



Direct and inverse elastic scattering from anisotropic media

Gang Bao^{a,*}, Guanghui Hu^b, Jiguang Sun^{c,d}, Tao Yin^a^a School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China^b Beijing Computational Science Research Center, Beijing 100094, China^c Department of Mathematical Sciences, Michigan Technological University, Houghton, MI 49931, USA^d College of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China

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ABSTRACT

Assume that a time-harmonic elastic wave is incident onto a penetrable anisotropic elastic body embedded in a homogeneous isotropic background medium. The scattering problem is reduced to a truncated domain. Uniqueness and existence of weak solutions are proved by applying the Fredholm alternative and using properties of the Dirichlet-to-Neumann map in both two and three dimensions. The Fréchet derivative of the near-field solution operator with respect to the boundary of the scatterer is derived. As an application, a descent algorithm is designed for recovering the interface from the near-field data of one or several incident directions and frequencies. Numerical examples in 2D are presented to show the validity and accuracy of the algorithm.

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R É S U M É

Nous avons supposé qu'une onde élastique harmonique en temps est incidente sur un objet élastique anisotrope pénétrable, qui s'intègre dans un milieu homogène isotrope. Le problème de diffusion se réduit sur un domaine tronqué. L'existence et l'unicité de la solution faible du problème direct qui se pose en dimension deux ou en dimension trois se démontre en appliquant l'alternative de Fredholm et en utilisant les propriétés de l'opérateur Dirichlet-to-Neumann. Nous avons calculé la dérivée de Fréchet d'un opérateur de near-field par rapport au bord de la dispersion. Pour l'application, un algorithme de la méthode de gradient est implémenté pour reconstruire l'interface à partir des données near-field d'une ou plusieurs directions et fréquences incidents. Des exemples en dimension deux sont présentés pour montrer la validité et la précision de l'algorithme.

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* Corresponding author.

E-mail addresses: baog@zju.edu.cn (G. Bao), hu@csrc.ac.cn (G. Hu), jiguangs@mtu.edu (J. Sun), taoyin@zju.edu.cn (T. Yin).

1. Introduction

Time-harmonic elastic scattering problems arise from many mechanic systems and engineering structures, in which the linear elasticity theory ([1]) provides an essential tool for analysis and design. For an infinite background medium, the boundary value problem for the Lamé system can be reduced to an equivalent system on a bounded domain. For instance, the finite element method for the scattering problems usually requires a strongly elliptic variational formulation with a nonlocal boundary condition (see e.g., [2,3]). To truncate the unbounded domain, one needs to derive the so-called Dirichlet-to-Neumann map (or non-reflecting boundary condition, transparent boundary operator) on an artificial boundary as a replacement of the Kupradze radiation condition at infinity. In the literature, the DtN map in elastodynamics have been used by some physicists and engineers for simulation ([4–10]). However, properties of the transparent operator have not been sufficiently investigated yet. These properties are fundamental for the strong ellipticity of the variational formulation and the well-posedness (existence, uniqueness and stability) of the model problem. We refer to [11–13] for the treatment of the time-harmonic Helmholtz and Maxwell equations. Although a nonlocal boundary condition closely related to the DtN map was utilized in [14], mapping properties of the non-reflecting operator in Sobolev spaces were not involved there. In a recent paper [15], a special sesquilinear form, which corresponds to a special choice of the elastic parameters, has been employed to prove well-posedness of the elastic scattering problem. However, the approach of [15] relies heavily on the boundary condition of the scatterer and applies only to a rigid impenetrable elastic body in two dimensions.

This paper is concerned with both direct and inverse scattering from an anisotropic elastic body in a homogenous isotropic background medium. The first half is devoted to the well-posedness in a more general setting. We propose an equivalent variational formulation on a truncated bounded domain, and show the uniqueness and existence of weak solutions for both inhomogeneous penetrable anisotropic bodies and impenetrable scatterers with various boundary conditions. In contrast to the Helmholtz equation, the real part of the DtN map is not negative-definite. Nevertheless, the resulting sesquilinear form may be shown to be strongly elliptic. Motivated by Betti's formula, we analyze the DtN map for the generalized stress operator which covers the usual stress operator and the special case discussed in [16,15]. To prove uniqueness, we verify the Rellich's identity in elasticity. Our proof should be of independent interest since it has generalized the arguments in [15] and [17, Lemma 5.8] for special cases. The Rellich's identity in periodic structures can be found in [18,16].

The second half of this paper is concerned with the inverse problem of reconstructing the shape of an unknown anisotropic body from near-field measurements of plane incident waves. It is well-known that the knowledge of DtN map cannot be used to uniquely identify general anisotropic elastic media; see e.g., the counter example presented in [5,19] based on the idea of transformation elastodynamics. In particular, two inclusions with different anisotropic elastic media may give rise to the same far-field measurements. In this paper the anisotropic medium is supposed to be homogeneous and known in advance, and the unknown inclusion needs to be recovered from the near-field data corresponding to a finite number of incident directions or frequencies. Relying on the variational arguments presented in the first half and those in [20] and [21], we derive the Fréchet derivative of the solution operator with respect to the scattering surface. A different approach based on the integral equation was used in acoustics [22] and in elasticity [23,24]. The shape derivative can be used to design a nonlinear optimization approach for shape recovery from the data of several incident directions and frequencies. We refer to [25] for the application of shape derivative in reconstructing targets from multistatic response measurements given point sources with resolution and stability analysis. Here, we employ a descent method to find the parameters of the unknown surface in a finite dimensional space. At each iteration step, the forward problem needs to be solved and the correctness of the parameters needs to be evaluated. Numerical examples in 2D are presented to show the validity and accuracy of our inversion algorithms. We refer to the review article [5] and the recent monograph [26] for various

inverse problems in elasticity and to [27–29,15,30,31] where iterative approaches using multi-frequency data were developed. For the imaging methods to reconstruct small inclusions in elastic medium, see [32,33] for example.

It is worth noting that there are still two open problems. Firstly, how to derive a frequency-dependent estimate of the solution for a star-shaped rigid scatterer? Readers are referred to e.g. [11] for a wavenumber-dependent estimate in the acoustic case, which was derived based on the use of a Rellich-type identity for the scalar Helmholtz equation. In linear elasticity, the lack of the negativity of the real part of the DtN map leads to essential difficulties in generalizing the arguments of [11]. Secondly, how to prove the well-posedness in a homogenous anisotropic background medium? A new radiation condition at infinity seems to be necessary, which should cover the Kupradze radiation condition as a special case. In this paper, the assumption of the isotropic background medium has considerably simplified our arguments. The far-field asymptotics of the Green's tensor for a transversely isotropic solid was recently analyzed in [34]. However, a radiation condition in the general case seems unavailable in the literature.

The remaining part of this paper is organized as following. In Section 2, we describe the forward scattering model in \mathbb{R}^N ($N = 2, 3$) and prove the unique solvability using variational arguments. Properties of the DtN map in two and three dimensions will be presented in Sections 2.3 and 2.4, respectively. In Section 3 we derive the Fréchet derivative and apply it to solve the inverse scattering problems. Numerical tests for both direct and inverse problems will be reported in Section 4.

2. Well-posedness of the direct scattering problems

2.1. Mathematical formulations

Suppose that a time-harmonic elastic wave u^{in} (with the time variation of the form $e^{-i\omega t}$ where $\omega > 0$ is a fixed frequency) is incident onto an anisotropic elastic body Ω embedded in an infinite homogeneous isotropic background medium in \mathbb{R}^N ($N = 2, 3$). It is assumed that Ω is a bounded Lipschitz domain and the exterior $\Omega^c := \mathbb{R}^N \setminus \overline{\Omega}$ of Ω is connected. In particular, Ω is allowed to consist of a finite number of disconnected bounded components. In linear elasticity, the spatially-dependent displacement vector $u(x) = (u_1, u_2, \dots, u_N)^\top(x)$, where $(\cdot)^\top$ means the transpose, is governed by the following reduced Lamé system

$$\sum_{j,k,l=1}^N \frac{\partial}{\partial x_j} \left(C_{ijkl}(x) \frac{\partial u_k(x)}{\partial x_l} \right) + \omega^2 \rho(x) u_i(x) = 0 \quad \text{in } \mathbb{R}^N, \quad i = 1, 2, \dots, N. \quad (2.1)$$

In (2.1), $u = u^{in} + u^{sc}$ is the total field and u^{sc} is the scattered field; $\mathcal{C} = (C_{ijkl})_{i,j,k,l=1}^N$ is a fourth-rank constitutive material tensor of the elastic medium which is physically referred to as the stiffness tensor; ρ is a complex-valued function with the real part $\text{Re } \rho > 0$ and imaginary part $\text{Im } \rho \geq 0$, denoting respectively the density and damping parameter of the elastic medium. The stiffness tensor satisfies the following symmetries for a generic anisotropic elastic material:

$$\text{major symmetry: } C_{ijkl} = C_{klij}, \quad \text{minor symmetries: } C_{ijkl} = C_{jikl} = C_{ijlk}, \quad (2.2)$$

for all $i, j, k, l = 1, 2, \dots, N$. By Hooke's law, the stress tensor σ relates to the stiffness tensor \mathcal{C} via the identity $\sigma(u) := \mathcal{C} : \nabla u$, where the action of \mathcal{C} on a matrix $A = (a_{ij})$ is defined as

$$\mathcal{C} : A = (\mathcal{C} : A)_{ij} = \sum_{k,l=1}^N C_{ijkl} a_{kl}.$$

Hence, the elliptic system in (2.1) can be restated as

$$\nabla \cdot (\mathcal{C} : \nabla u) + \omega^2 \rho u = 0 \quad \text{in } \mathbb{R}^N. \tag{2.3}$$

Note that in (2.1) we have assumed the continuity of the *stress vector* or *traction* (the normal component of the stress tensor) on $\partial\Omega$, i.e., $\mathcal{N}_{\mathcal{C}}^+ u = \mathcal{N}_{\mathcal{C}}^- u$ where

$$\begin{aligned} \mathcal{N}_{\mathcal{C}} u &:= \nu \cdot \sigma(u) \\ &= \left(\sum_{j,k,l=1}^N \nu_j C_{1jkl} \frac{\partial u_k}{\partial x_l}, \sum_{j,k,l=1}^N \nu_j C_{2jkl} \frac{\partial u_k}{\partial x_l}, \dots, \sum_{j,k,l=1}^N \nu_j C_{Njkl} \frac{\partial u_k}{\partial x_l} \right), \end{aligned}$$

with $\nu = (\nu_1, \nu_2, \dots, \nu_N)^\top \in \mathbb{S}^{N-1}$ denoting the exterior unit normal vector to $\partial\Omega$ and $(\cdot)^\pm$ the limits taken from outside and inside of Ω , respectively.

Since the elastic material in Ω^c is isotropic and homogeneous, one has

$$C_{ijkl}(x) = \lambda \delta_{i,j} \delta_{k,l} + \mu (\delta_{i,k} \delta_{j,l} + \delta_{i,l} \delta_{j,k}), \quad x \in \Omega^c. \tag{2.4}$$

That is, the stiffness tensor of the background medium is characterized by the Lamé constants λ and μ which satisfy $\mu > 0, N\lambda + 2\mu > 0$. Hence, the stress tensor in Ω^c takes the simple form

$$\sigma(u) = \lambda \mathbf{I} \operatorname{div} u + 2\mu \epsilon(u), \quad \epsilon(u) := \frac{1}{2} (\nabla u + \nabla u^\top),$$

where \mathbf{I} stands for the $N \times N$ identity matrix. Assuming that $\rho(x) \equiv \rho_0$ in Ω^c , the Lamé system (2.1) reduces to the time-harmonic Navier equation

$$\Delta^* u + \omega^2 \rho_0 u = 0 \quad \text{in } \Omega^c, \quad \Delta^* u := \mu \Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u. \tag{2.5}$$

Moreover, the surface traction $\mathcal{N}_{\mathcal{C}} u$ on $\partial\Omega$ takes the more explicit form $\mathcal{N}_{\mathcal{C}} u = T_{\lambda,\mu} u$, where

$$T_{\lambda,\mu} u := 2\mu \partial_\nu u + \lambda \nu \operatorname{div} u + \mu \nu^\perp (\partial_2 u_1 - \partial_1 u_2), \quad \nu = (\nu_1, \nu_2)^\top, \tag{2.6}$$

with $\nu^\perp := (-\nu_2, \nu_1)^\top$ in two dimensions, and

$$T_{\lambda,\mu} u := 2\mu \partial_\nu u + \lambda \nu \operatorname{div} u + \mu \nu \times \operatorname{curl} u, \quad \nu = (\nu_1, \nu_2, \nu_3)^\top, \tag{2.7}$$

in three dimensions. Here and also in what follows, we write $T_{\lambda,\mu} u = Tu$ to drop the dependance of $T_{\lambda,\mu}$ on the Lamé constants λ and μ of the background medium. Denote by

$$k_s := \omega \sqrt{\rho_0 / \mu}, \quad k_p := \omega \sqrt{\rho_0 / (\lambda + 2\mu)}$$

the shear and compressional wave numbers of the background material, respectively.

Since the domain Ω^c is unbounded, an appropriate radiation condition at infinity must be imposed on u^{sc} to ensure well-posedness of the scattering problem. The scattered field in Ω^c can be decomposed into the sum of the compressional (longitudinal) part u_p^{sc} and the shear (transversal) part u_s^{sc} as follows (in three dimensions):

$$u^{sc} = u_p^{sc} + u_s^{sc}, \quad u_p^{sc} = -\frac{1}{k_p^2} \operatorname{grad} \operatorname{div} u^{sc}, \quad u_s^{sc} = \frac{1}{k_s^2} \operatorname{curl} \operatorname{curl} u^{sc}. \tag{2.8}$$

In two dimensions, the shear part of the scattered field should be modified as

$$u_s^{sc} = \frac{1}{k_s^2} \overrightarrow{\text{curl}} \text{curl} u^{sc}, \tag{2.9}$$

where the two-dimensional operators curl and $\overrightarrow{\text{curl}}$ are defined respectively by

$$\text{curl} v = \partial_1 v_2 - \partial_2 v_1, \quad v = (v_1, v_2)^\top, \quad \overrightarrow{\text{curl}} f := (\partial_2 f, -\partial_1 f)^\top.$$

It then follows from the decompositions in (2.8) and (2.9) that

$$(\Delta + k_\alpha^2) u_\alpha^{sc} = 0, \quad \alpha = p, s, \quad \text{div} u_s^{sc} = 0$$

and

$$\text{curl} u_p^{sc} = 0 \quad \text{in } 3D, \quad \overrightarrow{\text{curl}} u_p^{sc} = 0 \quad \text{in } 2D.$$

The scattered field is required to satisfy the Kupradze radiation condition (see e.g. [35])

$$\lim_{r \rightarrow \infty} r^{\frac{N-1}{2}} \left(\frac{\partial u_p^{sc}}{\partial r} - ik_p u_p^{sc} \right) = 0, \quad \lim_{r \rightarrow \infty} r^{\frac{N-1}{2}} \left(\frac{\partial u_s^{sc}}{\partial r} - ik_s u_s^{sc} \right) = 0, \quad r = |x| \tag{2.10}$$

uniformly with respect to all $\hat{x} = x/|x| \in \mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$. The radiation conditions in (2.10) lead to the P-part (longitudinal part) u_p^∞ and the S-part (transversal part) u_s^∞ of the far-field pattern of u^{sc} , given by the asymptotic behavior as $|x| \rightarrow +\infty$

$$u^{sc}(x) = \frac{\exp(ik_p|x|)}{|x|^{\frac{N-1}{2}}} u_p^\infty(\hat{x}) + \frac{\exp(ik_s|x|)}{|x|^{\frac{N-1}{2}}} u_s^\infty(\hat{x}) + \mathcal{O}(|x|^{-\frac{N+1}{2}}), \tag{2.11}$$

where, with some normalization, u_p^∞ and u_s^∞ are the far-field patterns of u_p^{sc} and u_s^{sc} , respectively. We define the far-field pattern u^∞ of the scattered field u^{sc} as the sum of u_p^∞ and u_s^∞ , that is, $u^\infty := u_p^\infty + u_s^\infty$. Since u_p^∞ is normal to \mathbb{S}^{N-1} and u_s^∞ is tangential to \mathbb{S}^{N-1} , it holds the relations

$$u_p^\infty(\hat{x}) = (u^\infty(\hat{x}) \cdot \hat{x}) \hat{x}, \quad u_s^\infty(\hat{x}) = \begin{cases} \hat{x} \times u^\infty(\hat{x}) \times \hat{x} & \text{in } 3D, \\ (\hat{x}^\perp \cdot u^\infty(\hat{x})) \hat{x}^\perp & \text{in } 2D. \end{cases}$$

Throughout this paper we make the following assumptions:

- (A1) There exists $R > 0$ such that $\Omega \subset B_R := \{x \in \mathbb{R}^N : |x| < R\}$ and that u^{in} satisfies the Navier equation (2.5) in B_R .
- (A2) The stiffness tensor \mathcal{C} satisfies the uniform Legendre ellipticity condition

$$\sum_{i,j,k,l=1}^N \mathcal{C}_{ijkl}(x) a_{ij} a_{kl} \geq c_0 \sum_{i,j=1}^N |a_{ij}|^2, \quad a_{ij} = a_{ji}, \quad c_0 > 0, \tag{2.12}$$

for all $x \in \Omega$. In other words, $(\mathcal{C}(x) : A) : A \geq c_0 \|A\|^2$ for all symmetry matrices $A = (a_{ij})_{i,j=1}^N \in \mathbb{R}^{N \times N}$. Here $\|A\|$ means the Frobenius norm of the matrix A .

- (A3) $\|\rho\|_{L^\infty(\Omega)} < \infty$, and $\|\mathcal{C}_{ijkl}\|_{L^\infty(\Omega)} < \infty$ for all $1 \leq i, j, k, l \leq N$.

Remark 2.1. The incident wave u^{in} is allowed to be a linear combination of pressure and shear plane waves of the form

$$u^{in}(x, d) = c_p d \exp(ik_p x \cdot d) + c_s d^\perp \exp(ik_s x \cdot d), \quad c_p, c_s \in \mathbb{C}, \tag{2.13}$$

with $d \in \mathbb{S}^{N-1}$ being the incident direction and $d^\perp \in \mathbb{S}^{N-1}$ satisfying $d^\perp \cdot d = 0$. It also can be elastic point source waves satisfying the equation

$$\Delta^* u^{in}(\cdot; y) + \omega^2 \rho_0 u^{in}(\cdot; y) = \delta(\cdot - y) \mathbf{a} \quad \text{in } \mathbb{R}^N \setminus \{y\},$$

where $y \in \mathbb{R}^N \setminus \overline{B}_R$ represents the location of the source and $\mathbf{a} \in \mathbb{C}^N$ denotes the polarization direction. An explicit expression of $u^{in}(\cdot; y)$ is given by $u^{in}(\cdot; y) = \Pi(\cdot, y)\mathbf{a}$ where Π is the free-space Green’s tensor to the Navier equation given by

$$\Pi(x, y) = \frac{1}{\mu} \Phi_{k_s}(x, y)\mathbf{I} + \frac{1}{\rho_0 \omega^2} \text{grad}_x \text{grad}_x^\top [\Phi_{k_s}(x, y) - \Phi_{k_p}(x, y)], \quad x \neq y. \tag{2.14}$$

Here Φ_k ($k = k_p, k_s$) is the fundamental solution to the Helmholtz equation $(\Delta + k^2)u = 0$ in \mathbb{R}^N . It is well-known that

$$\Phi_k(x; y) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x - y|), & N = 2, \\ \frac{e^{ik|x - y|}}{4\pi|x - y|}, & N = 3, \end{cases} \quad x \neq y, \tag{2.15}$$

with $H_0^{(1)}(\cdot)$ being the Hankel function of the first kind of order zero.

