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# DETERMINATION OF SINGULAR TIME-DEPENDENT COEFFICIENTS FOR WAVE EQUATIONS FROM FULL AND PARTIAL DATA

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Abstract. We study the problem of determining uniquely a time-dependent singular potential $q$, appearing in the wave equation $\partial_t^2 u - \Delta_x u + q(t,x)u = 0$ in $Q = (0,T) \times \Omega$ with $T > 0$ and $\Omega$ a $\mathcal{C}^2$ bounded domain of $\mathbb{R}^n$, $n \geq 2$. We start by considering the unique determination of some general singular time-dependent coefficients. Then, by weakening the singularities of the set of admissible coefficients, we manage to reduce the set of data that still guaranties unique recovery of such a coefficient. To our best knowledge, this paper is the first claiming unique determination of unbounded time-dependent coefficients, which is motivated by the problem of determining general nonlinear terms appearing in nonlinear wave equations.

## 1. Introduction.

We fix $\Omega$ a $\mathcal{C}^2$ bounded domain of $\mathbb{R}^n$, $n \geq 2$, $\Sigma = (0,T) \times \partial\Omega$, $Q = (0,T) \times \Omega$ with $0 < T < \infty$. Then, we introduce the wave equation

$$\partial_t^2 u - \Delta_x u + q(t,x)u = 0, \quad (t,x) \in Q, \tag{1.1}$$

where the potential $q$ is assumed to be an unbounded real valued coefficient. In this paper we study the inverse problem of determining uniquely $q$ from observations of some solutions of (1.1) on $\partial Q$.

### 1.1. Obstruction to uniqueness and set of full data for our problem.

Let us first recall that $\partial Q = (\{0\} \times \overline{\Omega}) \cup \Sigma \cup (\{T\} \times \overline{\Omega})$. According to [45], for $T > \mathrm{Diam}(\Omega)$, measurements of solutions of (1.1) restricted to lateral boundary $\Sigma$ determines uniquely a time-independent potential $q$. Due to domain of dependence arguments, this result can not be extended to time-dependent coefficients (see [36, Subsection 1.1]) where, even for large values of the final time $T$, some additional information

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on the bottom $\{0\} \times \Omega$ and the top $\{T\} \times \Omega$ of $Q$, for solutions $u$ of (1.1), can not be avoided. In this context, we introduce the set of data

$$C_q = \{(u_{|\Sigma}, u_{|t=0}, \partial_t u_{|t=0}, \partial_\nu u_{|\Sigma}, u_{|t=T}, \partial_t u_{|t=T}) : \ u \in L^2(Q),\ \Box u + qu = 0\},$$

where $\nu$ is the outward unit normal vector to $\partial\Omega$, $\partial_\nu = \nu \cdot \nabla_x$ the normal derivative and $\Box := \partial_t^2 - \Delta_x$. We recall that [28] proved that, for $q \in L^\infty(Q)$, the data $C_q$ determines uniquely $q$. From now on we will refer to $C_q$ as the set of full data for our problem and we mention that [35, 36, 37] proved recovery of bounded time-dependent coefficients $q$ from partial data corresponding to partial knowledge of the set $C_q$. The goal of the present paper is to prove recovery of singular time-dependent coefficients $q$ from full and partial data.

1.2. **Motivations.** Our inverse problem can be seen as the determination of some unstable properties such as some rough time evolving density of an inhomogeneous medium from disturbances generated on the boundary and at initial time, and measurements of the response. Moreover, singular time-dependent coefficients can be associated to some unstable time-evolving phenomenon that can not be described by the wave equation with bounded time-dependent coefficients or time independent coefficients.

Let us also observe that, according to [13, 30], for parabolic equations the recovery of nonlinear terms, appearing in some suitable nonlinear equations, can be reduced to the determination of time-dependent coefficients. In this context, the information that allows to recover the nonlinear term is transferred, through a linearization process, to a time-dependent coefficient depending explicitly on some solutions of the nonlinear problem. In contrast to parabolic equations, due to the weak regularity of solutions, it is not clear that this process allows to transfer the recovery of nonlinear terms, appearing in a nonlinear wave equation, to a bounded time-dependent coefficient. Thus, in order to expect an application of the strategy set by [13, 30] to the recovery of nonlinear terms for nonlinear wave equations, it seems important to consider recovery of singular time-dependent coefficients.

1.3. **Known results.** The problem of determining coefficients appearing in hyperbolic equations has attracted a lot of attention over the last decades. The recovery of a time-independent potential $q$ from measurement on the full boundary $\Sigma$ has been addressed in [45] and extended to partial boundary measurements by [18, 42]. We recall that several authors considered this problem by using the so called boundary control method introduced by [2] and extended to Riemannian manifold in [3]. We refer to [31] for a review of the boundary control method for recovery of time-independent coefficients and we mention its recent extension to non smooth coefficients derived in [39]. We recall also that the stability issue for this problem has been studied by [6, 7, 34, 49].

Some authors considered also the problem of determining time-dependent coefficients appearing in wave equations. Without being exhaustive, we mention here the work of [4, 8, 19, 28, 44, 46, 47, 48, 51] and we refer to [37] for the description of these results. In addition to these works, we mention the papers [35, 36, 37], where the first author proved uniqueness and stability in the recovery of several time-dependent coefficients from partial knowledge of the full set of data $C_q$. More recently, [38] proved unique determination of such coefficients on Riemannian manifolds. We mention also the work of [50] who determined some information about time-dependent coefficients from the Dirichlet-to-Neumann map on a cylinder-like Lorentzian manifold related to the wave equation. We refer to the work [12, 14, 22, 23, 40] for determination of time-dependent coefficients appearing in fractional diffusion, parabolic and Schrödinger equations.

