = a_{j-\alpha,1} = a_{j-2\alpha,2} = a_{j-3\alpha,3} = 0$ for all $j = 3\alpha, 3\alpha+1, \cdots, 4\alpha-1$. Setting $n + \alpha m = 3\alpha, 3\alpha+1, \ldots, 4\alpha-1$ in (3.10), we get for such j that

$$\begin{aligned} &a_{j,0} + c_1 a_{j-\alpha,1} + {c_1}^2 a_{j-2\alpha,2} + {c_1}^3 a_{j-3\alpha,3} = 0, \\ &a_{j,0} + c_2 a_{j-\alpha,1} + {c_2}^2 a_{j-2\alpha,2} + {c_2}^3 a_{j-3\alpha,3} = 0. \end{aligned}$$

Setting $l = j - \alpha$ in (3.7) and (3.8) and making use of (3.13), we obtain

$$\begin{aligned} &a_{j-\alpha,1} + 2c_1 a_{j-2\alpha,2} + 3{c_1}^2 a_{j-3\alpha,3} = 0, \\ &a_{j-\alpha,1} + 2c_2 a_{j-2\alpha,2} + 3{c_2}^2 a_{j-3\alpha,3} = 0. \end{aligned}$$

For fixed $j \in \{3\alpha, 3\alpha+1, \cdots, 4\alpha-1\}$, the previous relations can be written as the system

$$\begin{pmatrix} 1 & c_1 & {c_1}^2 & {c_1}^3 \\ 1 & c_2 & {c_2}^2 & {c_2}^3 \\ 0 & 1 & 2c_1 & 3{c_1}^2 \\ 0 & 1 & 2c_2 & 3{c_2}^2 \end{pmatrix} \begin{pmatrix} a_{j,0} \\ a_{j-\alpha,1} \\ a_{j-2\alpha,2} \\ a_{j-3\alpha,3} \end{pmatrix} = 0. \tag{3.14}$$

It is not difficult to check that the determinant of the matrix on the left hand side of (3.14) is $-(c_1 - c_2)^4 \neq 0$, implying that $a_{j,0} = a_{j-\alpha,1} = a_{j-2\alpha,2} = a_{j-3\alpha,3} = 0$. Hence, it holds that

$$a_{n,m} = 0 \quad \text{if} \quad n + \alpha m < 4\alpha.$$

Step 5: Induction arguments. We make the induction hypothesis $a_{n,m} = 0$ for all $n + \alpha m < M$ for some $M \geqslant 4\alpha, M \in \mathbb{N}$. We need to prove that

$$a_{n,m} = 0 \quad \text{if} \quad n + \alpha m = M. \tag{3.15}$$

We first claim that

$$a_{n,m} = 0, \quad \text{if} \quad n + \alpha m = M, m \geqslant 4. \tag{3.16}$$

Let $n' = n, m' = m - 4 \geqslant 0$. Then we see i

$$n' + \alpha(m' + 4) = M, \quad n', m' \geqslant 0.$$

One can readily prove that

$$n' + 4 + \alpha m' < M, \quad n' + 2 + \alpha(m' + 2) < M, \quad n' + 2 + \alpha m' < M, \quad n' + \alpha(m' + 2) < M.$$

Therefore, by induction hypothesis,

$$a_{n'+4,m'} = a_{n'+2,m'+2} = a_{n'+2,m'} = a_{n',m'+2} = 0.$$

Using (3.4), we get the relation

$$a_{n',m'+4} = 0, \text{ if } \quad n' + \alpha(m' + 4) = M, \quad n', m' \geqslant 0,$$

which proves (3.16).
To proceed with the proof we set $l = M$ in (3.10) to obtain

$$a_{M,0} + c_1 a_{M-\alpha,1} + {c_1}^2 a_{M-2\alpha,2} + {c_1}^3 a_{M-3\alpha,3} = 0,$$
$$a_{M,0} + c_2 a_{M-\alpha,1} + {c_2}^2 a_{M-2\alpha,2} + {c_2}^3 a_{M-3\alpha,3} = 0,$$