Let $H^1(B_R)$ denote the Sobolev space of scalar functions on B_R . In the following we state the uniqueness and existence of weak solutions to our scattering problem in the energy space $X_R := (H^1(B_R))^N$.

Theorem 2.2. *Under the assumptions (A1)–(A3) there exists a unique solution $u \in X_R$ to the scattering problem (2.1), (2.5) and (2.10).*

The proof of **Theorem 2.2** will depend on the Fredholm alternative together with properties of the Dirichlet-to-Neumann mapping on $\Gamma_R := \partial B_R$. As a consequence, we also obtain the well-posedness of the scattering problem due to an impenetrable elastic body with various kinds of boundary conditions.

Corollary 2.3. *Consider the time-harmonic elastic scattering from an impenetrable bounded elastic body Ω with Lipschitz boundary embedded in a homogeneous isotropic medium. Suppose that the total field satisfies one of the following boundary conditions on $\partial\Omega$:*

- (i) *The first kind (Dirichlet) boundary condition: $u = 0$;*
- (ii) *The second kind (Neumann) boundary condition: $Tu = 0$;*
- (iii) *The third kind boundary condition: $\nu \cdot u = 0, \nu \times Tu = 0$ in $3D$, $\nu \cdot u = \nu^\perp \cdot Tu = 0$ in $2D$;*
- (iv) *The fourth kind boundary condition: $\nu \times u = 0, \nu \cdot Tu = 0$ in $3D$, $\nu^\perp \cdot u = \nu \cdot Tu = 0$ in $2D$;*
- (v) *Robin boundary condition: $Tu - i\eta u = 0, \quad \eta \in \mathbb{C}, \text{Re}(\eta) > 0$.*

Then the scattered field $u^{sc} = u - u^{in}$ is uniquely solvable in $(H_{loc}^1(\mathbb{R}^N \setminus \overline{\Omega}))^N$.

The variational approach for proving **Theorem 2.2** can be easily adapted to treat the boundary value problems in **Corollary 2.3**. We omit the details for simplicity and refer to [16] for the proof in unbounded periodic structures.

Remark 2.4. Using integral equation methods, well-posedness of the boundary value problems in [Corollary 2.3](#) has been investigated in Kupradze [\[35,36\]](#) for scatterers with C^2 -smooth boundaries. The variational arguments presented here have thus relaxed the regularity of the boundary to be Lipschitz.

2.2. Variational formulation with transparent boundary operator

Let $R > 0$ be specified in assumption (A1). By the first Betti’s formula, it follows that for $u, v \in X_R$,

$$\begin{aligned}
 & - \int_{B_R} [\nabla \cdot (\mathcal{C} : \nabla u) + \omega^2 \rho u] \cdot \bar{v} \, dx \\
 & = \int_{B_R} [(\mathcal{C} : \nabla u) : \nabla \bar{v} - \omega^2 \rho u \cdot \bar{v}] \, dx - \int_{\Gamma_R} T u \cdot \bar{v} \, ds.
 \end{aligned}
 \tag{2.16}$$

Below we introduce the Dirichlet-to-Neumann (DtN) map in a homogeneous isotropic background medium, allowing us to reduce the scattering problem on a bounded domain.

Definition 2.5. For any $w \in (H^{1/2}(\Gamma_R))^N$, the DtN map \mathcal{T} acting on w is defined as

$$\mathcal{T} w := (T v^{sc})|_{\Gamma_R},$$

where $v^{sc} \in (H^1_{loc}(\mathbb{R}^N \setminus \overline{B_R}))^N$ is the unique radiating solution to the boundary value problem

$$\Delta^* v^{sc} + \omega^2 \rho_0 v^{sc} = 0 \quad \text{in } \mathbb{R}^N \setminus \overline{B_R}, \quad v^{sc} = w \quad \text{on } \Gamma_R.
 \tag{2.17}$$

Remark 2.6. The DtN map \mathcal{T} is well-defined, since the Dirichlet-kind boundary value problem [\(2.17\)](#) is uniquely solvable in $(H^1_{loc}(\mathbb{R}^N \setminus \overline{B_R}))^N$; see [Remarks 2.12 and 2.16](#) for the explicit expressions in terms of special functions.

To obtain an equivalent variational formulation of [\(2.1\)](#), we shall apply Betti’s identity [\(2.16\)](#) to a solution $u = u^{in} + u^{sc}$ in B_R and use the relation

$$T u = T u^{in} + T u^{sc} = T u^{in} + \mathcal{T} u^{sc} = f + \mathcal{T} u, \quad f := (T u^{in} - \mathcal{T} u^{in})|_{\Gamma_R}.$$

Then the variational formulation reads as follows: find $u = (u_1, \dots, u_N) \in X_R$ such that

$$a(u, v) = \int_{\Gamma_R} f \cdot \bar{v} \, ds \quad \text{for all } v = (v_1, v_2, \dots, v_N)^T \in X_R,
 \tag{2.18}$$

where the sesquilinear form $a(\cdot, \cdot) : X_R \times X_R \rightarrow \mathbb{C}$ is defined by

$$a(u, v) := \int_{B_R} \left\{ \sum_{i,j,k,l=1}^N C_{ijkl} \frac{\partial u_k}{\partial x_l} \frac{\partial \bar{v}_i}{\partial x_j} - \omega^2 \rho u_i \bar{v}_i \right\} dx - \int_{\Gamma_R} T u \cdot \bar{v} \, ds.
 \tag{2.19}$$

Remark 2.7. The variational problem [\(2.18\)](#) and the scattering problem [\(2.1\)](#), [\(2.5\)](#), [\(2.10\)](#) are equivalent in the following sense. If $u^{sc} \in (H^1_{loc}(\mathbb{R}^N))^N$ is a solution of the scattering problem [\(2.1\)](#), [\(2.5\)](#) and [\(2.10\)](#), then the restriction of the total field u to B_R , i.e., $u|_{B_R}$, satisfies the variational problem [\(2.18\)](#). Conversely, a solution $u \in X_R$ of [\(2.18\)](#) can be extended to a solution $u = u^{in} + u^{sc}$ of the Lamé system in $|x| > R$, where u^{sc} is defined as the unique radiating solution to the isotropic Lamé system in $|x| > R$ satisfying the Dirichlet boundary value $u^{sc} = u - u^{in}$ on Γ_R .

In the following lemma, we show properties of the DtN map \mathcal{T} which play an essential role in our uniqueness and existence proofs. The two and three dimensional proofs will be carried out in the subsequent Sections 2.3 and 2.4, respectively.

Lemma 2.8.

- (i) \mathcal{T} is a bounded operator from $(H^{1/2}(\Gamma_R))^N$ to $(H^{-1/2}(\Gamma_R))^N$.
- (ii) The operator $-\mathcal{T}$ can be decomposed into the sum of a positive operator \mathcal{T}_1 and a compact operator \mathcal{T}_2 , that is, $-\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ on $(H^{1/2}(\Gamma_R))^N$.

Let X'_R denote the dual of X_R with respect to the inner product of $(L^2(B_R))^N$. By the boundedness of ρ, C_{ijkl} (see Assumption (A3)) and \mathcal{T} , there exists a continuous linear operator $\mathcal{A} : X_R \rightarrow X'_R$ associated with the sesquilinear form a such that

$$a(u, v) = \langle \mathcal{A}u, v \rangle \quad \text{for all } v \in X_R. \tag{2.20}$$

Here and henceforth the notation $\langle \cdot, \cdot \rangle$ denotes the duality between X'_R and X_R . By Assumption (A1) and Lemma 2.8 (ii), there exists $\mathcal{F} \in X'_R$ such that

$$\int_{\Gamma_R} f \cdot \bar{v} \, ds = \langle \mathcal{F}, v \rangle \quad \text{for all } v \in X_R.$$

Hence the variational formulation (2.18) can be written as an operator equation of finding $u \in X_R$ such that

$$\mathcal{A}u = \mathcal{F} \quad \text{in } X'_R.$$

Below we recall the definition of strong ellipticity.

Definition 2.9. A bounded sesquilinear form $a(\cdot, \cdot)$ on some Hilbert space X is called strongly elliptic if there exists a compact form $q(\cdot, \cdot)$ such that

$$|\operatorname{Re} a(u, u)| \geq C \|u\|_X^2 - q(u, u) \quad \text{for all } u \in X.$$

The following theorem establishes the strong ellipticity of the sesquilinear form a defined by (2.19).

Theorem 2.10. *The sesquilinear form $a(\cdot, \cdot)$ is strongly elliptic over X_R under the Assumption (A2). Moreover, the operator $\mathcal{A} : X_R \rightarrow X'_R$ defined by (2.20) is a Fredholm operator with index zero.*

Proof. We may rewrite the form a as the sum $a = a_1 + a_2$, where the sesquilinear forms a_j ($j = 1, 2$) are defined as

$$a_1(u, v) := \int_{B_R} \left\{ \sum_{i,j,k,l=1}^N C_{ijkl} \frac{\partial u_k}{\partial x_l} \frac{\partial \bar{v}_i}{\partial x_j} \right\} dx + \int_{\Gamma_R} \mathcal{T}_1 u \cdot \bar{v} \, ds,$$

$$a_2(u, v) := -\omega^2 \int_{B_R} \rho u \cdot \bar{v} \, dx + \int_{\Gamma_R} \mathcal{T}_2 u \cdot \bar{v} \, ds,$$

Note that \mathcal{T}_j ($j = 1, 2$) are the operators given by Lemma 2.8. It is seen from the uniform Legendre ellipticity condition and Lemma 2.8 (ii) that a_1 is coercive over X_R . The compact embedding of X_R into $(L^2(B_R))^N$

and the compactness of \mathcal{T}_2 give the compactness of the form a_2 . Hence $a(\cdot, \cdot)$ is strongly elliptic over X_R and thus \mathcal{A} is a Fredholm operator with index zero. \square

Proof of Theorem 2.2. Using Theorem 2.10 and applying the Fredholm alternative, we only need to prove the uniqueness of our scattering problem. Letting $u^{in} \equiv 0$ (which implies that $f = 0$ in X'_R) and taking the imaginary part of (2.18) with $v = u^{sc}$ we get

$$\operatorname{Im} \int_{\Gamma_R} \mathcal{T} u^{sc} \cdot \overline{u^{sc}} \, ds = 0.$$

By the analogue of Rellich’s lemma in elasticity (see Lemmas 2.14 and 2.17 below) we obtain $u^{sc} \equiv 0$ in B_R . This proves the uniqueness and Theorem 2.2. \square

The remaining part of this section will be devoted to the proof of properties of the DtN map in a more general setting. We shall consider the *generalized* stress vector (cf. (2.6))

$$\tilde{T}_{\tilde{\lambda}, \tilde{\mu}} u := \begin{cases} (\mu + \tilde{\mu})\nu \cdot \operatorname{grad} u + \tilde{\lambda}\nu \operatorname{div} u - \tilde{\mu}\nu^\perp \operatorname{curl} u, & \text{if } N = 2, \\ (\mu + \tilde{\mu})\nu \cdot \operatorname{grad} u + \tilde{\lambda}\nu \operatorname{div} u + \tilde{\mu}\nu \times \operatorname{curl} u, & \text{if } N = 3, \end{cases} \tag{2.21}$$

where $\tilde{\lambda}, \tilde{\mu} \in \mathbb{R}$ satisfying $\tilde{\lambda} + \tilde{\mu} = \lambda + \mu$. In the present paper we assume that

$$\frac{(\lambda - \mu)(\lambda + 2\mu)}{\lambda + 3\mu} < \tilde{\lambda} < \lambda + 2\mu. \tag{2.22}$$

The assumption (2.22) will be used later for proving Lemma 2.13 (ii) and Lemma 2.17 (ii). We emphasize that the above condition (2.22) covers at least the following three cases:

- Case (i): $\tilde{\lambda} = \lambda, \tilde{\mu} = \mu$.
- Case (ii): $\tilde{\lambda} = \lambda + \mu, \tilde{\mu} = 0$.
- Case (iii): $\tilde{\lambda} = (\lambda + 2\mu)(\lambda + \mu)/(\lambda + 3\mu), \tilde{\mu} = \mu(\lambda + \mu)/(\lambda + 3\mu)$.

Note that the usual surface traction coincides with $\tilde{T}_{\tilde{\lambda}, \tilde{\mu}}$ in the case (i). Properties of the DtN map in case (ii) were analyzed in [18] on a line and in [15] on a circle.

The generalized DtN map $\tilde{\mathcal{T}}$ corresponding to (2.21) is defined as

$$\tilde{\mathcal{T}} w = (\tilde{T}_{\tilde{\lambda}, \tilde{\mu}} v^{sc})|_{\Gamma_R}, \quad w \in (H^{1/2}(\Gamma_R))^N,$$

where $v^{sc} \in (H^1_{loc}(\Omega^c))^N$ is the radiating solution to the isotropic homogeneous Navier equation (2.5) in $|x| \geq R$.

2.3. Properties of DtN map in 2D

In this section we verify Lemma 2.8 and the Rellich’s identity for the generalized DtN map $\tilde{\mathcal{T}}$ in \mathbb{R}^2 . For this purpose, the surface vector harmonics in \mathbb{R}^2 are needed. Denote by (r, θ_x) the polar coordinates of $x = (x_1, x_2)^\top \in \mathbb{R}^2$, and by $\hat{r}, \hat{\theta} \in \mathbb{S}^1$ the unit vectors under the polar coordinates such that

$$\hat{r} = (\cos \theta, \sin \theta)^\top, \quad \hat{\theta} = (-\sin \theta, \cos \theta)^\top, \quad \theta \in [0, 2\pi).$$

Let \mathbf{P}_n and \mathbf{S}_n be the surface vector harmonics in two-dimensions defined as

$$\mathbf{P}_n(\hat{x}) := e^{in\theta_x} \hat{r}, \quad \mathbf{S}_n(\hat{x}) := e^{in\theta_x} \hat{\theta}, \quad \hat{x} = x/|x| \in \mathbb{S}^1. \tag{2.23}$$

Below we shall derive a series representation of the generalized DtN map. The solution v^{sc} can be split into the sum of a pressure part with vanishing curl and a shear part with vanishing divergence, that is,

$$v^{sc} = \text{grad } \psi_p + \overrightarrow{\text{curl}} \psi_s \quad \text{in } |x| \geq R, \tag{2.24}$$

where ψ_p and ψ_s are both scalar functions. It then follows that

$$\Delta \psi_\alpha + k_\alpha^2 \psi_\alpha = 0, \quad \lim_{r \rightarrow \infty} r^{1/2} \left(\frac{\partial \psi_\alpha}{\partial r} - ik_\alpha \psi_\alpha \right) = 0, \quad \alpha = p, s. \tag{2.25}$$

The solutions of (2.25) can be expressed as

$$\psi_\alpha(x) = \sum_{n=-\infty}^{\infty} \frac{H_n^{(1)}(k_\alpha r)}{H_n^{(1)}(k_\alpha R)} \psi_\alpha^n e^{in\theta_x}, \quad r = |x| \geq R, \quad \alpha = p, s, \tag{2.26}$$

where $\psi_\alpha^n \in \mathbb{C}$ stand for the Fourier coefficients of $\psi_\alpha|_{\Gamma_R}$ and $H_n^{(1)}$ is the Hankel function of the first kind of order n . Set

$$t_\alpha := k_\alpha R, \quad \gamma_\alpha := \frac{H_n^{(1)'}(t_\alpha)}{H_n^{(1)}(t_\alpha)}, \quad \beta_\alpha := \frac{H_n^{(1)''}(t_\alpha)}{H_n^{(1)}(t_\alpha)}, \quad \alpha = p, s.$$

Let (\cdot, \cdot) be the L^2 inner product on the unit circle given by

$$(u, v) := \frac{1}{2\pi} \int_0^{2\pi} u \cdot \bar{v} \, d\theta \quad \text{for all } u, v \in (L^2(\mathbb{S}^1))^2.$$

Due to the orthogonality relations between \mathbf{P}_n and \mathbf{S}_n , it is easy to derive from (2.24) and (2.26) that

$$(v^{sc}|_{\Gamma_R}, \mathbf{P}_n) = \frac{1}{R} [t_p \gamma_p \psi_p^n + in \psi_s^n], \quad (v^{sc}|_{\Gamma_R}, \mathbf{S}_n) = \frac{1}{R} [in \psi_p^n - t_s \gamma_s \psi_s^n].$$

Equivalently, the previous relations can be written in the matrix form

$$A_n \begin{bmatrix} \psi_p^n \\ \psi_s^n \end{bmatrix} = R \begin{bmatrix} (v^{sc}|_{\Gamma_R}, \mathbf{P}_n) \\ (v^{sc}|_{\Gamma_R}, \mathbf{S}_n) \end{bmatrix}, \quad A_n := \begin{bmatrix} t_p \gamma_p & in \\ in & -t_s \gamma_s \end{bmatrix}. \tag{2.27}$$

Lemma 2.11. *The matrix A_n is invertible for all $n \in \mathbb{Z}$ and $R > 0$. Its inverse is given by*

$$A_n^{-1} = \frac{1}{\Lambda_n} \begin{bmatrix} -t_s \gamma_s & -in \\ -in & t_p \gamma_p \end{bmatrix}, \quad \Lambda_n := \det(A_n) = n^2 - t_p t_s \gamma_p \gamma_s. \tag{2.28}$$

Proof. It's sufficient to prove that $\Lambda_n \neq 0$. We write Λ_n as

$$\Lambda_n = n^2 - I_n(t_p) I_n(t_s), \quad I_n(z) := z H_n^{(1)'}(z) / H_n^{(1)}(z).$$

Making use of the Wronskian identity for Bessel and Neumann functions (see, e.g., [12, Chapter 3.4]), it is easy to derive that

$$\text{Im}(I_n(z)) = \frac{2}{\pi |H_n^{(1)}(z)|^2} \quad \text{for all } n \in \mathbb{Z}, z > 0.$$