In all the above mentioned results, only the recovery of time-dependent coefficients, that are at least bounded, has been considered. There have been several works dealing with recovery of non-smooth coefficients appearing in elliptic equations such as [1, 11, 17, 21, 25, 26]. Nevertheless, to our best knowledge, except the present paper, there is no work in the mathematical literature dealing with the recovery of singular time-dependent coefficients $q$ even from the important set of full data $C_q$.

1.4. **Main results.** The main purpose of this paper is to prove the unique global determination of a time-dependent and unbounded coefficient $q$ from observations of solutions on $\partial Q$. More precisely, we would like to prove unique recovery of an unbounded coefficient $q \in L^{p_1}(0,T;L^{p_2}(\Omega))$, $p_1 > 1$, $p_2 \in [n,+\infty) \setminus \{2\}$, from partial knowledge of the full set of data $C_q$. We start by considering the recovery of a general coefficient $q$ from restriction of $C_q$ only on the bottom $t = 0$ and top $t = T$ of $Q$. More precisely, for $q \in L^{p_1}(0,T;L^{p_2}(\Omega))$, $p_1 > 1$, $p_2 \in [n,+\infty) \setminus \{2\}$, we consider the recovery of $q$ from the set of data

$$\begin{aligned} C_q(0) :=\{ &(u_{|\Sigma}, \partial_t u_{|t=0}, \partial_\nu u, u_{|t=T}, \partial_t u_{|t=T}) : \\ &u \in \mathcal{K}(Q), \Box u + qu = 0,\ u_{|t=0} = 0,\ u_{|\Sigma} \in H^1(\Sigma)\}, \end{aligned}$$

or the set of data

$$\begin{aligned} C_q(T) :=\{ &(u_{|\Sigma}, u_{t=0}, \partial_t u_{|t=0}, \partial_\nu u, u_{|t=T}) : \\ &u \in \mathcal{K}(Q),\ \Box u + qu = 0,\ u_{|\Sigma} \in H^1(\Sigma)\}, \end{aligned}$$

where $\mathcal{K}(Q) = \mathcal{C}([0,T];H^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega))$. In addition, assuming that $T > \mathrm{Diam}(\Omega)$, we prove the recovery of $q$ from

$$\begin{aligned} C_q(0,T) := \{ &(u_{|\Sigma}, \partial_t u_{|t=0}, \partial_\nu u, u_{|t=T}) : \\ &u \in \mathcal{K}(Q),\ \Box u + qu = 0,\ u_{|t=0} = 0,\ u_{|\Sigma} \in H^1(\Sigma)\}. \end{aligned}$$

Our first main result can be stated as follows

**Theorem 1.1.** *Let $p_1 \in (1,+\infty)$, $p_2 \in [n,+\infty) \setminus \{2\}$ and let $q_1,\ q_2 \in L^{p_1}(0,T;L^{p_2}(\Omega))$. Then, either of the following conditions:*

$$C_{q_1}(0) = C_{q_2}(0), \tag{1.2}$$

$$C_{q_1}(T) = C_{q_2}(T), \tag{1.3}$$

*implies that $q_1 = q_2$. Moreover, assuming that $T >$ Diam$(\Omega)$, the condition*

$$C_{q_1}(0,T) = C_{q_2}(0,T) \tag{1.4}$$

*implies that $q_1 = q_2$.*

We consider also the recovery of a time-dependent and unbounded coefficient $q$ from restriction of the data $C_q$ on the lateral boundary $\Sigma$. Namely, for all $\omega \in \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n :\ |x| = 1\}$ we recall that the $\omega$-shadowed and $\omega$-illuminated faces of $\partial\Omega$ corresponds to the portions

$$\partial\Omega_{+,\omega} := \{x \in \partial\Omega :\ \nu(x)\cdot\omega > 0\}, \quad \partial\Omega_{-,\omega} := \{x \in \partial\Omega :\ \nu(x)\cdot\omega \le 0\}.$$

Here and in the remaining parts of this paper, for all $\ell \in \mathbb{N}^*$, $\cdot$ is the scalar product in $\mathbb{R}^\ell$ given by

$$x \cdot y = x_1 y_1 + \ldots + x_\ell y_\ell, \quad x = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell,\ y = (y_1, \ldots, y_\ell) \in \mathbb{R}^\ell.$$

We consider also the parts of $\Sigma$ associated to these two portions of $\partial\Omega$ defined by $\Sigma_{\pm,\omega} := (0,T) \times \partial\Omega_{\pm,\omega}$. We set $\omega_0 \in \mathbb{S}^{n-1}$ and we introduce $V = (0,T) \times V'$ with $V'$ a neighborhood of $\partial\Omega_{-,\omega_0}$ in $\partial\Omega$. Then, we study the recovery of $q \in L^p(Q)$, $p > n+1$, from the data

$$C_q(T,V) = \{(u_{|\Sigma}, u_{|t=0}, \partial_t u_{|t=0}, \partial_\nu u_{|V}, u_{|t=T}) :\ u \in H^1(Q),\ \Box u + qu = 0\}$$

and the determination of a time-dependent coefficient $q \in L^\infty(0,T;L^p(\Omega))$, $p > n$, from the data

$$\begin{aligned} C_q(0,T,V) = &\{(u_{|\Sigma}, \partial_t u_{|t=0}, \partial_\nu u_{|V}, u_{|t=T}) : \\ & u \in L^2(0,T;H^1(\Omega)),\ \Box u + qu = 0,\ u_{|t=0} = 0\}. \end{aligned}$$

We give a more rigorous definition of these sets in Section 2. Our two last results can be stated as follows.