where the relation (3.16) was again used. On the other hand, setting $l = M - \alpha$ in (3.7) and (3.8) and recalling the induction hypothesis, we see

$$a_{M-\alpha,1} + 2c_1 a_{M-2\alpha,2} + 3{c_1}^2 a_{M-3\alpha,3} = 0,$$
$$a_{M-\alpha,1} + 2c_2 a_{M-2\alpha,2} + 3{c_2}^2 a_{M-3\alpha,3} = 0.$$

Note that the coefficient matrix for the unknowns $a_{M,0}, a_{M-\alpha,1}, a_{M-2\alpha,2}$ and $a_{M-3\alpha,3}$ is the same as the 4-by-4 matrix on the left hand side of (3.14). Since the determinant of this matrix does not vanish, we obtain $a_{M,0} = a_{M-\alpha,1} = a_{M-2\alpha,2} = a_{M-3\alpha,3} = 0$. This together with (3.16) proves (3.15).

By induction, it holds that $a_{n,m} = 0$ for all $n, m \in \mathbb{N}$. In view of (3.3) and the condition $q_1 \neq q_2$, we obtain $a^{(1)}_{n,m} = a^{(2)}_{n,m} = 0$ for all $n, m \in \mathbb{N}$. Finally, we get $u_1 = u_2 \equiv 0$ in B_1 by the analyticity of u_j ($j = 1, 2$).

3.2. *Proof of lemma 3.1 when* $2 \leqslant \alpha_1 < \alpha_2$

We first observe that the powers of x_1 in the first summation on the left hand side of (3.5) and (3.6) are all greater than or equal to $\alpha_j - 1$, whereas those in the second summation start from zero. Hence, equating coefficients of the term x_1^l ($l < \alpha_j - 1$) in (3.5) and (3.6) yields

$$\sum_{n \geqslant 0, m \geqslant 1, n + \alpha_1 m = l + \alpha_1} a_{n,m} m c_1^{m-1} = 0, \quad l \leqslant \alpha_1 - 2, \tag{3.17}$$

$$\sum_{n \geqslant 0, m \geqslant 1, n + \alpha_2 m = l + \alpha_2} a_{n,m} m c_2^{m-1} = 0, \quad l \leqslant \alpha_2 - 2. \tag{3.18}$$

Analogously, equating coefficients of the term x_1^l for $l \geqslant \alpha_j - 1$, we obtain

$$\sum_{n+\alpha_1 m=l-\alpha_1+2,n\geqslant 1,m\geqslant 0} \alpha_1 n a_{n,m} c_1^{m+1} - \sum_{n+\alpha_1 m=l+\alpha_1,n\geqslant 0,m\geqslant 1} m a_{n,m} c_1^{m-1} = 0, l \geqslant \alpha_1 - 1, \tag{3.19}$$

$$\sum_{n+\alpha_2 m=l-\alpha_2+2,n\geqslant 1,m\geqslant 0} \alpha_2 n a_{n,m} c_2^{m+1} - \sum_{n+\alpha_2 m=l+\alpha_2,n\geqslant 0,m\geqslant 1} m a_{n,m} c_2^{m-1} = 0, l \geqslant \alpha_2 - 1. \tag{3.20}$$

From the Dirichlet boundary condition $u = 0$ on Γ, we obtain

$$\sum_{n\geqslant 0}\sum_{m\geqslant 0} a_{n,m} x_1^{n+\alpha_1 m} {c_1}^m = 0, \quad \sum_{n\geqslant 0}\sum_{m\geqslant 0} a_{n,m} x_1^{n+\alpha_2 m} c_2^m = 0,$$

which implies that

$$\sum_{n+\alpha_1 m=l,n\geqslant 0,m\geqslant 0} a_{n,m} {c_1}^m = 0, \quad l \geqslant 0, \tag{3.21}$$

$$\sum_{n+\alpha_2 m=l,n\geqslant 0,m\geqslant 0} a_{n,m} {c_2}^m = 0, \quad l \geqslant 0. \tag{3.22}$$