This implies that, for any fixed $n \in \mathbb{Z}$,

$$\begin{aligned} \operatorname{Re}(\Lambda_n) &= n^2 - \operatorname{Re}(I_n(t_P)) \operatorname{Re}(I_n(t_s)) + \operatorname{Im}(I_n(t_P)) \operatorname{Im}(I_n(t_s)), \\ \operatorname{Im}(\Lambda_n) &= -\operatorname{Re}(I_n(t_P)) \operatorname{Im}(I_n(t_s)) - \operatorname{Im}(I_n(t_P)) \operatorname{Re}(I_n(t_s)) \end{aligned}$$

cannot vanish simultaneously. Hence, $\Lambda_n \neq 0$. \square

Remark 2.12. The unique radiating solution v^{sc} to the boundary value problem (2.17) can be represented as the series (2.24) and (2.26), where the coefficients ψ_p^n and ψ_s^n are given by

$$\begin{bmatrix} \psi_p^n \\ \psi_s^n \end{bmatrix} = RA_n^{-1} \begin{bmatrix} (w, \mathbf{P}_n) \\ (w, \mathbf{S}_n) \end{bmatrix}.$$

Now, we turn to investigating the generalized stress vector (cf. (2.21))

$$\tilde{\mathcal{T}}v^{sc} = (\mu + \tilde{\mu}) \hat{\mathbf{r}} \cdot \operatorname{grad} v^{sc} + \tilde{\lambda} \hat{\mathbf{r}} \operatorname{div} v^{sc} - \tilde{\mu} \hat{\boldsymbol{\theta}} \operatorname{curl} v^{sc} \quad \text{on } |x| = R.$$

Inserting (2.24) into the previous identity and using the relations

$$\operatorname{grad} = \hat{\mathbf{r}} \frac{\partial}{\partial r} + \frac{1}{r} \hat{\boldsymbol{\theta}} \frac{\partial}{\partial \theta}, \quad \hat{\mathbf{r}} \cdot \operatorname{grad} = \frac{\partial}{\partial r}, \quad \overrightarrow{\operatorname{curl}} = -\hat{\boldsymbol{\theta}} \frac{\partial}{\partial r} + \frac{1}{r} \hat{\mathbf{r}} \frac{\partial}{\partial \theta},$$

we obtain via straightforward calculations that

$$\begin{aligned} \hat{\mathbf{r}} \cdot \tilde{\mathcal{T}}v^{sc} &= (\mu + \tilde{\mu}) \hat{\mathbf{r}} \cdot \frac{\partial}{\partial r} \left[\hat{\mathbf{r}} \frac{\partial \psi_p}{\partial r} + \frac{1}{r} \hat{\boldsymbol{\theta}} \frac{\partial \psi_p}{\partial \theta} - \hat{\boldsymbol{\theta}} \frac{\partial \psi_s}{\partial r} + \frac{1}{r} \hat{\mathbf{r}} \frac{\partial \psi_s}{\partial \theta} \right] + \tilde{\lambda} \operatorname{div} \operatorname{curl} \psi_p \\ &= (\mu + \tilde{\mu}) \left(\frac{\partial^2 \psi_p}{\partial r^2} - \frac{1}{r^2} \frac{\partial \psi_s}{\partial \theta} + \frac{1}{r} \frac{\partial^2 \psi_s}{\partial r \partial \theta} \right) + \tilde{\lambda} \Delta \psi_p, \\ \hat{\boldsymbol{\theta}} \cdot \tilde{\mathcal{T}}v^{sc} &= (\mu + \tilde{\mu}) \hat{\boldsymbol{\theta}} \cdot \frac{\partial}{\partial r} \left[\hat{\mathbf{r}} \frac{\partial \psi_p}{\partial r} + \frac{1}{r} \hat{\boldsymbol{\theta}} \frac{\partial \psi_p}{\partial \theta} - \hat{\boldsymbol{\theta}} \frac{\partial \psi_s}{\partial r} + \frac{1}{r} \hat{\mathbf{r}} \frac{\partial \psi_s}{\partial \theta} \right] - \tilde{\mu} \operatorname{curl} \overrightarrow{\operatorname{curl}} \psi_s \\ &= (\mu + \tilde{\mu}) \left(-\frac{1}{r^2} \frac{\partial \psi_p}{\partial \theta} + \frac{1}{r} \frac{\partial^2 \psi_p}{\partial r \partial \theta} - \frac{\partial^2 \psi_s}{\partial r^2} \right) + \tilde{\mu} \Delta \psi_s. \end{aligned}$$

This implies that

$$\begin{bmatrix} \left(\tilde{\mathcal{T}}v^{sc}|_{\Gamma_R}, \mathbf{P}_n \right) \\ \left(\tilde{\mathcal{T}}v^{sc}|_{\Gamma_R}, \mathbf{S}_n \right) \end{bmatrix} = \frac{1}{R^2} B_n \begin{bmatrix} \psi_p^n \\ \psi_s^n \end{bmatrix} \tag{2.29}$$

where

$$B_n := \begin{bmatrix} (\mu + \tilde{\mu}) t_p^2 \beta_p - \tilde{\lambda} t_p^2 & i(\mu + \tilde{\mu}) n (t_s \gamma_s - 1) \\ i(\mu + \tilde{\mu}) n (t_p \gamma_p - 1) & -(\mu + \tilde{\mu}) t_s^2 \beta_s - \tilde{\mu} t_s^2 \end{bmatrix}. \tag{2.30}$$

Combining (2.29) with (2.27) gives the relation

$$\begin{bmatrix} \left(\tilde{\mathcal{T}}(v^{sc}|_{\Gamma_R}), \mathbf{P}_n \right) \\ \left(\tilde{\mathcal{T}}(v^{sc}|_{\Gamma_R}), \mathbf{S}_n \right) \end{bmatrix} = W_n \begin{bmatrix} (v^{sc}|_{\Gamma_R}, \mathbf{P}_n) \\ (v^{sc}|_{\Gamma_R}, \mathbf{S}_n) \end{bmatrix}, \quad W_n := \frac{1}{R} B_n A_n^{-1}. \tag{2.31}$$

Properties of the two-dimensional DtN map are summarized in the subsequent two lemmas.

Lemma 2.13. Let $w = \sum_{n \in \mathbb{Z}} w_p^n \mathbf{P}_n + w_s^n \mathbf{S}_n \in (H^{1/2}(\Gamma_R))^2$. Then,

(i) The generalized DtN operator $\tilde{\mathcal{T}}$ takes the form

$$\tilde{\mathcal{T}}w = \sum_{n \in \mathbb{Z}} W_n \begin{bmatrix} w_p^n \\ w_s^n \end{bmatrix}$$

in the orthogonal basis $\{(\mathbf{P}_n, \mathbf{S}_n) : n \in \mathbb{Z}\}$. Moreover, $\tilde{\mathcal{T}}$ is a bounded linear operator from $(H^s(\Gamma_R))^2$ to $(H^{s-1}(\Gamma_R))^2$ for all $s \in \mathbb{R}$.

(ii) For sufficiently large $M > 0$, the real part of the operator

$$-\tilde{\mathcal{T}}_1 w := - \sum_{|n| \geq M} W_n \begin{bmatrix} w_p^n \\ w_s^n \end{bmatrix}$$

is positive over $(H^{1/2}(\Gamma_R))^2$, and $\tilde{\mathcal{T}} - \tilde{\mathcal{T}}_1$ is a compact operator.

Proof. (i) We only need to show the boundedness of $\tilde{\mathcal{T}}$. Recall that

$$\begin{aligned} \|w\|_{(H^s(\Gamma_R))^2} &= \left(\sum_{n \in \mathbb{Z}} (1 + |n|)^{2s} |w^n|^2 \right)^{1/2}, \quad w^n := [w_p^n, w_s^n]^\top, \\ \|\tilde{\mathcal{T}}w\|_{(H^{s-1}(\Gamma_R))^2} &= \left(\sum_{n \in \mathbb{Z}} (1 + |n|)^{2(s-1)} |W_n w^n|^2 \right)^{1/2}. \end{aligned}$$

Hence, it suffices to estimate the max norm of the matrix W_n bounded by

$$\|W_n\|_{\max} \leq C |n|, \tag{2.32}$$

for some constant $C > 0$ uniformly in all $n \in \mathbb{Z}$, so that $|W_n w^n|^2 \leq C^2 |n|^2 |w^n|^2$.

It holds that

$$\begin{aligned} &H_n^{(1)''}(z) \\ &= \left(H_{n-1}^{(1)}(z) - \frac{n}{z} H_n^{(1)}(z) \right)' \\ &= -H_n^{(1)}(z) + \frac{n-1}{z} H_{n-1}^{(1)}(z) + \frac{n}{z^2} H_n^{(1)}(z) - \frac{n}{z} \left(H_{n-1}^{(1)}(z) - \frac{n}{z} H_n^{(1)}(z) \right) \\ &= \frac{n^2 + n - z^2}{z^2} H_n^{(1)}(z) - \frac{1}{z} H_{n-1}^{(1)}(z) \\ &= \frac{n^2 + n - z^2}{z^2} H_n^{(1)}(z) - \frac{1}{z} \left(H_n^{(1)'}(z) + \frac{n}{z} H_n^{(1)}(z) \right) \\ &= \left(\frac{n^2}{z^2} - 1 \right) H_n^{(1)}(z) - \frac{1}{z} H_n^{(1)'}(z), \end{aligned}$$

giving rise to the identities

$$\beta_p = \frac{n^2}{t_p^2} - 1 - \frac{1}{t_p} \gamma_p, \quad \beta_s = \frac{n^2}{t_s^2} - 1 - \frac{1}{t_s} \gamma_s. \tag{2.33}$$

From the expressions of A_n^{-1} and B_n we get the entries $W_n^{(i,j)}$ of W_n , given by

$$\begin{aligned} W_n^{(1,1)} &= \frac{1}{R\Lambda_n} \left\{ -t_s \gamma_s \left[(\mu + \tilde{\mu}) t_p^2 \beta_p - \tilde{\lambda} t_p^2 \right] + n^2 (\mu + \tilde{\mu}) (t_s \gamma_s - 1) \right\} \\ &= \frac{1}{R\Lambda_n} \left[-(\mu + \tilde{\mu}) \Lambda_n + \omega^2 \rho_0 R^2 t_s \gamma_s \right], \\ W_n^{(2,2)} &= \frac{1}{R\Lambda_n} \left\{ n^2 (\mu + \tilde{\mu}) (t_p \gamma_p - 1) - t_p \gamma_p \left[(\mu + \tilde{\mu}) t_s^2 \beta_s + \tilde{\mu} t_s^2 \right] \right\} \\ &= \frac{1}{R\Lambda_n} \left[-(\mu + \tilde{\mu}) \Lambda_n + \omega^2 \rho_0 R^2 t_p \gamma_p \right], \\ W_n^{(1,2)} &= \frac{1}{R\Lambda_n} \left\{ -in \left[(\mu + \tilde{\mu}) t_p^2 \beta_p - \tilde{\lambda} t_p^2 \right] + int_p \gamma_p (\mu + \tilde{\mu}) (t_s \gamma_s - 1) \right\} \\ &= \frac{1}{R\Lambda_n} \left[-in(\mu + \tilde{\mu}) \Lambda_n + in\omega^2 \rho_0 R^2 \right], \\ W_n^{(2,1)} &= \frac{1}{R\Lambda_n} \left\{ -in(\mu + \tilde{\mu}) t_s \gamma_s (t_p \gamma_p - 1) + in \left[(\mu + \tilde{\mu}) t_s^2 \beta_s + \tilde{\mu} t_s^2 \right] \right\} \\ &= \frac{1}{R\Lambda_n} \left[in(\mu + \tilde{\mu}) \Lambda_n - in\omega^2 \rho_0 R^2 \right], \end{aligned}$$

in which we have used (2.33) and the fact that $\tilde{\lambda} + \tilde{\mu} = \lambda + \mu$.

From the series expansions of the Bessel and Neumann functions (see, e.g., [12, Chapter 3]) we know

$$H_n^{(1)}(z) = \frac{(n-2)!}{i\pi} \left(\frac{2}{z}\right)^{n-1} \left[(n-1) \left(\frac{2}{z}\right)^2 + 1 + O\left(\frac{1}{n}\right) \right], \quad n \rightarrow +\infty.$$

This implies that

$$\begin{aligned} \frac{H_{n-1}^{(1)}(z)}{H_n^{(1)}(z)} &= \frac{z}{2n-4} \frac{1 + (n-2) \left(\frac{2}{z}\right)^2 + O\left(\frac{1}{n}\right)}{1 + (n-1) \left(\frac{2}{z}\right)^2 + O\left(\frac{1}{n}\right)} \\ &= \left[\frac{z}{2n} + O\left(\frac{1}{n^2}\right) \right] \left[1 + O\left(\frac{1}{n}\right) \right] \\ &= \frac{z}{2n} + O\left(\frac{1}{n^2}\right). \end{aligned} \tag{2.34}$$

The asymptotic behavior (2.34) together with the relation $H_n^{(1)'} = -n/z H_n^{(1)} + H_{n-1}^{(1)}$ leads to

$$\frac{H_n^{(1)'}(z)}{H_n^{(1)}(z)} = -\frac{n}{z} + \frac{z}{2n} + O\left(\frac{1}{n^2}\right), \quad n \rightarrow +\infty.$$

Since $H_{-n}^{(1)}(z) = (-1)^n H_n^{(1)}(z)$, we obtain as $|n| \rightarrow \infty$ that

$$\gamma_\alpha = \frac{H_n^{(1)'}(t_\alpha)}{H_n^{(1)}(t_\alpha)} = -\frac{|n|}{t_\alpha} + \frac{t_\alpha}{2|n|} + O\left(\frac{1}{n^2}\right), \quad \alpha = p, s, \tag{2.35}$$

$$\Lambda_n = \frac{R^2(k_p^2 + k_s^2)}{2} + O\left(\frac{1}{|n|}\right) = \frac{R^2 \rho_0 \omega^2 (\lambda + 3\mu)}{2\mu(\lambda + 2\mu)} + O\left(\frac{1}{|n|}\right). \tag{2.36}$$

Inserting (2.35) and (2.36) into the expression of $W_n^{(i,j)}$ yields

$$\begin{aligned} W_n^{(1,1)} &= -\frac{2\mu(\lambda + 2\mu)}{R(\lambda + 3\mu)}|n| + O(1), \\ W_n^{(2,2)} &= -\frac{2\mu(\lambda + 2\mu)}{R(\lambda + 3\mu)}|n| + O(1), \\ W_n^{(1,2)} &= \frac{i[(\mu + \tilde{\mu})(\lambda + 3\mu) - 2\mu(\lambda + 2\mu)]}{R(\lambda + 3\mu)}|n| + O(1), \\ W_n^{(2,1)} &= -\frac{i[(\mu + \tilde{\mu})(\lambda + 3\mu) - 2\mu(\lambda + 2\mu)]}{R(\lambda + 3\mu)}|n| + O(1), \end{aligned}$$

from which the estimate (2.32) follows directly.

(ii) Define $\tilde{W}_n := -(W_n + W_n^*)/2$, where $(\cdot)^*$ means the conjugate transpose of a matrix. For sufficiently large $|n|$, we have

$$\begin{aligned} \tilde{W}_n^{(1,1)} &= \frac{2\mu(\lambda + 2\mu)}{R(\lambda + 3\mu)}|n| + O(1) > 0, \\ \det(\tilde{W}_n) &= \frac{4\mu^2(\lambda + 2\mu)^2 - [(\lambda - \tilde{\lambda})(\lambda + 3\mu) + 2\mu^2]^2}{R^2(\lambda + 3\mu)^2}n^2 + O(n). \end{aligned}$$

Under the assumption (2.22) on $\tilde{\lambda}$, we see

$$4\mu^2(\lambda + 2\mu)^2 - [(\lambda - \tilde{\lambda})(\lambda + 3\mu) + 2\mu^2]^2 > 0,$$

implying that $\det(\tilde{W}_n) > 0$ for sufficiently large $|n|$. Hence, there exists $M > 0$ such that \tilde{W}_n is positive definite over \mathbb{C}^2 for all $|n| \geq M$. This proves the positivity of the operator $-\text{Re } \tilde{\mathcal{T}}_1$ defined in Lemma 2.13. Finally, $\tilde{\mathcal{T}} - \tilde{\mathcal{T}}_1$ is compact since it is a finite dimensional operator over $(H^{1/2}(\Gamma_R))^2$. \square

Below we verify the analogue of Rellich’s lemma in plane elasticity. It was used in the uniqueness proof of Theorem 2.2.

Lemma 2.14. *Let u^{sc} be a radiating solution to the Navier equation (2.5) in $|x| \geq R$. Suppose that*

$$\text{Im} \left(\int_{\Gamma_R} \tilde{\mathcal{T}}(u^{sc}|_{\Gamma_R}) \cdot \overline{u^{sc}} \, ds \right) = 0.$$

Then $u^{sc} \equiv 0$ in $|x| \geq R$.