**Theorem 1.2.** *Let $p \in (n+1, +\infty)$ and let $q_1,\ q_2 \in L^p(Q)$. Then, the condition*

$$C_{q_1}(T,V) = C_{q_2}(T,V) \tag{1.5}$$

*implies that $q_1 = q_2$.*

**Theorem 1.3.** *Let $p \in (n, +\infty)$ and let $q_1,\ q_2 \in L^\infty(0,T;L^p(\Omega))$. Then, the condition*

$$C_{q_1}(0,T,V) = C_{q_2}(0,T,V) \tag{1.6}$$

*implies that $q_1 = q_2$.*

To our best knowledge the results of Theorem 1.1, 1.2 and 1.3 are the first claiming unique determination of unbounded time-dependent coefficients for the wave equation. In Theorem 1.1, we prove recovery of coefficients $q$, that can admit some quite important singularities, by making restriction on the set of full data $C_q$ on the bottom $t = 0$ and the top $t = T$ of $Q$. While, in Theorem 1.2 and 1.3, we consider less singular time-dependent coefficients, in order to restrict the data on the lateral boundary $\Sigma = (0,T) \times \partial\Omega$.

We mention also that the uniqueness result of Theorem 1.3 is stated with data close to the one of [35, 36], who established determination of general class of bounded time-dependent potentials from, what seems to be, the weakest conditions so far. More precisely, the result of [35, 36] differs from Theorem 1.3 by the restriction on the Dirichlet input ([35, 36] consider Dirichlet boundary conditions restricted to a neighborhood of the $\omega_0$-shadowed face, while in Theorem 1.3 we consider the Dirichlet input on the full boundary).

In the present paper we consider two different approaches which depend mainly on the restriction that we make on the set of full data $C_q$. For Theorem 1.1, we use geometric optics solutions corresponding to oscillating solutions of the form

$$u(t,x) = \sum_{j=1}^{N} a_j(t,x) e^{i\lambda\psi_j(t,x)} + R_\lambda(t,x), \quad (t,x) \in Q, \tag{1.7}$$

with $\lambda > 1$ a large parameter, $R_\lambda$ a remainder term that admits a decay with respect to the parameter $\lambda$ and $\psi_j$, $j = 1,..,N$, real valued. For $N = 1$, these solutions correspond to a classical tool for proving determination of time independent or time-dependent coefficients (e.g. [4, 5, 6, 8, 44, 46, 45]). In a similar way to [38], we consider in Theorem 1.1 solutions of the form (1.7) with $N = 2$ in order to be able to restrict the data at $t = 0$ and $t = T$ while avoiding a "reflection". It seems that in the approach set so far for the construction of solutions of the form (1.7), the decay of the remainder term $R_\lambda$ relies in an important way to the fact that the coefficient $q$ is bounded (or time independent). In this paper, we prove how this construction can be extended to unbounded time-dependent coefficients.

The approach used for Theorem 1.1 allows in a quite straightforward way to restrict the data on the bottom $t = 0$ and on the top $t = T$ of $Q$. Nevertheless, it is not clear how one can extend this approach to restriction on the lateral boundary $\Sigma$ without requiring additional smoothness or geometrical assumptions. Indeed, this construction works quite well for restricting data on a flat part of $\partial Q$ but it seems to require more complex local construction and some geometrical assumptions for restricting the data on more general boundary. For this reason, in order to consider restriction on $\Sigma$, we use a different approach where the oscillating solutions (1.7) are replaced by exponentially growing and exponentially decaying solutions. The idea of this approach, which is inspired by [7, 35, 36, 37] (see also [10, 33] for elliptic equations), consists of combining results of density of products of solutions with Carleman estimates with linear weight in order to be able to restrict at the same time the data on the bottom $t = 0$, on the top $t = T$ and on the lateral boundary $\Sigma$ of $Q$. For the construction of these solutions, we use Carleman estimates in negative order Sobolev space. To our best knowledge this is the first extension of this approach to singular time-dependent coefficients.

1.5. **Outline.** This paper is organized as follows. In Section 2, we start with some preliminary results and we define the set of data $C_q(0)$, $C_q(T)$, $C_q(0,T)$, $C_q(T,V)$ and $C_q(0,T,V)$. In Section 3, we prove Theorem 1.1 by mean of geometric optics solutions of the form (1.7). Then, Section 4 and Section 5 are respectively devoted to the proof of Theorem 1.2 and Theorem 1.3.