Since $\alpha_1 < \alpha_2$, the proof in this section is more complicated than previous subsection. We shall still apply the induction argument to prove that $a_{n,m} = 0$ for all $n + \alpha m = j$, $j \in \mathbb{N}$. Below we carry out the proof under the assumption that $c_1 \neq 0$. If $c_1 = 0$, we have $c_2 \neq 0$ by assumption. Then the interface can be locally parameterized by the function given in (2.1) with $\alpha_1 = \alpha_2$. Hence, the vanishing of u_j in B_1 follows from the same arguments used in section 3.1, where the case $c_1 = 0$ is covered.

Step 1: Prove $a_{j,0} = 0$ for all $j = \alpha_1, \alpha_1 + 1, \ldots, \alpha_2 - 1$. This follows from (3.22).

Step 2: Prove $a_{j,0} = a_{j-\alpha_1,1} = 0$ when $j = \alpha_1, \alpha_2, \ldots, min(\alpha_2 - 1, 2\alpha_1 - 1)$.
Setting $j = \alpha_1, \ldots 2\alpha_1 - 1$ in (3.21), we obtain

$$a_{j,0} + a_{j-\alpha_1,1} c_1 = 0. \tag{3.23}$$

This together with the condition $c_1 \neq 0$ and the fact that $a_{l,0} = 0$ for $l = \alpha_1, \alpha_1 + 1, \ldots, \alpha_2 - 1$ (see Step 1) gives the desired result.

Step 3: Induction arguments. Assuming that

$$a_{n,m} = 0 \quad \text{for all} \quad n + m\alpha_1 < M, \quad M > \min(\alpha_2 - 1, 2\alpha_1 - 1),$$

we will prove that

$$a_{n,m} = 0 \quad \text{for all} \quad n + m\alpha_1 = M. \tag{3.24}$$

Setting $l = M$ in (3.22) gives

$$\sum_{n+m\alpha_2=M,n\geqslant 0,m\geqslant 0} a_{n,m} c_2^m = 0. \tag{3.25}$$

Since $\alpha_1 < \alpha_2$, the indices n, m appearing in the above summation fulfill $n + m\alpha_1 < M$ if $m \neq 0$. By induction hypothesis, this implies that

$$a_{n,m} = 0, \quad \text{if } n + m\alpha_2 = M, m \neq 0.$$

When $m = 0$, it follows from (3.25) that $a_{M,0} = 0$. Now, it remains to prove

$$a_{n,m} = 0 \quad \text{for all } n + \alpha_1 m = M, \quad m \neq 0 \tag{3.26}$$

in the following cases.

3.2.1. Case 1: $M \leqslant 2\alpha_1 - 1$. Setting $l = M$ in (3.21), we obtain

$$a_{M,0} + a_{M-\alpha_1,1}c_1 = 0,$$

which together with $a_{M,0} = 0$ and $c_1 \neq 0$ leads to $a_{M-\alpha_1,1} = 0$. This proves (3.26) when $M \leqslant 2\alpha_1 - 1$.

3.2.2. Case 2: $2\alpha_1 \leqslant M \leqslant 3\alpha_1 - 1$. Letting $l = M$ in (3.21) and using again the fact that $a_{M,0} = 0$, we obtain

$$a_{M-\alpha_1,1}c_1 + a_{M-2\alpha_1,2}{c_1}^2 = 0. \tag{3.27}$$

Setting $l = M - \alpha_1$ in (3.19) and making use of the induction hypothesis

$$a_{n,m} = 0, \quad \text{for all } n + \alpha_1 m = l - \alpha_1 + 2 < M,$$

we obtain

$$a_{M-\alpha_1,1} + 2a_{M-2\alpha_1,2}c_1 = 0. \tag{3.28}$$

Combining (3.27) and (3.28) leads to $a_{M-\alpha_1,1} = a_{M-2\alpha_1,2} = 0$, which proves (3.26).