Proof. Assume that u^{sc} can be decomposed into the form of (2.24) and (2.26) with the coefficients $\Psi_n = (\psi_p^n, \psi_s^n)^\top \in \mathbb{C}^2$. It follows from (2.27) and (2.29) that

$$\begin{aligned} \int_{\Gamma_R} \tilde{\mathcal{T}}(u^{sc}|_{\Gamma_R}) \cdot \overline{u^{sc}} \, ds &= \sum_{n \in \mathbb{Z}} (R^{-2}B_n \Psi_n, R^{-1}A_n \Psi_n) \\ &= R^{-3} \sum_{n \in \mathbb{Z}} (A_n^* B_n \Psi_n, \Psi_n). \end{aligned} \tag{2.37}$$

Using again the relations in (2.33), straightforward calculations show that

$$\begin{aligned}
 A_n^* B_n &= \begin{bmatrix} t_p \bar{\gamma}_p & -in \\ -in & -t_s \bar{\gamma}_s \end{bmatrix} \begin{bmatrix} (\mu + \tilde{\mu}) t_p^2 \beta_p - \tilde{\lambda} t_p^2 & i(\mu + \tilde{\mu}) n (t_s \gamma_s - 1) \\ i(\mu + \tilde{\mu}) n (t_p \gamma_p - 1) & -(\mu + \tilde{\mu}) t_s^2 \beta_s - \tilde{\mu} t_s^2 \end{bmatrix} \\
 &=: \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.
 \end{aligned}
 \tag{2.38}$$

Recalling $\tilde{\lambda} + \tilde{\mu} = \lambda + \mu$ and making use of the relations

$$t_\alpha^2 \beta_\alpha = n^2 - t_\alpha^2 - t_\alpha \gamma_\alpha, \quad \text{Im}(\bar{\gamma}_\alpha) = -\frac{2}{|H_n^{(1)}(t_\alpha)|^2 \pi t_\alpha} < 0, \quad \alpha = p, s,$$

we obtain

$$\begin{aligned}
 \text{Im}(a_{11}) &= -\text{Im}(\bar{\gamma}_p)(\lambda + 2\mu)t_p^3 = \frac{2\omega^2 R^2}{\pi |H_n^{(1)}(k_p R)|^2} > 0, \\
 \text{Im}(a_{22}) &= -\text{Im}(\bar{\gamma}_s)\mu t_s^3 = \frac{2\omega^2 R^2}{\pi |H_n^{(1)}(k_s R)|^2} > 0, \\
 a_{12} &= \bar{a}_{21}.
 \end{aligned}$$

This implies that

$$\text{Im}(A_n^* B_n) = \frac{(A_n^* B_n) - (A_n^* B_n)^*}{2i} = \frac{2\omega^2 R^2}{\pi} \begin{bmatrix} 1/|H_n^{(1)}(k_p R)|^2 & 0 \\ 0 & 1/|H_n^{(1)}(k_s R)|^2 \end{bmatrix}.$$

Now, we conclude from (2.37) and (2.38) that

$$0 = \frac{2\omega^2}{\pi R} \sum_{n \in \mathbb{Z}} \left(\left| \frac{\psi_p^n}{H_n^{(1)}(k_p R)} \right|^2 + \left| \frac{\psi_s^n}{H_n^{(1)}(k_s R)} \right|^2 \right)$$

implying that $\psi_s^n = \psi_p^n = 0$ for all $n \in \mathbb{Z}$. Therefore, $u^{sc} \equiv 0$ in $|x| \geq R$. \square

2.4. Properties of DtN map in 3D

The aim of this section is to derive properties of the generalized DtN map in 3D, following the lines in the previous section. Denote by (r, θ, ϕ) the spherical coordinates of $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$. The coordinate $\theta \in [0, \pi]$ corresponds to the angle from the z -axis, whereas $\phi \in [0, 2\pi)$ corresponds to the polar angle in the (x, y) -plane. Let

$$\begin{aligned}
 \hat{r} &= (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)^\top, \\
 \hat{\theta} &= (-\sin \theta, \cos \theta, 0)^\top, \\
 \hat{\phi} &= (\cos \theta \cos \phi, \sin \theta \cos \phi, -\sin \phi)^\top
 \end{aligned}$$

be the unit vectors in the spherical coordinates. In 3D, we need the nm -th spherical harmonic functions

$$Y_{nm}(\hat{x}) := Y_{nm}(\theta, \phi) = \sqrt{\frac{(2n+1)(n-|m|)!}{4\pi(n+|m|)!}} P_n^{|m|}(\cos \theta) e^{im\phi}, \quad \hat{x} := x/|x| \in \mathbb{S}^2$$

for all $n \in \mathbb{N}$ and $m = -n, \dots, n$, where P_n^m is the m -th associated Legendre function of order n . Let u_{nm} and \mathbf{V}_{nm} be the vector spherical harmonics defined as

$$u_{nm}(\hat{\mathbf{x}}) := \frac{\nabla_{\mathbb{S}^2} Y_{nm}(\hat{\mathbf{x}})}{\sqrt{\delta_n}}, \quad \mathbf{V}_{nm}(\hat{\mathbf{x}}) := \hat{\mathbf{x}} \times u_{nm}(\hat{\mathbf{x}}), \tag{2.39}$$

where $\delta_n := n(n + 1)$ and $\nabla_{\mathbb{S}^2}$ denotes the surface gradient on \mathbb{S}^2 . They form a complete orthonormal basis in the L^2 -tangent space of the unit sphere

$$L_T^2(\mathbb{S}^2) := \{ \varphi \in (L^2(\mathbb{S}^2))^3 : \hat{\mathbf{x}} \cdot \varphi(\hat{\mathbf{x}}) = 0 \}, \tag{2.40}$$

and satisfy the following equations for any $f(r) \in C^1(\mathbb{R}^+)$:

$$\text{curl}(f(r)\mathbf{V}_{nm}) = -\frac{\sqrt{\delta_n}f(r)}{r}Y_{nm}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial(rf(r))}{\partial r}u_{nm}, \tag{2.41}$$

$$\hat{\mathbf{r}} \times \text{curl}(f(r)\mathbf{V}_{nm}) = -\frac{1}{r}\frac{\partial(rf(r))}{\partial r}\mathbf{V}_{nm}, \tag{2.42}$$

$$\hat{\mathbf{r}} \times \text{curl}(f(r)Y_{nm}\hat{\mathbf{r}}) = \frac{\sqrt{\delta_n}f(r)}{r}u_{nm}, \tag{2.43}$$

$$\text{div}(f(r)u_{nm}) = -\frac{\sqrt{\delta_n}f(r)}{r}Y_{nm}. \tag{2.44}$$

Denote

$$\sum_{m,n} := \sum_{n=0}^{\infty} \sum_{m=-n}^n .$$

As done in 2D, we split a radiating solution v^{sc} to the Navier equation (2.5) into its compressional and shear parts,

$$v^{sc} = \text{grad } \psi_p + \boldsymbol{\psi}_s, \quad \text{div } \boldsymbol{\psi}_s = 0, \tag{2.45}$$

where ψ_p is a scalar function satisfying

$$\Delta\psi_p + k_p^2 = 0, \quad \lim_{r \rightarrow \infty} r \left(\frac{\partial\psi_p}{\partial r} - ik_p\psi_p \right) = 0, \tag{2.46}$$

and the vector function $\boldsymbol{\psi}_s$ fulfills

$$\text{curl curl } \boldsymbol{\psi}_s - k_s^2\boldsymbol{\psi}_s = 0, \quad \lim_{r \rightarrow \infty} r \left(\frac{\partial\boldsymbol{\psi}_s}{\partial r} - ik_s\boldsymbol{\psi}_s \right) = 0. \tag{2.47}$$

The solutions of (2.46) and (2.47) in $|x| \geq R$ can be expressed as

$$\psi_p = \sum_{m,n} \frac{h_n^{(1)}(k_p r)}{h_n^{(1)}(k_p R)} \psi_p^{nm} Y_{nm}(\theta, \phi), \tag{2.48}$$

$$\boldsymbol{\psi}_s = \sum_{m,n} \left\{ \frac{h_n^{(1)}(k_s r)}{h_n^{(1)}(k_s R)} \psi_{s,1}^{nm} \mathbf{V}_{nm}(\theta, \phi) + \text{curl} \left[\frac{h_n^{(1)}(k_s r)}{h_n^{(1)}(k_s R)} \psi_{s,2}^{nm} \mathbf{V}_{nm}(\theta, \phi) \right] \right\}, \tag{2.49}$$

where $\psi_p^{nm}, \psi_{s,j}^{nm} (j = 1, 2) \in \mathbb{C}$ and $h_n^{(1)}$ is the spherical Bessel function of the third kind of order n . A direct calculation implies that

$$\begin{aligned}
 v^{sc}(x) &= \sum_{m,n} \frac{h_n^{(1)}(k_s r)}{h_n^{(1)}(k_s R)} \psi_{s,1}^{nm} \mathbf{V}_{nm}(\theta, \phi) \\
 &+ \sum_{m,n} \left\{ \frac{\sqrt{\delta_n} h_n^{(1)}(k_p r)}{r h_n^{(1)}(k_p R)} \psi_p^{nm} - \left[\frac{h_n^{(1)}(k_s r)}{r h_n^{(1)}(k_s R)} + \frac{k_s h_n^{(1)'}(k_s r)}{h_n^{(1)}(k_s R)} \right] \psi_{s,2}^{nm} \right\} u_{nm}(\theta, \phi) \\
 &+ \sum_{m,n} \left\{ \frac{k_p h_n^{(1)'}(k_p r)}{h_n^{(1)}(k_p R)} \psi_p^{nm} - \frac{\sqrt{\delta_n} h_n^{(1)}(k_s r)}{r h_n^{(1)}(k_s R)} \psi_{s,2}^{nm} \right\} Y_{nm}(\theta, \phi) \hat{\mathbf{r}}.
 \end{aligned} \tag{2.50}$$

Analogous to the 2D case, we set

$$t_\alpha := k_\alpha R, \quad \gamma_\alpha := \frac{h_n^{(1)'}(t_\alpha)}{h_n^{(1)}(t_\alpha)}, \quad \beta_\alpha := \frac{h_n^{(1)''}(t_\alpha)}{h_n^{(1)}(t_\alpha)}, \quad \alpha = p, s. \tag{2.51}$$

Due to the orthogonality relations for u_{nm}, \mathbf{V}_{nm} and $Y_{nm} \hat{\mathbf{r}}$ we derive from (2.50) that

$$\begin{aligned}
 (v^{sc}|_{\Gamma_R}, \mathbf{V}_{nm}) &= \psi_{s,1}^{nm}, \\
 (v^{sc}|_{\Gamma_R}, u_{nm}) &= \frac{1}{R} \left[\sqrt{\delta_n} \psi_p^{nm} - (1 + t_s \gamma_s) \psi_{s,2}^{nm} \right], \\
 (v^{sc}|_{\Gamma_R}, Y_{nm} \hat{\mathbf{r}}) &= \frac{1}{R} \left(t_p \gamma_p \psi_p^{nm} - \sqrt{\delta_n} \psi_{s,2}^{nm} \right).
 \end{aligned}$$

In other words,

$$A_n \begin{bmatrix} \psi_{s,1}^{nm} \\ \psi_{s,2}^{nm} \\ \psi_p^{nm} \end{bmatrix} = R \begin{bmatrix} (v^{sc}|_{\Gamma_R}, \mathbf{V}_{nm}) \\ (v^{sc}|_{\Gamma_R}, u_{nm}) \\ (v^{sc}|_{\Gamma_R}, Y_{nm} \hat{\mathbf{r}}) \end{bmatrix}, \quad A_n := \begin{bmatrix} R & 0 & 0 \\ 0 & -1 - t_s \gamma_s & \sqrt{\delta_n} \\ 0 & -\sqrt{\delta_n} & t_p \gamma_p \end{bmatrix}. \tag{2.52}$$

Lemma 2.15. *The matrix A_n is invertible for all $n \geq 0, R > 0, k_p > 0$ and $k_s > 0$. Its inverse is given by*

$$A_n^{-1} = \begin{bmatrix} \frac{1}{R} & 0 & 0 \\ 0 & \frac{t_p \gamma_p}{\Lambda_n} & -\frac{\sqrt{\delta_n}}{\Lambda_n} \\ 0 & \frac{\sqrt{\delta_n}}{\Lambda_n} & \frac{-1 - t_s \gamma_s}{\Lambda_n} \end{bmatrix}, \quad \Lambda_n := \delta_n - t_p \gamma_p (1 + t_s \gamma_s). \tag{2.53}$$

Proof. It is sufficient to prove that $\det(A_n) \neq 0$, or equivalently, $\Lambda_n \neq 0$. Setting $I_n(z) := z h_n^{(1)'}(z) / h_n^{(1)}(z)$, we have $\Lambda_n = \delta_n - I_n(t_p) - I_n(t_p) I_n(t_s)$. Recalling from [13, Theorem 2.6.1] that

$$1 \leq -\operatorname{Re} I_n(z) \leq n + 1, \quad 0 < \operatorname{Im} I_n(z) = \frac{1}{z |h_n^{(1)}(z)|^2} \leq z \quad \text{for all } z > 0, \tag{2.54}$$

we obtain

$$\operatorname{Im}(\Lambda_n) = -\operatorname{Im} I_n(t_p) (1 + \operatorname{Re}(I_n(t_s))) - \operatorname{Re} I_n(t_p) \operatorname{Im} I_n(t_s) > 0. \quad \square$$

The equation (2.52) implies the following remark.

Remark 2.16. The unique radiating solution v^{sc} to the boundary value problem (2.17) can be represented in the form of (2.24) and (2.26), where the coefficients $\psi_p^{n,m}$ and $\psi_{s,j}^{n,m}$ ($j = 1, 2$) are given by

$$\begin{bmatrix} \psi_{s,1}^{nm} \\ \psi_{s,2}^{nm} \\ \psi_p^{nm} \end{bmatrix} = RA_n^{-1} \begin{bmatrix} (w, \mathbf{V}_{nm}) \\ (w, u_{nm}) \\ (w, Y_{nm} \hat{\mathbf{r}}) \end{bmatrix}.$$

We now consider the generalized stress operator

$$\tilde{\mathcal{T}}v^{sc} = (\mu + \tilde{\mu})\hat{\mathbf{r}} \cdot \text{grad } v^{sc} + \tilde{\lambda}\hat{\mathbf{r}} \text{ div } v^{sc} + \tilde{\mu}\hat{\mathbf{r}} \times \text{curl } v^{sc}, \tag{2.55}$$

where $\tilde{\lambda}, \tilde{\mu} \in \mathbb{R}$ satisfying $\tilde{\lambda} + \tilde{\mu} = \lambda + \mu$. Using the notation introduced in (2.51), the first and second terms on the right hand side of (2.55) can be rewritten respectively as

$$\begin{aligned} (\hat{\mathbf{r}} \cdot \text{grad } v^{sc})|_{\Gamma_R} &= \left(\frac{\partial v^{sc}}{\partial r} \right) \Big|_{\Gamma_R} \\ &= \sum_{m,n} \frac{t_s \gamma_s}{R} \psi_{s,1}^{nm} \mathbf{V}_{nm}(\theta, \phi) \\ &\quad + \sum_{m,n} \frac{1}{R^2} \left[\sqrt{\delta_n} (t_p \gamma_p - 1) \psi_p^{nm} + (1 - t_s \gamma_s - t_s^2 \beta_s) \psi_{s,2}^{nm} \right] u_{nm}(\theta, \phi) \\ &\quad + \sum_{m,n} \frac{1}{R^2} \left[t_p^2 \beta_p \psi_p^{nm} + \sqrt{\delta_n} (1 - t_s \gamma_s) \psi_{s,2}^{nm} \right] Y_{nm}(\theta, \phi) \hat{\mathbf{r}}, \end{aligned}$$

and

$$(\hat{\mathbf{r}} \text{ div } v^{sc})|_{\Gamma_R} = (\hat{\mathbf{r}} \Delta \psi_p) |_{\Gamma_R} = (-k_p^2 \psi_p \hat{\mathbf{r}}) |_{\Gamma_R} = - \sum_{m,n} \frac{t_p^2}{R^2} \psi_p^{nm} Y_{nm}(\theta, \phi) \hat{\mathbf{r}}.$$

Since $h_n^{(1)}(k_s r) \mathbf{V}_{nm}(\theta, \phi)$ is a radiating solution of (2.47) and

$$\hat{\mathbf{r}} \times \mathbf{V}_{nm} = \hat{\mathbf{r}} (\hat{\mathbf{r}} \cdot u_{nm}) - u_{nm} (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) = -u_{nm},$$

the third term of $\tilde{\mathcal{T}}v^{sc}$ in (2.55) takes the form

$$\begin{aligned} (\hat{\mathbf{r}} \times \text{curl } v^{sc})|_{\Gamma_R} &= (\hat{\mathbf{r}} \times \text{curl } \psi_s) |_{\Gamma_R} \\ &= \sum_{m,n} \hat{\mathbf{r}} \times \text{curl} \left[\frac{h_n^{(1)}(k_s r)}{h_n^{(1)}(k_s R)} \psi_{s,1}^{nm} \mathbf{V}_{nm}(\theta, \phi) \right] \\ &\quad + \sum_{m,n} \left\{ \hat{\mathbf{r}} \times \text{curl} \text{curl} \left[\psi_{s,2}^{nm} \frac{h_n^{(1)}(k_s r)}{h_n^{(1)}(k_s R)} \mathbf{V}_{nm}(\theta, \phi) \right] \right\} \\ &= - \sum_{m,n} \frac{1}{R} (1 + t_s \gamma_s) \psi_{s,1}^{nm} \mathbf{V}_{nm}(\theta, \phi) \\ &\quad - \sum_{m,n} \frac{t_s^2}{R^2} \psi_{s,2}^{nm} u_{nm}(\theta, \phi). \end{aligned}$$

Therefore,

$$\begin{aligned} \left(\tilde{\mathcal{T}}v^{sc}|_{\Gamma_R}, \mathbf{V}_{nm}\right) &= \frac{1}{R} (\mu t_s \gamma_s - \tilde{\mu}) \psi_{s,1}^{nm}, \\ \left(\tilde{\mathcal{T}}v^{sc}|_{\Gamma_R}, u_{nm}\right) &= \frac{1}{R^2} \sqrt{\delta_n} (\mu + \tilde{\mu}) (t_p \gamma_p - 1) \psi_p^{nm} \\ &\quad + \frac{1}{R^2} [(\mu + \tilde{\mu}) (1 - t_s \gamma_s - t_s^2 \beta_s) - \tilde{\mu} t_s^2] \psi_{s,2}^{nm}, \\ \left(\tilde{\mathcal{T}}v^{sc}|_{\Gamma_R}, Y_{nm} \hat{\mathbf{r}}\right) &= \frac{1}{R^2} \left\{ [(\mu + \tilde{\mu}) t_p^2 \beta_p - \tilde{\lambda} t_p^2] \psi_p^{nm} + \sqrt{\delta_n} (\mu + \tilde{\mu}) (1 - t_s \gamma_s) \psi_{s,2}^{nm} \right\}. \end{aligned}$$

Set the matrices B_n to be

$$\begin{bmatrix} R(\mu t_s \gamma_s - \tilde{\mu}) & 0 & 0 \\ 0 & (\mu + \tilde{\mu}) (1 - t_s \gamma_s - t_s^2 \beta_s) - \tilde{\mu} t_s^2 & \sqrt{\delta_n} (\mu + \tilde{\mu}) (t_p \gamma_p - 1) \\ 0 & \sqrt{\delta_n} (\mu + \tilde{\mu}) (1 - t_s \gamma_s) & (\mu + \tilde{\mu}) t_p^2 \beta_p - \tilde{\lambda} t_p^2 \end{bmatrix}, \tag{2.56}$$

and define $W_n := 1/RB_nA_n^{-1}$. Then we obtain

$$\begin{bmatrix} \left(\tilde{\mathcal{T}}v^{sc}|_{\Gamma_R}, \mathbf{V}_{nm}\right) \\ \left(\tilde{\mathcal{T}}v^{sc}|_{\Gamma_R}, u_{nm}\right) \\ \left(\tilde{\mathcal{T}}v^{sc}|_{\Gamma_R}, Y_{nm} \hat{\mathbf{r}}\right) \end{bmatrix} = B_n \begin{bmatrix} \psi_{s,1}^{nm} \\ \psi_{s,2}^{nm} \\ \psi_p^{nm} \end{bmatrix} = W_n \begin{bmatrix} (v^{sc}|_{\Gamma_R}, \mathbf{V}_{nm}) \\ (v^{sc}|_{\Gamma_R}, u_{nm}) \\ (v^{sc}|_{\Gamma_R}, Y_{nm} \hat{\mathbf{r}}) \end{bmatrix}. \tag{2.57}$$

The above identity links the generalized stress operator $\tilde{\mathcal{T}}v^{sc}|_{\Gamma_R}$ and $v^{sc}|_{\Gamma_R}$ in the coordinate system $(\mathbf{V}_{nm}, u_{nm}, Y_{nm} \hat{\mathbf{r}})$ of the vector space $(L^2(\mathbb{S}^2))^3$. Below we shall investigate properties of the three dimensional DtN map $\tilde{\mathcal{T}}$ using (2.57).