2. **Preliminary results.** In this section we give a rigorous definition of the set of data $C_q(T,V)$, $C_q(0,T,V)$ and we introduce some properties of (1.1) for any $q \in L^{p_1}(Q)$, with $p_1 > n+1$, or, for $q \in L^\infty(0,T;L^{p_2}(\Omega))$, with $p_2 > n$. We start by considering some results that require some functional spaces borrowed from [36]. We introduce the space

$$H_\Box(Q) = \{u \in H^1(Q) :\ \Box u = (\partial_t^2 - \Delta_x)u \in L^2(Q)\},$$

$$H_{\Box,*}(Q) = \{u \in L^2(0,T;H^1(\Omega)) :\ \Box u = (\partial_t^2 - \Delta_x)u \in L^2(Q)\},$$

with the norm

$$\|u\|^2_{H_\Box(Q)} = \|u\|^2_{H^1(Q)} + \left\|(\partial_t^2 - \Delta_x)u\right\|^2_{L^2(Q)},$$

$$\|u\|^2_{H_{\Box,*}(Q)} = \|u\|^2_{L^2(0,T;H^1(\Omega))} + \left\|(\partial_t^2 - \Delta_x)u\right\|^2_{L^2(Q)}.$$

We define also the space

$$S = \{u \in H^1(Q) :\ (\partial_t^2 - \Delta_x)u = 0\}$$

$$(\text{resp } S_* = \{u \in L^2(0,T;H^1(\Omega)) :\ (\partial_t^2 - \Delta_x)u = 0\})$$

considered as a subset of $H^1(Q)$ (resp $L^2(0,T;H^1(\Omega))$). According to [36, Proposition 4], the maps

$$\tau_0 w = (w_{|\Sigma}, w_{|t=0}, \partial_t w_{|t=0}), \quad \tau_1 w = (\partial_\nu w_{|\Sigma}, w_{|t=T}, \partial_t w_{|t=T}), \quad w \in \mathcal{C}^\infty(\overline{Q}),$$

can be extended continuously to $\tau_0 : H_{\Box,*}(Q) \to H^{-3}(0,T;H^{-\frac{1}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$, $\tau_1 : H_{\Box,*}(Q) \to H^{-3}(0,T;H^{-\frac{3}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$. For all $w \in \mathcal{C}^\infty(\overline{Q})$ we fix also

$$\tau_0 w = (\tau_{0,1}w, \tau_{0,2}w, \tau_{0,3}w), \quad \tau_1 w = (\tau_{1,1}w, \tau_{1,2}w, \tau_{1,3}w),$$

with

$$\tau_{0,1}w = w_{|\Sigma},\ \tau_{0,2}w = w_{|t=0},\ \tau_{0,3}w = \partial_t w_{|t=0},$$
$$\tau_{1,1}w = \partial_\nu w_{|\Sigma},\ \tau_{1,2}w = w_{|t=T},\ \tau_{1,3}w = \partial_t w_{|t=T}.$$

Using these operators, we define

$$\mathcal{H} := \{\tau_0 u :\ u \in H_\Box(Q)\} \subset H^{-3}(0,T;H^{-\frac{1}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega),$$

$$\mathcal{H}_* := \{(\tau_{0,1}u, \tau_{0,3}u) :\ u \in H_{\Box,*}(Q),\ \tau_{0,2}u = 0\} \subset H^{-3}(0,T;H^{-\frac{1}{2}}(\partial\Omega)) \times H^{-4}(\Omega).$$

In view of [36, Proposition 1], it is clear that the restriction of $\tau_0$ to $S$ (resp $S_*$) is one to one and onto. Therefore, we can consider on $\mathcal{H}$ (resp $\mathcal{H}_*$) the norm

$$\|(f,v_0,v_1)\|_{\mathcal{H}} = \left\|(\tau_{0|S})^{-1}(f,v_0,v_1)\right\|_{H^1(Q)}, \quad (f,v_0,v_1) \in \mathcal{H},$$

$$(\text{resp } \|(f,v_1)\|_{\mathcal{H}_*} = \left\|(\tau_{0|S_*})^{-1}(f,0,v_1)\right\|_{L^2(0,T;H^1(\Omega))}, \quad (f,v_1) \in \mathcal{H}_*).$$

We consider now the initial boundary value problem (IBVP in short)

$$\left\{\begin{array}{rl} \partial_t^2 v - \Delta_x v + qv = F(t,x), & (t,x) \in Q, \\ v(0,x) = v_0(x), \quad \partial_t v(0,x) = v_1(x), & x \in \Omega \\ v(t,x) = 0, & (t,x) \in \Sigma. \end{array}\right. \tag{2.1}$$

We have the following well-posedness result for this IBVP when $q$ is unbounded.

**Proposition 1.** *Let $p_1 \in (1,+\infty)$ and $p_2 \in [n,+\infty) \setminus \{2\}$. For $q \in L^{p_1}(0,T;L^{p_2}(\Omega))$, $v_0 \in H_0^1(\Omega)$, $v_1 \in L^2(\Omega)$ and $F \in L^{p_1}(0,T;L^2(\Omega))$, problem (2.1) admits a unique solution $v \in \mathcal{C}([0,T];H_0^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega))$ which satisfies*

$$\|v\|_{\mathcal{C}([0,T];H_0^1(\Omega))} + \|v\|_{\mathcal{C}^1([0,T];L^2(\Omega))} \le C(\|v_0\|_{H^1(\Omega)} + \|v_1\|_{L^2(\Omega)} + \|F\|_{L^{p_1}(0,T;L^2(\Omega))}), \tag{2.2}$$

*with $C$ depending only on $p_1$, $p_2$, $n$, $T$, $\Omega$ and any $M \ge \|q\|_{L^{p_1}(0,T;L^{p_2}(\Omega))}$.*