3.2.3. Case 3: $3\alpha_1 \leqslant M \leqslant 4\alpha_1 - 1$. As done in previous cases, setting $l = M$ in (3.21) and using $a_{M,0} = 0$ gives

$$a_{M-\alpha_1,1}c_1 + a_{M-2\alpha_1,2}{c_1}^2 + a_{M-3\alpha_1,3}{c_1}^3 = 0. \tag{3.29}$$

Setting $l = M - \alpha_1$ in (3.19). Recalling from the induction hypothesis that

$$a_{n,m} = 0, \quad \text{if } n + \alpha_1 m = l - \alpha_1 + 2 < M,$$

we obtain

$$a_{M-\alpha_1,1} + 2a_{M-2\alpha_1,2}c_1 + 3a_{M-3\alpha_1,3}{c_1}^2 = 0. \tag{3.30}$$

Next we will show $a_{M-\alpha_1,1} = 0$. Write $N = M - \alpha_1 + \alpha_2$ for notational simplicity.
If $N \leqslant 2\alpha_2 - 2$, setting $l = N - \alpha_2$ in (3.18) gives $a_{M-\alpha_1,1} = 0$.
If $N \geqslant 2\alpha_2 - 1$, we have the relation

$$M - \alpha_1 + \alpha_2 - 2\alpha_2 + 2 = M - \alpha_1 - \alpha_2 + 2 < M.$$

Letting $l = M - \alpha_1$ in (3.20), we can obtain

$$\sum_{n+\alpha_2 m = N, n \geqslant 0, m \geqslant 1} a_{n,m} m {c_2}^{m-1} = 0. \tag{3.31}$$

Since $\alpha_2 > \alpha_1$, it holds that

$$n + \alpha_1 m < M \quad \text{for all} \quad n + \alpha_2 m = N, m \geqslant 2,$$

implying that

$$a_{n,m} = 0 \quad \text{for all} \quad n + \alpha_2 m = N, m \geqslant 2$$

due to the induction hypothesis. Hence, it follows from (3.31) that $a_{M-\alpha_1,1} = 0$. Combining this with (3.29) and (3.30) and the fact that $c_1 \neq 0$, we obtain $a_{M-2\alpha_1,2} = a_{M-3\alpha_1,3} = 0$, which proves (3.26).

3.2.4. Case 4: $M \geqslant 4\alpha_1$. We first prove that

$$a_{n,m+4} = 0, \quad \text{if } n + \alpha_1(m+4) = M, n \geqslant 0, m \geqslant 0. \tag{3.32}$$

Supposing that indices $n, m \geqslant 0$ in (3.4) satisfy the relation $n + \alpha_1(m+4) = M$. Then we have

$$\begin{aligned} &n + 4 + \alpha_1 m < M, \quad n + 2 + \alpha_1(m+2) < M, \\ &n + 2 + \alpha_1 m < M, \quad n + \alpha_1(m+2) < M. \end{aligned}$$

By induction hypothesis, we see

$$a_{n+4,m} = a_{n+2,m+2} = a_{n+2,m} = a_{n,m+2} = 0.$$

Hence, the relation (3.32) follows from (3.4). To prove (3.26) we only need to verify

$$a_{M-\alpha_1,1} = a_{M-2\alpha_1,2} = a_{M-3\alpha_1,3} = 0. \tag{3.33}$$

Analogously to case 3, we will show that $a_{M-\alpha_1,1} = 0$ by setting $N = M - \alpha_1 + \alpha_2$.
If $N \leqslant 2\alpha_2 - 2$, letting $l = N$ in (3.18) leads to $a_{M-\alpha_1,1} = 0$.
If $N \geqslant 2\alpha_2 - 1$, letting $l = M - \alpha_1$ in (3.20) and noting that

$$M - \alpha_1 + \alpha_2 - 2\alpha_2 + 2 = M - \alpha_1 - \alpha_2 + 2 < M,$$

we obtain

$$\sum_{n+\alpha_2 m = N, n \geqslant 0, m \geqslant 1} a_{n,m} m {c_2}^{m-1} = 0. \tag{3.34}$$