Lemma 2.17.

- (i) $\tilde{\mathcal{T}}$ is a bounded linear operator from $(H^s(\Gamma_R))^3$ to $(H^{s-1}(\Gamma_R))^3$ for all $s \in \mathbb{R}$.
- (ii) The matrix $-\text{Re } W_n$ is positive definite for sufficiently large $n > 0$. Hence $\tilde{\mathcal{T}}$ is the sum of a positive operator and a compact operator over $(H^{1/2}(\Gamma_R))^3$.
- (iii) Lemma 2.14 remains valid for the generalized DtN map $\tilde{\mathcal{T}}$ in 3D.

Proof. (i) We only need to show that the max norm of the matrix W_n is bounded by

$$\|W_n\|_{\max} = R^{-1} \|B_n A_n^{-1}\|_{\max} \leq C n, \tag{2.58}$$

for some constant $C > 0$ uniformly in all $n > 0$, where the matrices A_n and B_n are given by (2.53) and (2.56), respectively. For this purpose we need to derive the asymptotics of each entry $W_n^{(i,j)}$ ($1 \leq i, j \leq 3$) of W_n . In three dimensions, it holds that

$$\begin{aligned} &h_n^{(1)''}(z) \\ &= \left(h_{n-1}^{(1)}(z) - \frac{n+1}{z} h_n^{(1)}(z) \right)' \\ &= -h_n^{(1)}(z) + \frac{n-1}{z} h_{n-1}^{(1)}(z) + \frac{n+1}{z^2} h_n^{(1)}(z) - \frac{n+1}{z} \left(h_{n-1}^{(1)}(z) - \frac{n+1}{z} h_n^{(1)}(z) \right) \end{aligned}$$

$$\begin{aligned} &= \frac{(n+1)^2 + n + 1 - z^2}{z^2} h_n^{(1)}(z) - \frac{2}{z} h_{n-1}^{(1)}(z) \\ &= \frac{(n+1)^2 + n + 1 - z^2}{z^2} h_n^{(1)}(z) - \frac{2}{z} \left(h_n^{(1)'}(z) + \frac{n+1}{z} h_n^{(1)}(z) \right) \\ &= \left(\frac{\delta_n}{z^2} - 1 \right) h_n^{(1)}(z) - \frac{2}{z} h_n^{(1)'}(z), \end{aligned}$$

implying that

$$\beta_p = \frac{\delta_n}{t_p^2} - 1 - \frac{2}{t_p} \gamma_p, \quad \beta_s = \frac{\delta_n}{t_s^2} - 1 - \frac{2}{t_s} \gamma_s. \tag{2.59}$$

Note that the relations in (2.59) differ from those in two dimensions; cf. (2.33). Using the expressions of B_n and A_n , we obtain the entries of W_n via straightforward calculations

$$\begin{aligned} W_n^{(1,1)} &= \frac{\mu t_s \gamma_s - \tilde{\mu}}{R}, \\ W_n^{(1,2)} &= W_n^{(2,1)} = W_n^{(1,3)} = W_n^{(3,1)} = 0, \\ W_n^{(2,2)} &= \frac{1}{R \Lambda_n} [(\mu + \tilde{\mu})(t_p t_s \gamma_p \gamma_s - \delta_n + t_p \gamma_p) + \mu t_s^2 t_p \gamma_p], \\ W_n^{(3,3)} &= \frac{1}{R \Lambda_n} [t_p^2 (\lambda + 2\mu)(1 + t_s \gamma_s) + 2(\mu + \tilde{\mu})(t_p t_s \gamma_p \gamma_s - \delta_n + t_p \gamma_p)], \\ W_n^{(2,3)} &= \frac{1}{R \Lambda_n} [\sqrt{\delta_n}(\mu + \tilde{\mu})(\delta_n - t_p t_s \gamma_p \gamma_s - t_p \gamma_p) - \mu t_s^2 \sqrt{\delta_n}], \\ W_n^{(3,2)} &= \frac{1}{R \Lambda_n} [\sqrt{\delta_n}(\mu + \tilde{\mu})(\delta_n - t_p t_s \gamma_p \gamma_s - t_p \gamma_p) - (\lambda + 2\mu) t_p^2 \sqrt{\delta_n}], \end{aligned}$$

in which we have used the relation (2.59) and the fact that $\tilde{\lambda} + \tilde{\mu} = \lambda + \mu$. Now, we need to derive the asymptotics of $W_n^{(i,j)}$ ($1 \leq i, j \leq 3$) as n tends to infinity. From the series expansions of the spherical Bessel and Neumann functions we know

$$h_n^{(1)}(z) = \frac{1}{i} 1 \cdot 3 \cdots (2n-1) \left[\frac{1}{z^{n+1}} + \frac{1}{2z^{n-1}(2n-1)} + O\left(\frac{1}{n^2}\right) \right], \quad n \rightarrow +\infty.$$

Then

$$\begin{aligned} \frac{h_{n-1}^{(1)}(z)}{h_n^{(1)}(z)} &= \frac{1}{2n-1} \frac{\frac{1}{z^n} + \frac{1}{2z^{n-2}(2n-3)} + O\left(\frac{1}{n^2}\right)}{\frac{1}{z^{n+1}} + \frac{1}{2z^{n-1}(2n-1)} + O\left(\frac{1}{n^2}\right)} \\ &= \left[\frac{1}{2n} + O\left(\frac{1}{n^2}\right) \right] \left[z + O\left(\frac{1}{n}\right) \right] \\ &= \frac{z}{2n} + O\left(\frac{1}{n^2}\right), \end{aligned}$$

which further leads to

$$\frac{h_n^{(1)'}(z)}{h_n^{(1)}(z)} = \frac{z}{2n} - \frac{n+1}{z} + O\left(\frac{1}{n^2}\right), \quad n \rightarrow +\infty.$$

Therefore, as $n \rightarrow +\infty$,

$$\begin{aligned} W_n^{(1,1)} &= -\frac{\mu}{R}n - \frac{\mu + \tilde{\mu}}{R} + O\left(\frac{1}{n}\right), \\ W_n^{(2,2)} &= -\frac{2\mu(\lambda + 2\mu)}{R(\lambda + 3\mu)}n + O(1), \\ W_n^{(3,3)} &= -\frac{2\mu(\lambda + 2\mu)}{R(\lambda + 3\mu)}n + O(1), \\ W_n^{(2,3)} &= \frac{[(\mu + \tilde{\mu})(\lambda + 3\mu) - 2\mu(\lambda + 2\mu)]}{R(\lambda + 3\mu)}n + O(1), \\ W_n^{(3,2)} &= \frac{[(\mu + \tilde{\mu})(\lambda + 3\mu) - 2\mu(\lambda + 2\mu)]}{R(\lambda + 3\mu)}n + O(1). \end{aligned}$$

This proves (2.58) and thus the first assertion.

(ii) Set $\tilde{W}_n := -(W_n + W_n^*)/2$ for $n \geq 0$. For sufficiently large $n > 0$, we have

$$\begin{aligned} \tilde{W}_n^{(1,1)} &= \frac{\mu}{R}n + \frac{\mu + \tilde{\mu}}{R} + O\left(\frac{1}{n}\right) > 0, \\ \tilde{W}_n^{(1,1)}\tilde{W}_n^{(2,2)} &= \frac{2\mu^2(\lambda + 2\mu)}{R^2(\lambda + 3\mu)}n^2 + O(n) > 0, \\ \det(\tilde{W}_n) &= \tilde{W}_n^{(1,1)} \left(\frac{4\mu^2(\lambda + 2\mu)^2 - [(\lambda - \tilde{\lambda})(\lambda + 3\mu) + 2\mu^2]^2}{R^2(\lambda + 3\mu)^2}n^2 + O(n) \right). \end{aligned}$$

Recalling the assumption (2.22) on $\tilde{\lambda}$ we see

$$4\mu^2(\lambda + 2\mu)^2 - [(\lambda - \tilde{\lambda})(\lambda + 3\mu) + 2\mu^2]^2 > 0.$$

This implies that $\det \tilde{W}_n$ is positive definite over \mathbb{C}^3 for sufficiently large n . The proof of the second assertion is complete.

(iii) Assume that a radiating solution v^{sc} to the Navier equation (2.5) admits the series expansion (2.45), (2.48) and (2.49) with the vector coefficient $\Psi^{nm} := (\psi_{s,1}^{nm}, \psi_{s,2}^{nm}, \psi_p^{nm})^\top \in \mathbb{C}^3$. Making use of (2.52) and the first relation in (2.57), we get

$$\begin{aligned} \int_{\Gamma_R} \tilde{T}(v^{sc}|_{\Gamma_R}) \cdot \overline{v^{sc}} ds &= \sum_{m,n} \langle R^{-2}B_n^{nm}, R^{-1}A_n\Psi^{nm} \rangle \\ &= R^{-3} \sum_{m,n} \langle A_n^*B_n\Psi^{nm}, \Psi^{nm} \rangle. \end{aligned}$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product over \mathbb{C}^3 . Hence,

$$\sum_{m,n} \langle \text{Im}(A_n^*B_n)\Psi^{nm}, \Psi^{nm} \rangle = 0. \tag{2.60}$$

To evaluate the product of A_n^* and B_n we need the identities (cf. (2.54), (2.59))

$$\text{Im}(t_\alpha\gamma_\alpha) = 1/(t_\alpha|h_n^{(1)}(t_\alpha)|^2) > 0, \quad t_\alpha^2\beta_\alpha = \delta_n - t_\alpha^2 - 2t_\alpha\gamma_\alpha, \quad \alpha = p, s. \tag{2.61}$$

Since

$$A_n^* = \begin{bmatrix} R & 0 & 0 \\ 0 & -1 - t_s \bar{\gamma}_s & -\sqrt{\delta_n} \\ 0 & \sqrt{\delta_n} & t_p \bar{\gamma}_p \end{bmatrix},$$

direct calculations show that

$$\text{Im}(A_n^* B_n) = R^2 \begin{bmatrix} \mu \text{Im}(t_s \gamma_s) & 0 & 0 \\ 0 & \omega^2 \text{Im}(t_s \gamma_s) & 0 \\ 0 & 0 & \omega^2 \text{Im}(t_s \gamma_s) \end{bmatrix}.$$

This together with (2.60) and the first relation in (2.61) yields $|\Psi^{nm}| = 0$ for all $n \geq 0, m = -n, \dots, n$. Therefore, $v^{sc} \equiv 0$ in $|x| \geq R$. \square

3. Reconstruction of multiple anisotropic obstacles

In this section, we consider the inverse scattering problem of reconstructing the support of multiple unknown anisotropic obstacles from near-field measurement data. We first derive the Fréchet derivative of the near-field solution operator, which maps the boundaries of several disconnected scatterers to the measurement data. Then, as an application, we design an iterative approach to the inverse problem using the data of one or several incident directions and frequencies.

3.1. Fréchet derivative of the solution operator

Suppose that $\Omega = \cup_{i=1}^{N_0} \Omega_j$ is a union of several disconnected bounded components $\Omega_j \subset \mathbb{R}^N$. Each component Ω_j is supposed to be occupied by an anisotropic elastic obstacle with constant density $\rho_j > 0$ and constant stiffness tensor $\mathcal{C}_j = (C_{j,klmn})_{k,l,m,n=1}^N$. Assume that the boundary Γ_j of Ω_j is C^2 . Let $\Omega_0 := B_R \setminus \bar{\Omega}$. Denote by $\rho_0 > 0$ and $\mathcal{C}_0 = (C_{0,klmn})_{k,l,m,n=1}^N$ the density and stiffness tensor of the homogeneous isotropic background medium. Set

$$u := \begin{cases} u_j, & x \in \Omega_j, \\ u^{sc} + u^{in}, & x \in \mathbb{R}^N \setminus \bar{\Omega}. \end{cases} \tag{3.62}$$

We assume there is an a priori information that the unknown elastic scatterers $\Omega_j, j = 1, \dots, N_0$, are embedded in the region B_R for some $R > 0$. The variational formulation for the forward scattering problem in the truncated domain B_R reads as follows: find $u \in X_R := (H^1(B_R))^N$ such that

$$a(u, v) = \int_{\Gamma_R} f \cdot \bar{v} \, ds \quad \text{for all } v \in X_R, \quad f := (Tu^{in} - \mathcal{T}u^{in})|_{\Gamma_R}, \tag{3.63}$$

where

$$a(u, v) := \sum_{j=0}^{N_0} A_{\Omega_j}(u, v) - \int_{\Gamma_R} \mathcal{T}u \cdot \bar{v} \, ds$$

$$A_{\Omega_j}(u, v) := \int_{\Omega_j} \left(\sum_{k,l,m,n=1}^N C_{j,klmn} \frac{\partial u_m}{\partial x_n} \frac{\partial \bar{v}_k}{\partial x_l} - \rho_j \omega^2 u \cdot \bar{v} \right) dx, \quad j = 0, 1, \dots, N_0.$$

Here \mathcal{T} is the DtN map introduced in the previous section. We study the following inverse problem:

(IP): Determine the boundaries $\Gamma_1, \dots, \Gamma_{N_0}$ from knowledge of multi-frequency near-field measurements $u|_{\Gamma_R}$ corresponding to the incident plane wave (2.13) with one or several incident directions.

Let $u \in X_R$ be the unique solution to the variational problem (3.63). Since each boundary Γ_j is C^2 , we have $u \in (H^2(B_R))^N$. In this paper we define the near-field solution operator \mathcal{J} as

$$\mathcal{J} : (\Gamma_1, \dots, \Gamma_{N_0}) \rightarrow u|_{\Gamma_R}. \tag{3.64}$$

The mapping \mathcal{J} is obviously nonlinear. To define the Fréchet derivative of \mathcal{J} with respect to the boundary $\Gamma = \cup_{j=1}^{N_0} \Gamma_j$, we assume that the function

$$h_j = (h_{j,1}, \dots, h_{j,N})^\top \in (C^1(\Gamma_j))^N, \quad \|h_j\|_{(C^1(\Gamma_j))^N} \ll 1$$

is a small perturbation of Γ_j . The perturbed boundary is given by

$$\Gamma_{j,h} := \{y \in \mathbb{R}^N : y = x + h_j(x), x \in \Gamma_j\}.$$

Definition 3.1. The solution operator \mathcal{J} is called Fréchet differentiable at Γ if there exists a linear bounded operator $\mathcal{J}'_\Gamma : (C^1(\Gamma_1))^N \times \dots \times (C^1(\Gamma_{N_0}))^N \rightarrow (L^2(\Gamma_R))^N$ such that

$$\begin{aligned} \|\mathcal{J}(\Gamma_{1,h}, \dots, \Gamma_{N_0,h}) - \mathcal{J}(\Gamma_1, \dots, \Gamma_{N_0}) - \mathcal{J}'_\Gamma(h_1, \dots, h_{N_0})\|_{(L^2(\Gamma_R))^N} \\ = o\left(\sum_{j=1}^{N_0} \|h_j\|_{(C^1(\Gamma_j))^N}\right). \end{aligned}$$

The operator \mathcal{J}'_Γ is called the Fréchet derivative of \mathcal{J} at Γ .

Given $h_j \in (C^1(\Gamma_j))^N$, there exists an extension of h_j , which we still denote by h_j , such that $h_j \in (C^1(\mathbb{R}^N))^N$, $\|h_j\|_{(C^1(\mathbb{R}^N))^N} \leq c\|h_j\|_{(C^1(\Gamma_j))^N}$ and $\text{supp}(h_j) \subset K_j$, where K_j is a domain satisfying $\Gamma_j \subset K_j \subset\subset B_R \setminus \overline{(\cup_{i=1, i \neq j}^{N_0} \Omega_i)}$. Define the functions

$$h(x) := \sum_{j=1}^{N_0} h_j(x), \quad y = \xi^h(x) = x + h(x), \quad x \in \mathbb{R}^N.$$

For small perturbations, ξ^h is a diffeomorphism between Γ_j and $\Gamma_{j,h}$. The inverse map of ξ^h is denoted by η^h . Corresponding to Ω_j ($j = 0, 1, \dots, N_0$), we define

$$\Omega_{j,h} := \{y \in \mathbb{R}^N : y = \xi^h(x), x \in \Omega_j\}, \quad j = 1, 2, \dots, N_0, \quad \Omega_{0,h} := B_R \setminus \overline{\cup_{j=1}^{N_0} \Omega_{j,h}}.$$

The differentiability of \mathcal{J} at Γ is stated as following.