*Proof.* According to the second part of the proof of [43, Theorem 8.1, Chapter 3], [43, Remark 8.2, Chapter 3] and [43, Theorem 8.3, Chapter 3], the proof of this proposition will be completed if we show that for any $v \in W^{2,\infty}(0,T;H_0^1(\Omega))$ solving (2.1) the a priori estimate (2.2) holds true. Without lost of generality we assume that $v$ is real valued. From now on we consider this estimate. We define the energy $E(t)$ at time $t \in [0,T]$ by

$$E(t) := \int_\Omega \left(|\partial_t v(t,x)|^2 + |\nabla_x v(t,x)|^2\right) dx.$$

Multiplying (2.1) by $\partial_t v$ and integrating by parts we get

$$E(t) - E(0) = -2\int_0^t \int_\Omega q(s,x)v(s,x)\partial_t v(s,x)dxds + 2\int_0^t \int_\Omega F(s,x)\partial_t v(s,x)dxds. \tag{2.3}$$

On the other hand, we have

$$\left|\int_0^t \int_\Omega q(s,x)v(s,x)\partial_t v(s,x)dxds\right| \le \int_0^t \|qv(s,\cdot)\|_{L^2(\Omega)} \|\partial_t v(s,\cdot)\|_{L^2(\Omega)}\, ds. \tag{2.4}$$

Let us observe that, since $v \in \mathcal{C}([0,T]; H^1(\Omega))$ and since, for $n \geq 3$, we have

$$2 < \frac{2p_2}{p_2 - 2} = \frac{2n}{n - 2\frac{n}{p_2}} \leq \frac{2n}{n-2},$$

the Sobolev embedding theorem implies

$$\|v(t,\cdot)\|_{L^{\frac{2p_2}{p_2-2}}(\Omega)} \leq C\, \|v(t,\cdot)\|_{H^1(\Omega)}, \quad t \in (0,T), \tag{2.5}$$

with $C$ depending only on $\Omega$, $p_2$, $n$. Moreover, since $v_{|\Sigma} = 0$, an application of the Poincaré inequality yields

$$\|v(t,\cdot)\|_{H^1(\Omega)} \leq C\, \|\nabla_x v(t,\cdot)\|_{L^2(\Omega)} \leq C E(t)^{\frac{1}{2}}, \quad t \in (0,T), \tag{2.6}$$

with $C$ depending only on $\Omega$, $p_2$, $n$. Therefore, combining (2.5)-(2.6) with the Hölder inequality, for all $s \in (0,T)$, we get

$$\|qv(s,\cdot)\|_{L^2(\Omega)} \leq \|q(s,\cdot)\|_{L^{p_2}(\Omega)}\, \|v(s,\cdot)\|_{L^{\frac{2p_2}{p_2-2}}(\Omega)} \leq C\, \|q(s,\cdot)\|_{L^{p_2}(\Omega)}\, E(s)^{\frac{1}{2}},$$

with $C$ depending only on $\Omega$, $p_2$, $n$. Thus, from (2.4), we get

$$\begin{aligned}\left|\int_0^t\int_\Omega q(s,x)v(s,x)\partial_t v(s,x)dxds\right| &\leq C\int_0^t \|q(s,\cdot)\|_{L^{p_2}(\Omega)}\, E(s)ds \\ &\leq \|q\|_{L^{p_1}(0,T;L^{p_2}(\Omega))}\left(\int_0^t E(s)^{\frac{p_1}{p_1-1}}ds\right)^{\frac{p_1-1}{p_1}}.\end{aligned} \tag{2.7}$$

In the same way, an application of the Hölder inequality yields

$$\begin{aligned}\left|\int_0^t\int_\Omega F(s,x)\partial_t v(s,x)dxds\right| &\leq \|F\|_{L^{p_1}(0,T;L^2(\Omega))}\left(\int_0^t E(s)^{\frac{p_1}{2(p_1-1)}}ds\right)^{\frac{p_1-1}{p_1}} \\ &\leq \|F\|^2_{L^{p_1}(0,T;L^2(\Omega))} + \left(\int_0^t E(s)^{\frac{p_1}{2(p_1-1)}}ds\right)^{\frac{2(p_1-1)}{p_1}}.\end{aligned}$$

Then, using the Cauchy-Schwarz inequality, we get

$$\left|\int_0^t\int_\Omega F(s,x)\partial_t v(s,x)dxds\right|$$

$$\leq \|F\|^2_{L^{p_1}(0,T;L^2(\Omega))} + T^{\frac{(p_1-1)}{p_1}}\left(\int_0^t E(s)^{\frac{p_1}{(p_1-1)}}ds\right)^{\frac{(p_1-1)}{p_1}}.$$

Combining this estimate with (2.3)-(2.7), we deduce that

$$E(t) \leq E(0) + \|F\|^2_{L^{p_1}(0,T;L^2(\Omega))} + C\left(\int_0^t E(s)^{\frac{p_1}{p_1-1}}ds\right)^{\frac{p_1-1}{p_1}},$$

where $C$ depends only on $T$, $\Omega$, $p_1$, $p_2$, $n$ and any $M \geq \|q\|_{L^{p_1}(0,T;L^{p_2}(\Omega))}$. By taking the power $\frac{p_1}{p_1-1}$ on both side of this inequality, we get

$$E(t)^{\frac{p_1}{p_1-1}} \leq C\left(\|v_0\|_{H^1(\Omega)} + \|v_1\|_{L^2(\Omega)} + \|F\|_{L^{p_1}(0,T;L^2(\Omega))}\right)^{\frac{2p_1}{p_1-1}} + C\int_0^t E(s)^{\frac{p_1}{p_1-1}}ds.$$