Similar to the arguments in section 3.2.3, we can obtain using the induction hypothesis that

$$a_{n,m} = 0 \quad \text{for all} \quad n + \alpha_2 m = N, m \geqslant 2,$$

because $n + \alpha_1 m < M$ for such indices n and m. Therefore, we get $a_{M-\alpha_1,1} = 0$ from (3.34). Now, setting $l = M$ in (3.21), using (3.32) and the fact that $a_{M,0} = a_{M-\alpha_1,1} = 0$, we see

$$a_{M-2\alpha_1,2} {c_1}^2 + a_{M-3\alpha_1,3} {c_1}^3 = 0. \tag{3.35}$$

On the other hand, setting $l = M - \alpha_1$ in (3.19), using $a_{M-\alpha_1,1} = 0$ and the relations

$$a_{n,m} = 0 \quad \text{for all} \quad n + \alpha_1 m = l - \alpha_1 + 2 = M - 2\alpha_1 - 2 < M,$$

we deduce that

$$2a_{M-2\alpha_1,2} c_1 + 3a_{M-3\alpha_1,3} {c_1}^2 = 0. \tag{3.36}$$

Since $c_1 \neq 0$, we obtain $a_{M-2\alpha_1,2} = a_{M-3\alpha_1,3} = 0$ by combining (3.35) and (3.36). This finishes the proof of (3.26) when $M \geqslant 4\alpha_1$.

Finally, the relation (3.24) follows from (3.26) and the fact that $a_{M,0} = 0$. The proof of lemma 3.1 is thus complete under the assumption that $2 \leqslant \alpha_1 < \alpha_2$.

4. Concluding remarks

In this paper, existing results for strongly singular corners are generalized to the case of weakly singular point in 2D. We remark that lemma 3.1 does not hold true if the curve Γ is analytic. Counterexamples can be easily constructed when Γ is a line segment (see [8, remark 3.3]) or a circle. If $\Gamma \subset B_1$ is a circle of radius $R < 1$ centered at the origin, one may find interior transmission eigenvalues (ITEs) (or equivalently, q_1 and q_2) such that the coupling problem

$$\Delta u_j + q_j u_j = 0 \quad \text{in} \quad |x| < R, \quad u_1 = u_2, \quad \partial_\nu u_1 = \partial_\nu u_2 \quad \text{on} \quad |x| = R, \tag{4.1}$$

admits non-trivial solutions u_1 and u_2 in B_R (see e.g. [5]), which can be analytically extended to B_1. Our lemma 3.1 implies that, if Γ possesses a singular point, the non-trivial solutions u_j to (4.1) cannot be analytically extended onto $\overline{B}_1$. We refer to [3–5, 15, 16] for the existence of ITEs in inverse scattering theory. Note that all results of this paper carry over to variable potential functions which are constant in a small neighborhood of the singular point under question. The singular points considered here form only a subset of non-analytical points of Γ. We conjecture that lemma 3.1 remain valid under the weak assumption that Γ contains a single non-analytical point. However, the proof requires novel mathematical arguments and the progress along this direction will be reported in our forthcoming publications.

Acknowledgments

The work of G Hu is supported by the NSFC grant (No. 11671028) and NSAF grant (No. U1530401). The work of L Li and J Yang are partially supported by the National Science Foundation of China (61421062, 61520106004) and Microsoft Research of Asia.

ORCID iDs

Guanghui Hu https://orcid.org/0000-0002-8485-9896

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