Theorem 3.2. Let u (see (3.62)) be the unique solution of the variational problem (3.63), and let $h_j \in (C^1(\Gamma_j))^N$, $j = 1, \dots, N_0$, be sufficiently small perturbations. Then the solution operator \mathcal{J} is Fréchet differentiable at Γ . Further, the Fréchet derivative \mathcal{J}'_Γ is given by $\mathcal{J}'_\Gamma(h_1, \dots, h_{N_0}) = \tilde{u}_0|_{\Gamma_R}$, where \tilde{u}_0 together with \tilde{u}_j ($j = 1, \dots, N_0$) is the unique weak solution of the boundary value problem:

$$\nabla \cdot (\mathcal{C}_j : \nabla \tilde{u}_j) + \rho_j \omega^2 \tilde{u}_j = 0 \quad \text{in } \Omega_j, \quad j = 0, 1, \dots, N_0, \tag{3.65}$$

$$\tilde{u}_j - \tilde{u}_0 - f_j = 0 \quad \text{on } \Gamma_j, \quad j = 1, \dots, N_0, \tag{3.66}$$

$$\mathcal{N}_{\mathcal{C}}^- \tilde{u}_j - \mathcal{N}_{\mathcal{C}}^+ \tilde{u}_0 - g_j = 0 \quad \text{on } \Gamma_j, \quad j = 1, \dots, N_0, \tag{3.67}$$

$$\mathcal{T} \tilde{u}_0 - \mathcal{T} \tilde{u}_0 = 0 \quad \text{on } \Gamma_R, \tag{3.68}$$

where

$$f_j = -(h_j \cdot \nu) [\partial_\nu^- u_j - \partial_\nu^+ (u^{sc} + u^{in})] |_{\Gamma_j} \tag{3.69}$$

and the expressions of $g_j \in (H^{-1/2}(\Gamma_j))^N$ rely on the space dimensions. In 2D, we have

$$g_j = \omega^2 (h_j \cdot \nu) [\rho_j u_j^- - \rho_0 (u^{sc} + u^{in})^+] - \partial_\tau [((\sigma_j(u_j))^- - (\sigma_0(u^{sc} + u^{in}))^+) (h_{j,2}, -h_{j,1})^\top], \tag{3.70}$$

where $\partial_\tau = \nu^\perp \cdot \nabla$ is the tangential derivative. In 3D, it holds that

$$g_j = \omega^2 (h_j \cdot \nu) [\rho_j u_j^- - \rho_0 (u^{sc} + u^{in})^+] - \text{div}_{\Gamma_j} ((\mathbf{A}_j - \mathbf{A}_{0,j}) \times \nu), \tag{3.71}$$

where div_Γ is the surface divergence operator on Γ and $\mathbf{A}_j \in \mathbb{C}^{N \times N}$ are defined by

$$\mathbf{A}_{0,j} = (\sigma_0(u^{sc} + u^{in}))^+ M_{h_j}, \quad \mathbf{A}_j = (\sigma_j(u_j))^- M_{h_j}, \quad j = 1, \dots, N_0, \tag{3.72}$$

with

$$M_{h_j} := \begin{bmatrix} 0 & -h_{j,3} & h_{j,2} \\ -h_{j,3} & 0 & h_{j,1} \\ h_{j,2} & h_{j,1} & 0 \end{bmatrix}, \quad j = 1, \dots, N_0.$$

Proof. Set the space

$$\mathcal{H} := \{(v, w) \in (H^1(\Omega))^N \times (H^1(\Omega_0))^N : v = w \quad \text{on } \Gamma_j, \quad j = 1, \dots, N_0\}.$$

The variational problem of (3.65)–(3.68) can be formulated as the problem of finding $\tilde{u}_0 \in (H^1(\Omega_0))^N$, $\tilde{u}_j \in (H^1(\Omega_j))^N$ such that $\tilde{u}_j - \tilde{u}_0 = f_j$ on Γ_j , $j = 1, \dots, N_0$, and

$$\sum_{j=0}^{N_0} A_{\Omega_j}(\tilde{u}_j, v) - \int_{\Gamma_R} \mathcal{T} \tilde{u}_0 \cdot \bar{w} \, ds = \sum_{j=1}^{N_0} \int_{\Gamma_j} g_j \cdot \bar{w} \, ds \quad \text{for all } (v, w) \in \mathcal{H}. \tag{3.73}$$

It follows from the regularity of u that $f_j \in (H^{1/2}(\Gamma_j))^N$ and $g_j \in (H^{-1/2}(\Gamma_j))^N$. Let $\hat{f}_j \in (H^1(\Omega_j))^N$ be the trace lifting functions of f_j . Then the variational formulation (3.73) search for $\tilde{u}_0 \in (H^1(\Omega_0))^N$ and $\hat{u}_j = \tilde{u}_j - \hat{f}_j \in (H^1(\Omega_j))^N$ such that $\hat{u}_j = \tilde{u}_0$ on Γ_j , $j = 1, \dots, N_0$ and

$$\begin{aligned} & \sum_{j=0}^{N_0} A_{\Omega_j}(\hat{u}_j, v) - \int_{\Gamma_R} \mathcal{T} \tilde{u}_0 \cdot \bar{w} \, ds \\ &= \sum_{j=1}^{N_0} \int_{\Gamma_j} g_j \cdot \bar{w} \, ds - \sum_{j=1}^{N_0} A_{\Omega_j}(\hat{f}_j, v) \quad \text{for all } (v, w) \in \mathcal{H}. \end{aligned} \tag{3.74}$$

Applying Lemma 2.8 and Theorem 2.10, we see that the above variational equation (3.74) admits a unique solution. For the given functions $h_j \in (C^1(\Gamma_j))^N$, we extend them to B_R in the same way as before. Let J_{η^h} and J_{ξ^h} be the Jacobian matrices of the transforms η^h and ξ^h , respectively. It then follows that

$$J_{\xi^h} = I + \nabla h, \quad J_{\eta^h} = I - \nabla h + \mathcal{O}\left(\|h\|_{(C^1(B_R))^N}^2\right),$$

$$\det(J_{\xi^h}) = 1 + \nabla \cdot h + \mathcal{O}\left(\|h\|_{(C^1(B_R))^N}^2\right).$$

Consider the perturbed variational problem: find $u_h \in X_R$ such that

$$\sum_{j=0}^{N_0} A_{\Omega_j, h}(u_h, v_h) - \int_{\Gamma_R} \mathcal{T} u_h \cdot \overline{v_h} \, ds = \int_{\Gamma_R} f \cdot \overline{v_h} \, ds \quad \text{for all } v_h \in X_R. \tag{3.75}$$

Here $f = (Tu^{in} - \mathcal{T}u^{in})|_{\Gamma_R}$. Define $\widehat{u} = (\widehat{u}_1, \dots, \widehat{u}_N)^\top := (u_h \circ \xi^h)(x)$. Then we have

$$\begin{aligned} & \sum_{j=0}^{N_0} A_{\Omega_j, h}(u_h, v_h) \\ &= \sum_{j=0}^{N_0} \int_{\Omega_j} \sum_{k, l, m, n=1}^N C_{j,klmn} \nabla^\top \widehat{u}_m J_{\eta^h}(:, n) J_{\eta^h}(:, l)^\top \nabla \overline{\widehat{v}_k} \det(J_{\xi^h}) \, dx \\ & - \sum_{j=0}^{N_0} \rho_j \omega^2 \int_{\Omega_j} \widehat{u} \cdot \overline{\widehat{v}} \det(J_{\xi^h}) \, dx \end{aligned}$$

where $A(:, n)$ means the n -th column of the matrix A . From the stability of the direct scattering problem it follows that \widehat{u} converges to u in X_R as $\|h\|_{(C^1(B_R))^N} \rightarrow 0$. Let $w \in X_R$ be the solution of the variational problem

$$a(w, v) = \sum_{j=0}^{N_0} b_j(u, v, h) \quad \text{for all } v \in X_R,$$

where

$$\begin{aligned} & b_j(u, v, h) \\ &:= \int_{\Omega_j} \sum_{k, l, m, n=1}^N C_{j,klmn} \left[\frac{\partial u_m}{\partial x_n} \frac{\partial h^\top}{\partial x_l} \nabla \overline{v_k} + \nabla^\top u_m \frac{\partial h}{\partial x_n} \frac{\partial \overline{v_k}}{\partial x_l} - (\nabla \cdot h) \frac{\partial u_m}{\partial x_n} \frac{\partial \overline{v_k}}{\partial x_l} \right] dx \\ & + \rho_j \omega^2 \int_{\Omega_j} (\nabla \cdot h) u \cdot \overline{v} \, dx. \end{aligned} \tag{3.76}$$

Then it's easy to prove that

$$\sup_{v \in X_R} a(\widehat{u} - u - w, v) / \|v\|_{X_R} = o\left(\|h\|_{(C^1(B_R))^N}\right).$$

Applying the trace theorem it follows that $a(\widehat{u} - u - w) / \|h\|_{(C^1(B_R))^N}$ tends to zero in $(H^{1/2}(\Gamma_R))^N$ as $\|h\|_{(C^1(B_R))^N}$ tends to zero. By Definition 3.1, we get

$$\mathcal{J}'_{\Gamma}(h_1, \dots, h_{N_0}) = w|_{\Gamma_R}.$$

Hence, it only remains to prove that $w = \tilde{u}_0$ on Γ_R .

Below we are going to calculate $b_j(u, v, h)$ for $j = 0, 1, \dots, N_0$. Set $(v, w) \in \mathcal{H}$. Using integration by parts and the relation

$$(\nabla \cdot h)(u \cdot v) = \nabla \cdot [(u \cdot v)h] - (h \cdot \nabla u) \cdot v - (h \cdot \nabla v) \cdot u,$$

the last term of (3.76) can be written as

$$\begin{aligned} \rho_j \omega^2 \int_{\Omega_j} (\nabla \cdot h)u \cdot \bar{v} \, dx &= \rho_j \omega^2 \int_{\Gamma_j} (h \cdot \nu)(u \cdot \bar{v}) \, ds \\ &\quad - \rho_j \omega^2 \int_{\Omega_j} [(h \cdot \nabla u_j) \cdot \bar{v} + (h \cdot \nabla \bar{v}) \cdot u] \, dx \quad \text{for } j \geq 1. \end{aligned} \tag{3.77}$$

To compute the first integral on the right hand of (3.76), we need the identities

$$\begin{aligned} \frac{\partial u_m}{\partial x_n} \frac{\partial h^\top}{\partial x_l} \nabla \bar{v}_k &= \frac{\partial}{\partial x_l} \left(\frac{\partial u_m}{\partial x_n} h^\top \nabla \bar{v}_k \right) - \frac{\partial^2 u_m}{\partial x_l \partial x_n} (h^\top \nabla \bar{v}_k) - \frac{\partial u_m}{\partial x_n} h^\top \nabla \left(\frac{\partial \bar{v}_k}{\partial x_l} \right), \\ \nabla^\top u_m \frac{\partial h}{\partial x_n} \frac{\partial \bar{v}_k}{\partial x_l} &= \frac{\partial}{\partial x_n} (h^\top \nabla u_m) \frac{\partial \bar{v}_k}{\partial x_l} - \nabla \cdot \left(h \frac{\partial u_m}{\partial x_n} \frac{\partial \bar{v}_k}{\partial x_l} \right) \\ &\quad + (\nabla \cdot h) \frac{\partial u_m}{\partial x_n} \frac{\partial \bar{v}_k}{\partial x_l} + \frac{\partial u_m}{\partial x_n} h^\top \nabla \left(\frac{\partial \bar{v}_k}{\partial x_l} \right). \end{aligned}$$

Making use of the previous two identities and applying again the integration by parts, it follows for $j \geq 1$ that

$$\begin{aligned} &\int_{\Omega_j} \sum_{k,l,m,n=1}^N C_{j,klmn} \left[\frac{\partial u_m}{\partial x_n} \frac{\partial h^\top}{\partial x_l} \nabla \bar{v}_k + \nabla^\top u_m \frac{\partial h}{\partial x_n} \frac{\partial \bar{v}_k}{\partial x_l} - (\nabla \cdot h) \frac{\partial u_m}{\partial x_n} \frac{\partial \bar{v}_k}{\partial x_l} \right] dx \\ &= \int_{\Omega_j} \nabla \cdot [(h \cdot \nabla \bar{v}) \cdot \sigma_j(u) - h(\sigma_j(u) : \nabla \bar{v})] \, dx + \rho_j \omega^2 \int_{\Omega_j} (h \cdot \nabla \bar{v}) \cdot u \, dx \\ &\quad + \int_{\Omega_j} \sum_{k,l,m,n=1}^N C_{j,klmn} \frac{\partial}{\partial x_n} (h \cdot \nabla u_m) \frac{\partial \bar{v}_k}{\partial x_l} \, dx \\ &= \int_{\Gamma_j} [(h_j \cdot \nabla \bar{v}) \cdot (\nu \cdot \sigma_j(u_j)) - (h_j \cdot \nu)(\sigma_j(u_j) : \nabla \bar{v})] \, ds + \rho_j \omega^2 \int_{\Omega_j} (h \cdot \nabla \bar{v}) \cdot u \, dx \\ &\quad + \int_{\Omega_j} \sum_{k,l,m,n=1}^N C_{j,klmn} \frac{\partial}{\partial x_n} (h \cdot \nabla u_m) \frac{\partial \bar{v}_k}{\partial x_l} \, dx. \end{aligned} \tag{3.78}$$

Next, we proceed with the space dimensions. In two dimensions (i.e., $N = 2$), we have

$$(h_j \cdot \nabla \bar{v}) \cdot (\nu \cdot \sigma_j(u_j)) - (h_j \cdot \nu)(\sigma_j(u_j) : \nabla \bar{v}) = \sigma_j(u_j)(h_{j,2}, -h_{j,1})^\top \cdot \partial_\tau \bar{v} \quad \text{on } \Gamma_j.$$

Therefore, combining (3.76), (3.77) and (3.78) yields

$$b_j(u, v, h) = A_{\Omega_j}(h \cdot \nabla u, v) + \rho_j \omega^2 \int_{\Gamma_j} (h_j \cdot \nu)(u_j \cdot \bar{v}) ds - \int_{\Gamma_j} \partial_\tau [(\sigma_j(u_j))^- (h_{j,2}, -h_{j,1})^\top] \cdot \bar{v} ds,$$

for $j \geq 1$. When $j = 0$, we obtain in a similar manner that

$$b_0(u, v, h) = A_{\Omega_0}(h \cdot \nabla u, v) - \sum_{j=1}^{N_0} \rho_0 \omega^2 \int_{\Gamma_j} (h_j \cdot \nu)[(u^{sc} + u^{in}) \cdot \bar{w}] ds + \sum_{j=1}^{N_0} \int_{\Gamma_j} \partial_\tau [(\sigma_0(u^{sc} + u^{in}))^+ (h_{j,2}, -h_{j,1})^\top] \cdot \bar{w} ds.$$

Now define $\tilde{u} = w - h \cdot \nabla u$ and set $\tilde{u}_j := \tilde{u}|_{\Omega_j}$ for $j = 0, 1, \dots, N_0$. We conclude that $\tilde{u}_0|_{\Gamma_R} = w|_{\Gamma_R}$ and the formula (3.73) holds with such \tilde{u} . Furthermore, we have the transmission conditions

$$\tilde{u}_j - \tilde{u}_0 = -h_j \cdot [\nabla u_j - \nabla(u^{sc} + u^{in})] = -(h_j \cdot \nu)[\partial_\nu^- u_j - \partial_\nu^+ (u^{sc} + u^{in})] \quad \text{on } \Gamma_j,$$

since $u_j - u^{sc} = u^{in}$ on Γ_j . This proves the relation $\mathcal{J}'_\Gamma(h_1, \dots, h_{N_0}) = \tilde{u}_0|_{\Gamma_R}$ in two dimensions.

If $N = 3$, we recall the tangential gradient ∇_Γ for a scalar function u and the surface divergence div_Γ for a vector function v by

$$\nabla u = \nabla_\Gamma u + \nu \partial_\nu u, \quad \nabla \cdot v = \text{div}_\Gamma v + \nu \cdot \partial_\nu v. \tag{3.79}$$

In this case, the first integrand on the right hand side of (3.78) can be rewritten as

$$(h_j \cdot \nabla \bar{v}) \cdot (\nu \cdot \sigma_j(u_j)) - (h_j \cdot \nu)(\sigma_j(u_j) : \nabla \bar{v}) = \sum_{i=1}^3 (\mathbf{A}_j(i, :))^\top \cdot (\nu \times \nabla v_i) = \sum_{i=1}^3 \nabla_\Gamma v_i \cdot ((\mathbf{A}_j(i, :))^\top \times \nu),$$

where the matrix \mathbf{A}_j is given by (3.72). Hence, by integration by part we find

$$b_j(u, v, h) = A_{\Omega_j}(h \cdot \nabla u, v) + \rho_j \omega^2 \int_{\Gamma_j} (h_j \cdot \nu)(u_j \cdot \bar{v}) ds - \int_{\Gamma_j} \text{div}_{\Gamma_j}(\mathbf{A}_j \times \nu) \cdot \bar{v} ds \tag{3.80}$$

for $j \geq 1$. Analogously,

$$b_0(u, v, h) = A_{\Omega_0}(h \cdot \nabla u, w) - \sum_{j=1}^{N_0} \left\{ \rho_0 \omega^2 \int_{\Gamma_j} (h_j \cdot \nu)[(u^{sc} + u^{in}) \cdot \bar{w}] ds - \int_{\Gamma_j} \text{div}_{\Gamma_j}(\mathbf{A}_{0,j} \times \nu) \cdot \bar{w} ds \right\}. \tag{3.81}$$

From (3.80) and (3.81) we conclude the variational formulation (3.73) still holds with $\tilde{u} = w - h \cdot \nabla u$ in three dimensions. Moreover, we get $\tilde{u}_0|_{\Gamma_R} = w|_{\Gamma_R}$ and the transmission conditions (3.66) due to the fact that $\text{div}_{\Gamma_j} u_j = \text{div}_{\Gamma_j} (u^{sc} - u^{in})$. This completes the proof. \square

3.2. Inversion algorithm in 2D

In this subsection we design a descent algorithm for the inverse problem in two dimensions. Assume that Γ_l ($l = 1, 2, \dots, N_0$) is a star-shaped boundary that can be parameterized by $\gamma_l(\theta)$ as follows

$$\Gamma_l = \{x : x = \gamma^{(l)}(\theta) := (a_1^{(l)}, a_2^{(l)})^\top + r^{(l)}(\theta)(\cos \theta, \sin \theta)^\top, \theta \in [0, 2\pi]\}$$

where the function $r^{(l)}$ is 2π -periodic and twice continuously differentiable. Let the Fourier series expansion of $r^{(l)}$ be given by

$$r^{(l)}(\theta) = \alpha_0^{(l)} + \sum_{m=1}^{\infty} [\alpha_{2m-1}^{(l)} \cos(m\theta) + \alpha_{2m}^{(l)} \sin(m\theta)].$$

We approximate the unknown boundary Γ_l by the surface

$$\begin{aligned} \Gamma_M^{(l)} &= \{x : x = \gamma^{(l)}(\theta) := (a_1^{(l)}, a_2^{(l)})^\top + r_M^{(l)}(\theta)(\cos \theta, \sin \theta)^\top, \theta \in [0, 2\pi]\}, \\ r_M^{(l)}(\theta) &= \alpha_0^{(l)} + \sum_{m=1}^M [\alpha_{2m-1}^{(l)} \cos(m\theta) + \alpha_{2m}^{(l)} \sin(m\theta)], \end{aligned} \tag{3.82}$$

in a finite dimensional space. The function $r_M^{(l)}$ is a truncated series of $r^{(l)}$. For large M , the surface $\Gamma_M^{(l)}$ differs from Γ_l only in those high frequency modes of $m > M$. Evidently, there are totally $2M + 3$ unknown parameters for $\Gamma_M^{(l)}$, which we denote by

$$\Lambda^{(l)} = (\Lambda_1^{(l)}, \dots, \Lambda_{2M+3}^{(l)})^\top := (a_1^{(l)}, a_2^{(l)}, \alpha_0^{(l)}, \alpha_1^{(l)}, \alpha_2^{(l)}, \dots, \alpha_{2M-1}^{(l)}, \alpha_{2M}^{(l)})^\top \in \mathbb{C}^{2M+3}.$$

Assume that the measurement points $\{z_i\}_{i=1}^{N_{mea}}$ are uniformly distributed on Γ_R , that is, $z_i = R(\cos \theta_i, \sin \theta_i)^\top$ with $\theta_i = (i - 1)2\pi/N_{mea}$. We use the notation $u(\cdot, d, \omega)$ to denote the dependence of the total field on the incident direction d and frequency ω . It is supposed that the measured data are available over a finite number of frequencies $\omega_l \in [\omega_{min}, \omega_{max}]$ ($l = 1, 2, \dots, K$) and several incident directions d_j ($j = 1, \dots, N_{inc}$). Hence, we have the data set of the total field

$$U_{mea} := \{u(z_i, d_j, \omega_l) : i = 1, \dots, N_{mea}, j = 1, \dots, N_{inc}, l = 1, \dots, K\}.$$

Then we consider the following modified inverse problem:

(IP'): Determine the parameter vector $\Lambda^{(j)}$ of the boundary Γ_j , $j = 1, \dots, N_0$, from knowledge of the near-field data set U_{mea} .