Then, the Gronwall inequality implies

$$\begin{aligned}E(t)^{\frac{p_1}{p_1-1}} &\leq C\left(\|v_0\|_{H^1(\Omega)}+\|v_1\|_{L^2(\Omega)}+\|F\|_{L^{p_1}(0,T;L^2(\Omega))}\right)^{\frac{2p_1}{p_1-1}}e^{Ct}\\ &\leq C\left(\|v_0\|_{H^1(\Omega)}+\|v_1\|_{L^2(\Omega)}+\|F\|_{L^{p_1}(0,T;L^2(\Omega))}\right)^{\frac{2p_1}{p_1-1}}e^{CT}.\end{aligned}$$

From this last estimate one can easily deduce (2.2). □

Let us introduce the IBVP

$$\left\{\begin{array}{ll}\partial_t^2u-\Delta_x u+q(t,x)u=0, & \text{in } Q,\\ u(0,\cdot)=v_0,\quad \partial_t u(0,\cdot)=v_1, & \text{in } \Omega,\\ u=g, & \text{on } \Sigma.\end{array}\right. \tag{2.8}$$

Combining Proposition 1 with the arguments used in [37, Proposition 2.1], we obtain the following result of existence and uniqueness of solutions of the IBVP (2.8) for $(g,v_0,v_1)\in\mathcal{H}$ and $q\in L^p(Q)$, $p>n+1$.

**Proposition 2.** *Let $(g,v_0,v_1)\in\mathcal{H}$, $q\in L^p(Q)$, $p>n+1$. Then, the IBVP (2.8) admits a unique weak solution $u\in H^1(Q)$ satisfying*

$$\|u\|_{H^1(Q)}\leq C\,\|(g,v_0,v_1)\|_{\mathcal{H}} \tag{2.9}$$

*and the boundary operator $B_q:(g,v_0,v_1)\mapsto(\tau_{1,1}u_{|V},\tau_{1,2}u)$ is a bounded operator from $\mathcal{H}$ to*
*$H^{-3}(0,T;H^{-\frac{3}{2}}(V'))\times H^{-2}(\Omega)$.*

Applying these results, we fix

$$C_q(T,V)=\{(g,v_0,v_1,B_q(g,v_0,v_1)):\ (g,v_0,v_1)\in\mathcal{H}\}.$$

**Proposition 3.** *Let $(g,v_1)\in\mathcal{H}_*$ with $v_0=0$ and let $q\in L^\infty(0,T;L^p(\Omega))$, $p>n$. Then, the IBVP (2.8) admits a unique weak solution $u\in L^2(0,T;H^1(\Omega))$ satisfying*

$$\|u\|_{L^2(0,T;H^1(\Omega))}\leq C\,\|(g,v_1)\|_{\mathcal{H}_*} \tag{2.10}$$

*and the boundary operator $B_{q,*}:(g,v_1)\mapsto(\tau_{1,1}u_{|V},\tau_{1,2}u)$ is a bounded operator from $\mathcal{H}_*$ to*
*$H^{-3}(0,T;H^{-\frac{3}{2}}(V'))\times H^{-2}(\Omega)$.*

We define the set $C_q(0,T,V)$ by

$$C_q(0,T,V)=\{(g,v_1,B_{q,*}(g,v_1)):\ (g,v_1)\in\mathcal{H}_*\}.$$

Now let us define the sets $C_q(T)$, $C_q(0)$, $C_q(0,T)$ introduced before Theorem 1.1 for $q\in L^{p_1}(0,T;L^{p_2}(\Omega))$, $p_1>1$, $p_2\in[n,+\infty)\setminus\{2\}$. Note first that for any $u\in\mathcal{K}(Q)$ satisfying $\partial_t^2u-\Delta u+qu=0$, $u$ solves the IBVP

$$\left\{\begin{array}{rl}\partial_t^2u-\Delta_x u=G(t,x), & (t,x)\in Q,\\ u(0,x)=v_0(x),\quad \partial_t u(0,x)=v_1(x), & x\in\Omega\\ u(t,x)=g, & (t,x)\in\Sigma.\end{array}\right.$$

with $G=-qu\in L^1(0,T;L^2(\Omega))$, $v_0\in H^1(\Omega)$, $v_1\in L^2(\Omega)$, $g\in H^1(\Sigma)$, $g_{|t=0}=v_{0|\partial\Omega}$. Thus, applying [5, Theorem A.1] we deduce that $\partial_\nu u\in L^2(\Sigma)$ and we can consider the sets $C_q(T)$, $C_q(0)$, $C_q(0,T)$.

3. **Proof of Theorem 1.1.** The goal of this section is to prove Theorem 1.1. For this purpose, we consider special solutions $u_j$ of the equation

$$\partial_t^2 u_j - \Delta_x u_j + q_j u_j = 0 \tag{3.1}$$

taking the form

$$u_j = a_{j,1} e^{i\lambda\psi_{j,1}(t,x)} + a_{j,2} e^{i\lambda\psi_{j,2}(t,x)} + R_{j,\lambda} \tag{3.2}$$

with a large parameter $\lambda > 0$ and a remainder term $R_{j,\lambda}$ that admits some decay with respect to $\lambda$. The use of such solutions, also called oscillating geometric optics solutions, goes back to [45] who have proved unique recovery of time-independent coefficients. Since then, such approach has been used by various authors in different context including recovery of a bounded time-dependent coefficient by [38]. In this section we will prove how one can extend this approach, that has been specifically designed for the recovery of time-independent coefficients or bounded time-dependent coefficients, to the recovery of singular time-dependent coefficients.