The inverse problem can be formulated as the nonlinear operator equation

$$\mathcal{J}(\Lambda^{(1)}, \dots, \Lambda^{(N_0)}) = U_{mea}, \tag{3.83}$$

where \mathcal{J} is the solution operator for all incident directions d_j and frequencies ω_l . The data set can be rewritten as $U_{mea} = \cup_{i=1}^{N_{mea}} u_{mea}(z_i)$, where

$$u_{mea}(z_i) := \{u(z_i, d_j, \omega_l) : j = 1, \dots, N_{inc}, l = 1, \dots, K\}$$

is the data set at $z_i \in \Gamma_R$ over all d_j and ω_l . Let \mathcal{J}_i be the solution operator mapping the boundary to $u_{mea}(z_i)$, i.e., $\mathcal{J}_i(\Lambda^{(1)}, \dots, \Lambda^{(N_0)}) = u_{mea}(z_i)$.

To solve the problem (3.83), we consider the objective function

$$F(\Lambda^{(1)}, \dots, \Lambda^{(N_0)}) = \frac{1}{2} \|\mathcal{J}(\Lambda^{(1)}, \dots, \Lambda^{(N_0)}) - U_{mea}\|_{l^2}^2.$$

Then the inverse problem (IP') can be formulated as the minimization problem

$$\min_{\Lambda^{(1)}, \dots, \Lambda^{(N_0)} \in \mathbb{C}^{2M+3}} F(\Lambda^{(1)}, \dots, \Lambda^{(N_0)}).$$

To apply the descent method, it is necessary to compute the gradient of the objective function. A direct calculation yields that

$$\begin{aligned} & \frac{\partial F(\Lambda^{(1)}, \dots, \Lambda^{(N_0)})}{\partial \Lambda_n^{(l)}} \\ &= \operatorname{Re} \left\{ \sum_{i=1}^{N_{mea}} \frac{\partial \mathcal{J}_i(\Lambda^{(1)}, \dots, \Lambda^{(N_0)})}{\partial \Lambda_n^{(l)}} \cdot [\mathcal{J}_i(\Lambda^{(1)}, \dots, \Lambda^{(N_0)}) - u_{mea}(z_i)] \right\}. \end{aligned}$$

Set

$$\nabla_{\Lambda^{(l)}} F := \left(\frac{\partial F(\Lambda^{(1)}, \dots, \Lambda^{(N_0)})}{\partial \Lambda_1^{(l)}}, \dots, \frac{\partial F(\Lambda^{(1)}, \dots, \Lambda^{(N_0)})}{\partial \Lambda_{2M+3}^{(l)}} \right)^\top, \quad l = 1, 2, \dots, N_0.$$

The calculation of $\nabla_{\Lambda^{(l)}} F$ is based on Theorem 3.3 below, which is a consequence of Theorem 3.2.

Theorem 3.3. *Let u be the unique solution of the variational problem (3.73) with fixed incident direction and frequency. Then the operator \mathcal{J}_i is differentiable in $\Lambda_n^{(l)}$ and its derivatives are given by*

$$\frac{\partial \mathcal{J}_i(\Lambda^{(1)}, \dots, \Lambda^{(N_0)})}{\partial \Lambda_n^{(l)}} = \tilde{u}_0(z_i), \quad l = 1, \dots, N_0, \quad n = 1, \dots, 2M + 3, \quad i = 1, \dots, N_{mea},$$

where \tilde{u}_0 , together with \tilde{u}_j ($j = 1, \dots, N_0$), is the unique weak solution of the boundary value problem:

$$\begin{aligned} \nabla \cdot (\mathcal{C}_j : \nabla \tilde{u}_j) + \rho_j \omega^2 \tilde{u}_j &= 0 && \text{in } \Omega_j, \quad j = 0, 1, \dots, N_0, \\ \tilde{u}_j - \tilde{u}_0 - f_j &= 0 && \text{on } \Gamma_j, \quad j = 1, \dots, N_0, \\ \mathcal{N}_{\mathcal{C}}^- \tilde{u}_j - \mathcal{N}_{\mathcal{C}}^+ \tilde{u}_0 - g_j &= 0 && \text{on } \Gamma_j, \quad j = 1, \dots, N_0, \\ T \tilde{u}_0 - \mathcal{T} \tilde{u}_0 &= 0 && \text{on } \Gamma_R. \end{aligned}$$

Here, $f_j = g_j = 0$ for $j = 1, \dots, N_0, j \neq l$, and

$$\begin{aligned} f_l &= -(h_l \cdot \nu) [\partial_\nu^- u_l - \partial_\nu^+ (u^{sc} + u^{in})], \\ g_l &= \omega^2 (h_l \cdot \nu) [\rho_l u_l^- - \rho_0 (u^{sc} + u^{in})^+] \\ &\quad - \partial_\tau [((\sigma_l(u_l))^- - (\sigma_0(u^{sc} + u^{in}))^+) (h_{l,2}, -h_{l,1})^\top], \end{aligned}$$

where $u_l := u|_{\Omega_l}$ and the functions $h_{l,1}, h_{l,2}$ are defined in the following way relying on n :

$$h_{l,1}(\theta) = \begin{cases} 1, & n = 1, \\ 0, & n = 2, \\ \cos \theta, & n = 3, \\ \cos((n - 2)\theta/2) \cos \theta, & n = 4, 6, 8, \dots, 2M + 2, \\ \sin((n - 3)\theta/2) \cos \theta, & n = 5, 7, 9, \dots, 2M + 3, \end{cases}$$

$$h_{l,2}(\theta) = \begin{cases} 0, & n = 1, \\ 1, & n = 2, \\ \sin \theta, & n = 3, \\ \cos((n - 2)\theta/2) \sin \theta, & n = 4, 6, 8, \dots, 2M + 2, \\ \sin((n - 3)\theta/2) \sin \theta, & n = 5, 7, 9, \dots, 2M + 3. \end{cases}$$

We now propose an algorithm based on the descent method to reconstruct the coefficient vectors $\Lambda^{(l)}$, $l = 1, \dots, N_0$. We assume the number N_0 of the disconnected components is known in advance. For notational convenience we denote by $\Lambda^{(l,i,j,m)}$ the solution of the inverse problem at the i -th iteration step reconstructed from the data set at the frequency ω_m with the incident direction d_j . Our approach consists of the following steps:

- Step 1.** Collect the near-field data over all frequencies ω_m , $m = 1, \dots, K$ and all incident directions d_j , $j = 1, \dots, N_{inc}$.
- Step 2.** Set initial approximations $\Lambda^{(l,0,0,0)}$ for every $l = 1, \dots, N_0$.
- Step 3.** For all $l = 1, \dots, N_0$, update the coefficient vector by the iterative formula

$$\Lambda^{(l,i+1,j,m)} = \Lambda^{(l,i,j,m)} - \epsilon \nabla_{\Lambda^{(l,i,j,m)}} F, \quad i = 0, \dots, L - 1,$$

where ϵ and $L > 0$ are the step size and total number of iterations, respectively.

- Step 4.** For all $l = 1, \dots, N_0$, set $\Lambda^{(l,0,j+1,m)} = \Lambda^{(l,L,j,m)}$ and repeat Step 3 until the last incident directions $d_{N_{inc}}$ is reached.
- Step 5.** For all $l = 1, \dots, N_0$, set $\Lambda^{(l,0,0,m+1)} = \Lambda^{(l,L,N_{inc},m)}$. Repeat Step 3 from the smallest frequency ω_1 and end up with the highest frequency ω_K .

4. Numerical examples

In this section, we present several numerical examples in 2D to verify the efficiency and validity of the finite element method solving direct scattering problems and the reconstruction scheme for inverse scattering problems.

4.1. Numerical solutions to direct scattering problems

Firstly, we introduce an analytic solution to the elastic wave equation in a homogeneous anisotropic medium; see [37, Chapter 1.7.1] for the details. Such a solution will be used to verify the accuracy of our numerical scheme. For simplicity we assume that Ω consists of one component only, i.e., $N_0 = 1$.

In 2D, the symmetry of the stiffness tensor $\mathcal{C} = \{C_{ijkl}\}_{i,j,k,l=1}^2$ leads to at most 6 different elements of stiffness. Using the Voigt notation for tensor indices, i.e.,

$$\begin{array}{cccc}
 ij & = & 11 & 22 & 12, 21 \\
 & & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\
 \alpha & = & 1 & 2 & 3
 \end{array}$$

one can rewrite the stiffness tensor as

$$C_{ijkl} \Rightarrow C_{\alpha\beta} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}.$$

In particular, we have

$$C_{\alpha\beta} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix},$$

if the elastic medium is homogeneous isotropic with Lamé constants λ and μ .

In a homogeneous anisotropic medium, we consider the propagation of a plane wave which is perpendicular to a fixed unit vector $d = (d_1, d_2)^T \in \mathbb{S}^1$. The plane wave takes the form

$$u = p e^{i \frac{\omega}{v_p} x \cdot d}, \tag{4.84}$$

where $p = (p_1, p_2)^T$ and v_p are the polarization vector and phase velocity to be determined, respectively. Inserting the solution (4.84) into the elastic equation (2.3) gives

$$A_C p = \rho v_p^2 p,$$

where

$$\begin{aligned}
 A_C &= \left\{ \sum_{k,l=1}^2 C_{iklj} d_k d_l \right\}_{i,j=1}^2 \\
 &= \begin{bmatrix} C_{11} & C_{13} \\ C_{13} & C_{33} \end{bmatrix} d_1^2 + \begin{bmatrix} 2C_{13} & C_{12} + C_{33} \\ C_{12} + C_{33} & 2C_{23} \end{bmatrix} d_1 d_2 + \begin{bmatrix} C_{33} & C_{23} \\ C_{23} & C_{22} \end{bmatrix} d_2^2.
 \end{aligned} \tag{4.85}$$

It follows from the uniform Legendre ellipticity condition of \mathcal{C} that the matrix A_C is positive definite. Thus, the eigenvectors of A_C give the vector p with the corresponding eigenvalue ρv_p^2 .

In order to check whether our code provides the true solution, we consider the elastic transmission problem: *Given $f \in (H^{1/2}(\Gamma))^2$ and $g \in (H^{-1/2}(\Gamma))^2$, find $u \in (H^1(\Omega))^2$ and $u^{sc} \in (H^1_{loc}(\Omega^c))^2$ such that*

$$\nabla \cdot (\mathcal{C} : \nabla u) + \rho \omega^2 u = 0 \quad \text{in } \Omega, \tag{4.86}$$

$$\Delta^* u^{sc} + \rho_0 \omega^2 u^{sc} = 0 \quad \text{in } \Omega^c, \tag{4.87}$$

$$u - u^{sc} = f \quad \text{on } \partial\Omega, \tag{4.88}$$

$$\mathcal{N}_{\mathcal{C}}^- u - T_{\lambda, \mu} u^{sc} = g \quad \text{on } \partial\Omega, \tag{4.89}$$

and the scattered field u^{sc} satisfies the Kupradze radiation condition. If Ω is specified as a homogeneous isotropic medium characterized by the density $\rho_1 > 0$ and the Lamé constants λ_1 and μ_1 are such that $\mu_1 > 0$ and $\lambda_1 + \mu_1 > 0$, then the problem (4.86)–(4.89) is reduced to

$$\Delta_1^* u + \rho_1 \omega^2 u = 0 \quad \text{in } \Omega, \tag{4.90}$$

$$\Delta^* u^{sc} + \rho_0 \omega^2 u^{sc} = 0 \quad \text{in } \Omega^c, \tag{4.91}$$

$$u - u^{sc} = f \quad \text{on } \partial\Omega, \tag{4.92}$$

$$T_{\lambda_1, \mu_1} u - T_{\lambda, \mu} u^{sc} = g \quad \text{on } \partial\Omega, \tag{4.93}$$

where $\Delta_1^* := \mu_1 \Delta + (\lambda_1 + \mu_1) \text{grad div}$.

We modify the definition of the far-field pattern of the total displacement (2.11) as

$$u^\infty(\hat{x}) = u_p^\infty(\hat{x}) \hat{x} + u_s^\infty(\hat{x}) \hat{x}^\perp,$$

where $u_p^\infty(\hat{x}) = u^\infty(\hat{x}) \cdot \hat{x}$, $u_s^\infty(\hat{x}) = u^\infty(\hat{x}) \cdot \hat{x}^\perp$ are two scalar functions given by the asymptotic behavior

$$u^{sc} = \frac{\exp(ik_p x + i\pi/4)}{\sqrt{8\pi k_p |x|}} u_p^\infty(\hat{x}) \hat{x} + \frac{\exp(ik_s x + i\pi/4)}{\sqrt{8\pi k_s |x|}} u_s^\infty(\hat{x}) \hat{x}^\perp + \mathcal{O}(|x|^{-3/2}).$$

We decompose the scattered field into

$$u^{sc} = \text{grad } \Psi_p + \overrightarrow{\text{curl}} \Psi_s,$$

where

$$\Psi_p = \sum_{n \in \mathbb{Z}} \Psi_p^n H_n^{(1)}(k_p |x|) e^{in\theta_x}, \quad \Psi_s = \sum_{n \in \mathbb{Z}} \Psi_s^n H_n^{(1)}(k_s |x|) e^{in\theta_x}, \quad \Psi_p^n, \Psi_s^n \in \mathbb{C}.$$

Then it follows from the asymptotic behavior of Hankel functions that

$$u_p^\infty(\hat{x}) = u_p^\infty(\theta) = 4k_p \sum_{n \in \mathbb{Z}} \Psi_p^n e^{in(\theta - \pi/2)},$$

$$u_s^\infty(\hat{x}) = u_s^\infty(\theta) = -4k_s \sum_{n \in \mathbb{Z}} \Psi_s^n e^{in(\theta - \pi/2)}.$$

In numerical computations, the computational domains Ω and Ω_0 are discretized by uniform triangle elements and we employ piecewise linear basis functions to construct the finite element space of $(H^1(\Omega))^2$ and $(H^1(\Omega_0))^2$.

Example 1. In the first example, Ω is specified as a homogeneous isotropic medium and we consider the problem (4.90)–(4.93). Let f and g be such that the analytic solution of the above boundary value problem is given by

$$u(x) = \nabla J_0(k_{p,1}|x|), \quad x \in \Omega, \quad u^{sc}(x) = \nabla H_0^{(1)}(k_p|x|), \quad x \in \Omega^c,$$

where $k_{p,1} = \omega \sqrt{\rho_1 / (\lambda_1 + 2\mu_1)}$. We choose $\lambda_1 = 2$, $\mu_1 = 3$, $\rho_1 = 3$, $\lambda = 1$, $\mu = 2$, $\rho_0 = 1$ and the boundary $\partial\Omega$ is selected to be a circle

$$\partial\Omega = \{x \in \mathbb{R}^2 : |x| = 1\},$$

or a rounded-triangle-shaped curve

$$\partial\Omega = \{x \in \mathbb{R}^2 : x = (2 + 0.5 \cos 3t)(\cos t, \sin t)^\top, t \in [0, 2\pi)\}.$$

Table 1
Numerical errors for [Example 1](#) where Γ is a circle, $h_0 = 0.4304$ and $R = 2$.

ω	h	E_0	Order	E_1	Order
1	h_0	1.55E-2	–	2.29E-1	–
	$h_0/2$	3.97E-3	1.97	1.07E-1	1.10
	$h_0/4$	1.00E-3	1.99	5.22E-2	1.04
3	h_0	1.52E-1	–	6.66E-1	–
	$h_0/2$	3.81E-2	2.00	2.95E-1	1.17
	$h_0/4$	9.62E-3	1.99	1.43E-1	1.04

Table 2
Numerical errors for [Example 1](#) where Γ is a rounded-triangle-shaped curve, $h_0 = 1.1474$ and $R = 5$.

ω	h	E_0	Order	E_1	Order
1	h_0	3.03E-2	–	1.56E-1	–
	$h_0/2$	6.83E-3	2.15	7.16E-2	1.12
	$h_0/4$	1.79E-3	1.93	3.56E-2	1.01
3	h_0	5.22E-1	–	1.94E0	–
	$h_0/2$	1.32E-1	1.98	7.81E-1	1.31
	$h_0/4$	3.59E-2	1.88	3.75E-1	1.06

Table 3
Numerical errors for [Example 2](#) where Γ is a circle, $h_0 = 0.4304$ and $R = 2$.

ω	h	E_0	Order	E_1	Order
1	h_0	1.95E-2	–	2.46E-1	–
	$h_0/2$	5.11E-3	1.93	1.15E-1	1.10
	$h_0/4$	1.32E-3	1.95	5.60E-2	1.04
3	h_0	2.86E-1	–	1.27E0	–
	$h_0/2$	8.62E-2	1.73	5.15E-1	1.30
	$h_0/4$	2.32E-2	1.89	2.28E-1	1.18

Denote $U = (u, u^{sc})$ and $U_h = (u_h, u_h^{sc})$ the exact and numerical solutions, respectively. The numerical errors (see [Tables 1 and 2](#))

$$E_0 = \|U - U_h\|_{(L^2(\Omega))^2 \times (L^2(\Omega_0))^2}, \quad E_1 = \|U - U_h\|_{(H^1(\Omega))^2 \times (H^1(\Omega_0))^2}, \tag{4.94}$$

indicate the convergence order

$$E_0 = O(h^2), \quad E_1 = O(h), \tag{4.95}$$

where h denotes the finite element mesh size for discretizing our variational formulation.

Example 2. In this example, Ω is supposed to be a homogeneous anisotropic medium. Consider the problem [\(4.86\)–\(4.89\)](#) and let f and g be such that the analytic solution is given by

$$u(x) = pe^{i\frac{\omega}{v_p}x \cdot d}, \quad x \in \Omega, \quad u^{sc}(x) = \nabla H_0^{(1)}(k_p|x|), \quad x \in \Omega^c,$$

where $d = (\sqrt{2}/2, \sqrt{2}/2)^\top$ and ρv_p^2 is the first eigenvalue of the matrix A_C (see [\(4.85\)](#)). We choose

$$C_{11} = 10.5, C_{22} = 13, C_{33} = 4.75, C_{12} = 3.25, C_{13} = -0.65, C_{23} = -1.52, \\ \rho = 3, \lambda = 1, \mu = 2, \rho_0 = 1.$$

The boundary $\partial\Omega$ is selected to be a circle or a rounded-triangle-shaped curve given in [Example 1](#). In [Tables 3 and 4](#) we illustrate the numerical errors of E_0 and E_1 (see [\(4.94\)](#)) which also indicate the convergence order

Table 4
 Numerical errors for Example 2 where Γ is a rounded-triangle-shaped curve, $h_0 = 1.1474$ and $R = 5$.