3.1. **Oscillating geometric optics solutions.** Fixing $\omega \in \mathbb{S}^{n-1}$, $\lambda > 1$ and $a_{j,k} \in \mathcal{C}^\infty(\overline{Q})$, $j = 1, 2$, $k = 1, 2$, we consider solutions of (3.1) taking the form

$$u_1(t,x) = a_{1,1}(t,x)e^{-i\lambda(t+x\cdot\omega)} + a_{1,2}(t,x)e^{-i\lambda((2T-t)+x\cdot\omega)} + R_{1,\lambda}(t,x), \quad (t,x) \in Q, \tag{3.3}$$

$$u_2(t,x) = a_{2,1}(t,x)e^{i\lambda(t+x\cdot\omega)} + a_{2,2}(t,x)e^{i\lambda(-t+x\cdot\omega)} + R_{2,\lambda}(t,x), \quad (t,x) \in Q. \tag{3.4}$$

Here, the expression $a_{j,k}$, $j, k = 1, 2$, are independent of $\lambda$ and they are respectively solutions of the transport equation

$$\partial_t a_{j,k} + (-1)^k \omega \cdot \nabla_x a_{j,k} = 0, \quad (t,x) \in Q, \tag{3.5}$$

and the expression $R_{j,\lambda}$, $j = 1, 2$, solves respectively the IBVP

$$\left\{\begin{array}{rll} \partial_t^2 R_{1,\lambda} - \Delta_x R_{1,\lambda} + q_1 R_{1,\lambda} & = F_{1,\lambda}, & (t,x) \in Q, \\ R_{1,\lambda}(T,x) = 0, \quad \partial_t R_{1,\lambda}(T,x) & = 0, & x \in \Omega \\ R_{1,\lambda}(t,x) = 0, & & (t,x) \in \Sigma, \end{array}\right. \tag{3.6}$$

$$\left\{\begin{array}{rll} \partial_t^2 R_{2,\lambda} - \Delta_x R_{2,\lambda} + q_2 R_{2,\lambda} & = F_{2,\lambda}, & (t,x) \in Q, \\ R_{2,\lambda}(0,x) = 0, \quad \partial_t R_{2,\lambda}(0,x) & = 0, & x \in \Omega \\ R_{2,\lambda}(t,x) = 0, & & (t,x) \in \Sigma, \end{array}\right. \tag{3.7}$$

with $F_{j,\lambda} = -[(\Box + q_j)(u_j - R_{j,\lambda})]$. The main point in the construction of such solutions, also called oscillating geometric optics (GO in short) solutions, consists of proving the decay of the expression $R_{j,\lambda}$ with respect to $\lambda \to +\infty$. Actually, we would like to prove the following,

$$\lim_{\lambda\to+\infty} \|R_{j,\lambda}\|_{L^\infty(0,T;L^2(\Omega))} = 0. \tag{3.8}$$

For $q \in L^\infty(Q)$, the construction of GO solutions of the form (3.3)-(3.4), with $a_{j,k}$ satisfying (3.5) and $R_{j,\lambda}$ satisfying (3.6)-(3.8), has been proved in [38, Lemma 2.2]. The fact that $q$ is bounded plays an important role in the arguments of [38, Lemma 2.2]. For this reason we can not apply the result of [38] and we need to consider the following.

**Lemma 3.1.** *Let $q_j \in L^{p_1}(0,T;L^{p_2}(\Omega))$, $j=1,2$, $p_1>1$, $p_2\in[n,+\infty)\setminus\{2\}$. Then, we can find $u_j\in\mathcal{K}(Q)$ solving (3.1), of the form (3.3)-(3.4), with $R_{j,\lambda}$, $j=1,2$, satisfying (3.8) and the following estimate*

$$\sup_{\lambda>1}\max_{j=1,2}\|R_{j,\lambda}\|_{L^\infty(0,T;H^1(\Omega))}<\infty. \tag{3.9}$$

*Proof.* We will consider this result only for $j=2$, the proof for $j=1$ being similar by symmetry. Note first that, (3.5) implies that

$$\begin{aligned}F_{2,\lambda}(t,x)&=-e^{i\lambda(t+x\cdot\omega)}(\Box+q_2)a_{2,1}(t,x)-e^{i\lambda(-t+x\cdot\omega)}(\Box+q_2)a_{2,2}(t,x)\\&=H_\lambda(t,x)\\&=e^{i\lambda t}H_{1,\lambda}(t,x)+e^{-i\lambda t}H_{2,\lambda}(t,x),\end{aligned}$$

with

$$\|H_\lambda\|_{L^{p_1}(0,T;L^2(\Omega))}\leq\|(\Box+q_2)a_{2,1}\|_{L^{p_1}(0,T;L^2(\Omega))}+\|(\Box+q_2)a_{2,2}\|_{L^{p_1}(0,T;L^2(\Omega))}. \tag{3.10}$$