ω	h	E_0	Order	E_1	Order
1	h_0	9.08E-2	–	3.12E-1	–
	$h_0/2$	2.26E-2	1.97	1.30E-1	1.10
	$h_0/4$	5.87E-3	1.99	6.30E-2	1.04
3	h_0	1.82E0	–	5.78E0	–
	$h_0/2$	5.73E-1	2.00	2.04E0	1.17
	$h_0/4$	1.59E-1	1.99	7.35E-1	1.04

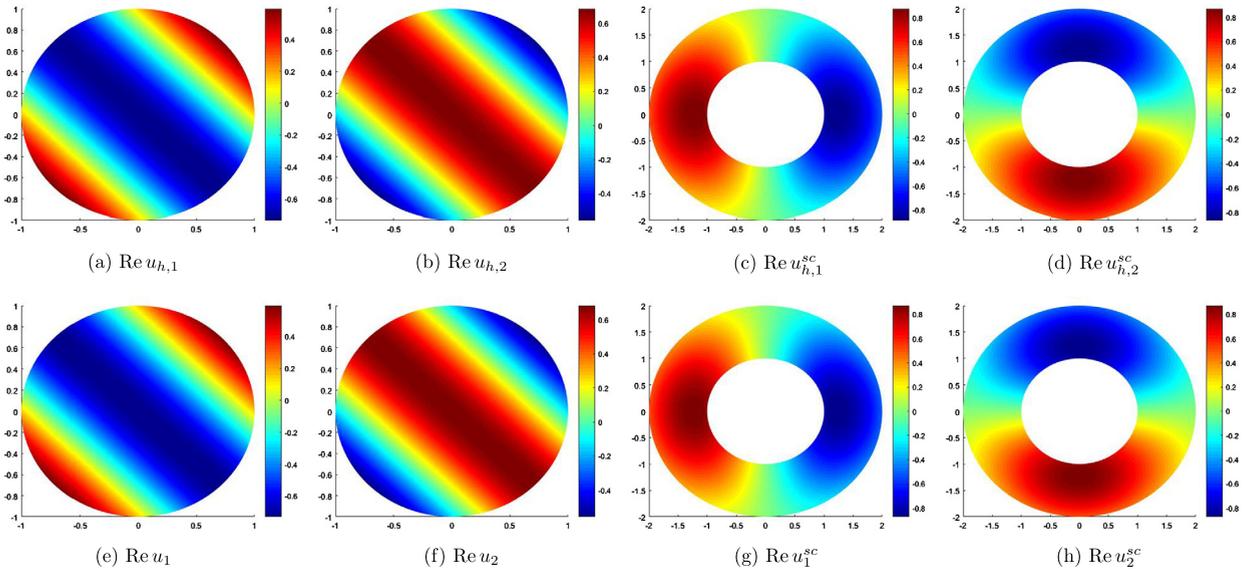


Fig. 1. Real parts of the numerical solutions $u_h = (u_{h,1}, u_{h,2})^\top$, $u_h^{sc} = (u_{h,1}^{sc}, u_{h,2}^{sc})^\top$ and exact solutions $u = (u_1, u_2)^\top$, $u^{sc} = (u_1^{sc}, u_2^{sc})^\top$ for Example 2. We set $\omega = 3$, $h = 0.1076$ and $R = 2$.

(4.95). We plot the numerical solutions in Figs. 1 and 2 from which it can be seen that they are in a good agreement with the exact ones. To compare the errors for far-field patterns, we observe that the exact far-field pattern takes the explicit form $u^\infty(\hat{x}) = 4k_p \hat{x}$. From Figs. 3 and 4 it can be seen that the numerical far-field patterns provide good approximations to the exact ones.

4.2. Numerical solutions to inverse scattering problems

We consider the reconstruction of multiple anisotropic elastic bodies in 2D using the inversion algorithm described in Section 3. Set $\rho_0 = 2000 \text{ Kg/m}^3$, $c_p = \sqrt{(\lambda + 2\mu)/\rho_0} = 3000 \text{ m/s}$, $c_s = \sqrt{\mu/\rho_0} = 1800 \text{ m/s}$ and $R = 5 \text{ m}$. The number of measurement points and iterations are taken as $N_{mea} = 64$, $L = 10$, respectively. For each frequency, we set the step size as $\epsilon = 0.005/k_p$. The boundary of the unknown anisotropic obstacles together with the initial guess are illustrated in Fig. 5, in which Obstacle 1 is kite-shaped and Obstacle 2 is an ellipse. The density of the anisotropic medium is selected as $\rho = 2400 \text{ Kg/m}^3$. We choose the stiffness tensor as

$$C_{klmn} \Rightarrow C_{\alpha\beta} = \begin{bmatrix} 6 & 8 & 2 \\ 8 & 21 & 10 \\ 2 & 10 & 30 \end{bmatrix} \times 10^{10} \text{ Pa.}$$

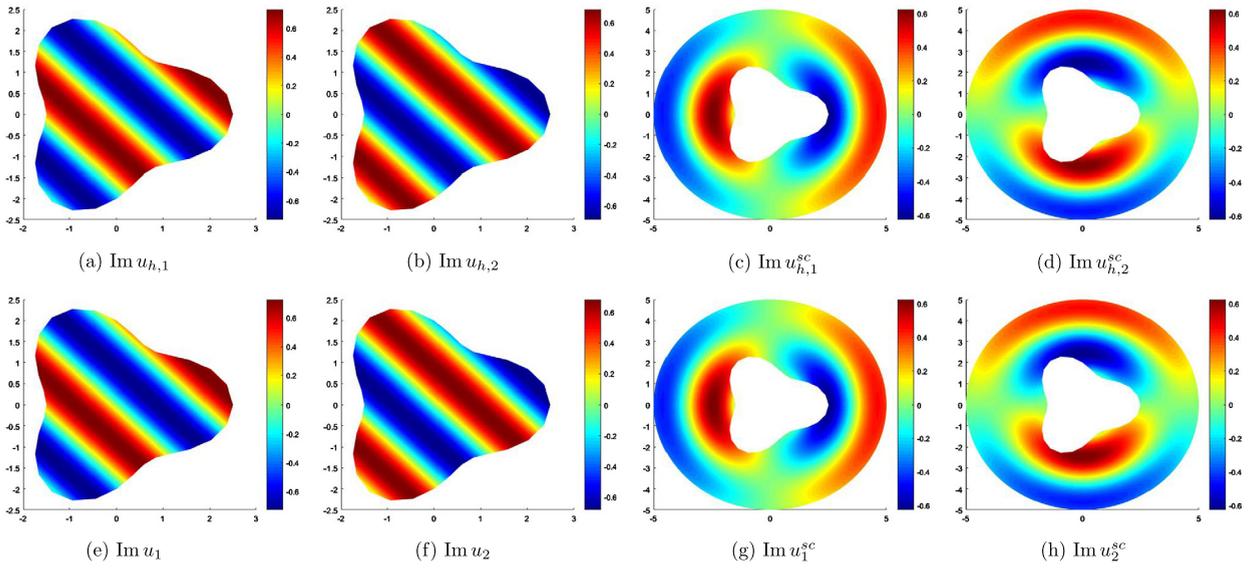


Fig. 2. Imaginary parts of the numerical solutions $u_h = (u_{h,1}, u_{h,2})^\top$, $u_h^{sc} = (u_{h,1}^{sc}, u_{h,2}^{sc})^\top$ and exact solutions $u = (u_1, u_2)^\top$, $u^{sc} = (u_1^{sc}, u_2^{sc})^\top$ for Example 2. We set $\omega = 3$, $h = 0.2869$ and $R = 5$.

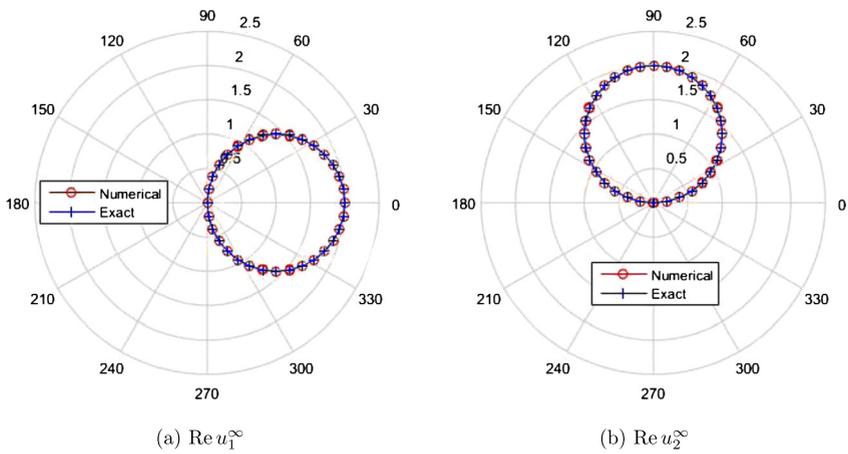


Fig. 3. Exact and numerical far-field pattern $u^\infty = (u_1^\infty, u_2^\infty)^\top$ for Example 2 when Γ is a circle.

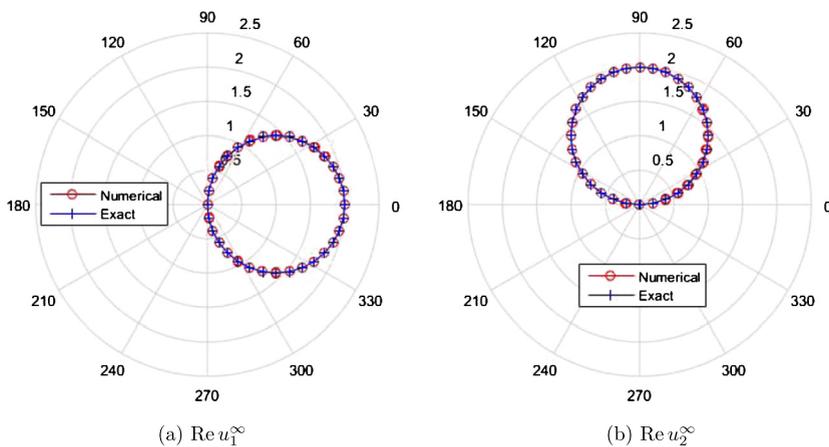


Fig. 4. Exact and numerical far-field pattern $u^\infty = (u_1^\infty, u_2^\infty)^\top$ for Example 2 when Γ is a rounded-triangle-shaped curve.

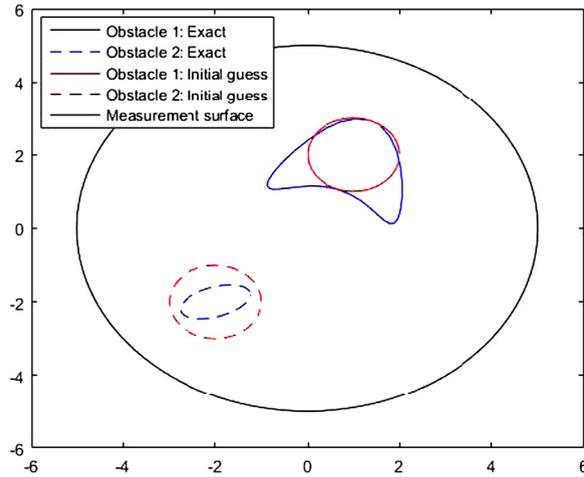


Fig. 5. The obstacles to be reconstructed and initial guess.

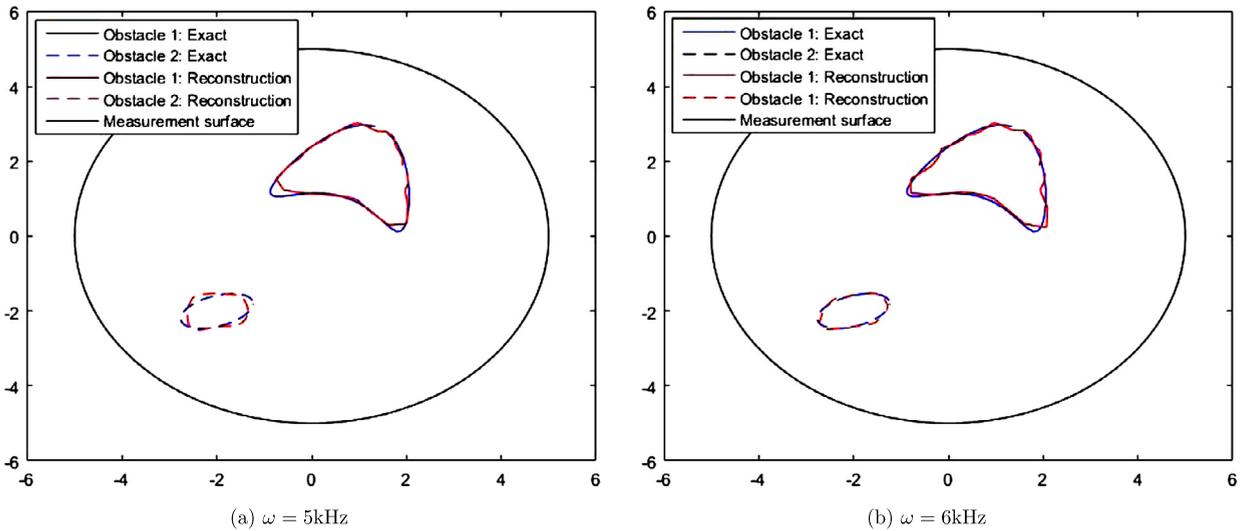


Fig. 6. Reconstruction results from four incident directions at distinct frequencies. We set $M = 10$.

To examine the reconstruction results, we compute the residual error RError_{i+1} , $i = 0, \dots, K$ of the total field where

$$\text{RError}_{i+1} = \frac{\|\mathcal{J}_N(\Lambda^{(1,L,N_{inc},i)}, \dots, \Lambda^{(N,L,N_{inc},i)}) - U_{mea}\|_{l_2}}{\|U_{mea}\|_{l_2}}.$$

In the first experiment, we use four incident plane waves (i.e., $N_{inc} = 4$) incited at two frequencies $\omega_1 = 5$ kHz and $\omega_2 = 6$ kHz (i.e. $K = 2$). The reconstruction results at each frequency are shown in Fig. 6. For different choice of M (see (3.82)), the residual errors listed in Table 5 indicates that the residual error decreases as frequency increases. Note that the errors corresponding to $M = 10$ and $M = 20$ are almost the same, because the underlying scatterers possess smooth boundaries.

In the second experiment, we use the data generated by one fixed direction $d = (-\sqrt{2}/2, \sqrt{2}/2)^\top$ (i.e., $N_{inc} = 1$) and by three distinct frequencies $\omega_1 = 5$ kHz, $\omega_2 = 6$ kHz and $\omega_3 = 7$ kHz (i.e., $K = 3$). In this case the number of iterations at each frequency is set as $L = 20$. The parameter M for truncating the Fourier series is taken as $M = 20$. The reconstruction results shown in Fig. 7 are very satisfactory.

Table 5
Change of residual reconstruction errors with respect to frequencies.

M	RError ₁	RError ₂	RError ₃
10	0.3042	0.0656	0.0521
20	0.3042	0.0738	0.0528

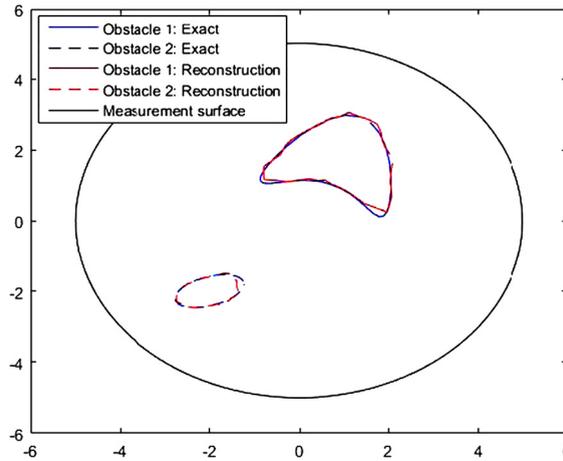


Fig. 7. Reconstruction result from the data of one incident direction and three frequencies.

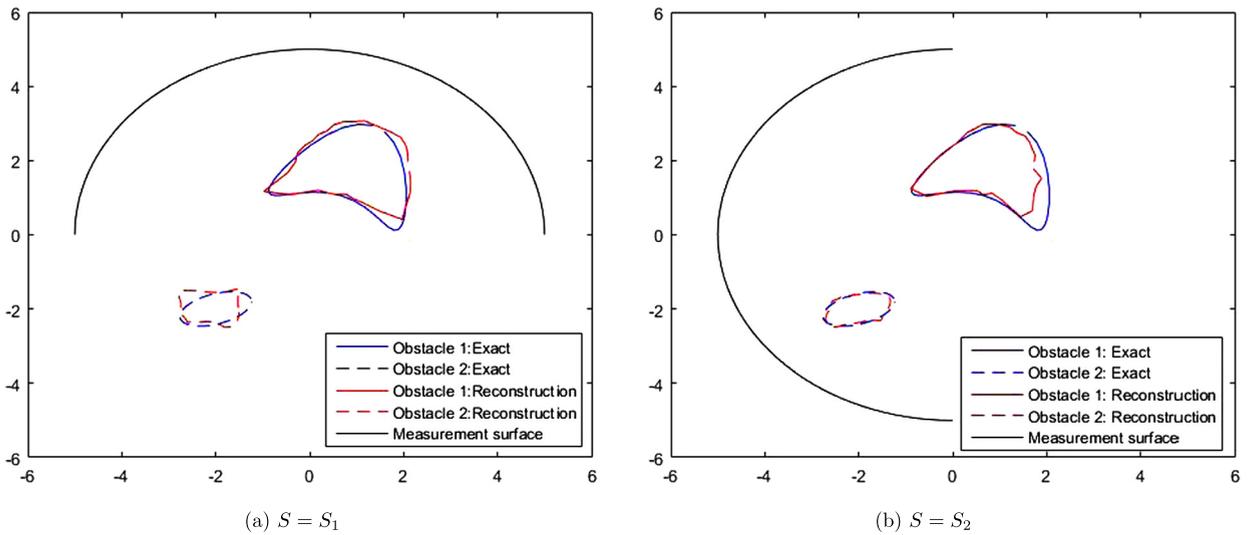


Fig. 8. Reconstruction results from limited-view measurement data.

Finally, we discuss the limited-view case. Assume that the measurement surface S is confined to an open arc in \mathbb{R}^2 . We shall consider the following two different curves

$$S_i := \{x \in \mathbb{R}^2 : x = R(\cos \theta, \sin \theta)^\top, \theta \in I_i\}, \quad i = 1, 2,$$

with $I_1 = [0, \pi]$ and $I_2 = [\pi/2, 3\pi/2]$. The other settings and parameters are the same as the second example. Due to the limited number of observation angles, it is expected to accurately reconstruct part of the elastic inclusion only. We present the numerical reconstructions in Fig. 8 and the residual reconstruction errors in Table 6. It can be seen from Fig. 8 that the interface within the scope of observation angles can be better reconstructed than the remaining part which cannot be ‘seen’.

Table 6

Change of residual reconstruction errors with respect to frequencies for limited-view case.

S	RError ₁	RError ₂	RError ₃	RError ₄
S_1	0.3275	0.0920	0.0667	0.0670
S_2	0.3474	0.1026	0.0541	0.0503

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