Thus, in light Proposition 1, we have $R_{2,\lambda}\in\mathcal{K}(Q)$ with

$$\begin{aligned}&\|R_{2,\lambda}\|_{\mathcal{C}^1([0,T];L^2(\Omega))}+\|R_{2,\lambda}\|_{\mathcal{C}([0,T];H^1(\Omega))}\\\leq&C(1+\|q_2\|_{L^{p_1}(0,T;L^{p_2}(\Omega))})(\|a_{2,1}\|_{W^{2,\infty}(Q)}+\|a_{2,2}\|_{W^{2,\infty}(Q)}).\end{aligned} \tag{3.11}$$

In particular, this proves (3.9). The only point that we need to check is the decay with respect to $\lambda$ given by (3.8). For this purpose, we consider $v(t,x):=\int_0^t R_{2,\lambda}(s,x)ds$ and we easily check that $v$ solves

$$\left\{\begin{array}{rl}\partial_t^2v-\Delta_xv=G_\lambda, & (t,x)\in Q,\\ v(0,x)=0,\quad \partial_tv(0,x)=0, & x\in\Omega\\ v(t,x)=0, & (t,x)\in\Sigma,\end{array}\right. \tag{3.12}$$

with

$$G_\lambda(t,x)=-\int_0^t q_2(s,x)R_{2,\lambda}(s,x)ds+\int_0^t H_\lambda(s,x)ds,\quad (t,x)\in Q.$$

Note first that, since $R_{2,\lambda}\in\mathcal{C}^1([0,T];L^2(\Omega))\cap\mathcal{C}([0,T];H^1(\Omega))$, we have $v\in\mathcal{C}^2([0,T];L^2(\Omega))\cap\mathcal{C}^1([0,T];H^1(\Omega))$. Moreover, using the fact that, by the Sobolev embedding theorem, $q_2R_{2,\lambda}\in L^{p_1}(0,T;L^2(\Omega))$, we deduce that $G_\lambda\in W^{1,p_1}(0,T;L^2(\Omega))\subset L^2(Q)$ and $v$ solves the elliptic boundary value problem

$$\left\{\begin{array}{ll}-\Delta_xv=S, & (t,x)\in(0,T)\times\Omega,\\ v(t,x)=0, & (t,x)\in(0,T)\times\partial\Omega,\end{array}\right. \tag{3.13}$$

with $S=-\partial_t^2v+G_\lambda\in L^2(0,T;L^2(\Omega))$. Then, from the elliptic regularity of solutions of (3.13), we deduce that $v\in L^2(0,T;H^2(\Omega))$ and it follows that $v\in H^2(Q)$. We define the energy $E(t)$ at time $t$ associated with $v$ and given by

$$E(t):=\int_\Omega\left(|\partial_tv|^2(t,x)+|\nabla_xv|^2(t,x)\right)dx\geq\int_\Omega|R_{2,\lambda}(t,x)|^2dx.$$

Multiplying (3.12) by $\overline{\partial_tv}$ and taking the real part, we find

$$\begin{aligned}E(t)=&-2\Re\left(\int_0^t\int_\Omega\left(\int_0^s q_2(\tau,x)R_{2,\lambda}(\tau,x)d\tau\right)\overline{\partial_tv(s,x)}dxds\right)\\&+2\Re\left(\int_0^t\int_\Omega\left(\int_0^s H_\lambda(\tau,x)d\tau\right)\overline{\partial_tv(s,x)}dxds\right).\end{aligned}$$

Applying Fubini's theorem, we obtain

$$\begin{aligned} E(t) &= -2\Re\left(\int_0^t\int_\Omega q_2(\tau,x)R_{2,\lambda}(\tau,x)\left(\overline{\int_\tau^t \partial_t v(s,x)ds}\right)dxd\tau\right) \\ &\quad + 2\Re\left(\int_0^t\int_\Omega\left(\int_0^s H_\lambda(\tau,x)d\tau\right)\overline{\partial_t v(s,x)}dxds\right) \\ &= -2\Re\left(\int_0^t\int_\Omega q_2(\tau,x)R_{2,\lambda}(\tau,x)(\overline{v(t,x)}-\overline{v(\tau,x)})dxd\tau\right) \\ &\quad + 2\Re\left(\int_0^t\int_\Omega\left(\int_0^s H_\lambda(\tau,x)d\tau\right)\overline{\partial_t v(s,x)}dxds\right). \end{aligned} \tag{3.14}$$

On the other hand, applying the Hölder inequality, we get

$$\begin{aligned} &\left|\int_0^t\int_\Omega q_2(\tau,x)R_{2,\lambda}(\tau,x)\overline{v(t,x)}dxd\tau\right| \\ &\leq \int_0^t \|\partial_t v(\tau,\cdot)\|_{L^2(\Omega)}\|q_2(\tau,\cdot)v(t,\cdot)\|_{L^2(\Omega)}d\tau \\ &\leq \left(\int_0^t \|\partial_t v(\tau,\cdot)\|_{L^2(\Omega)}\|q_2(\tau,\cdot)\|_{L^{p_2}(\Omega)}d\tau\right)\|v(t,\cdot)\|_{L^{\frac{2p_2}{p_2-2}}(\Omega)}. \end{aligned}$$

Combining this inequality with the estimates (2.5)-(2.6) and applying the Hölder inequality, we obtain

$$\begin{aligned} &\left|\int_0^t\int_\Omega q_2(\tau,x)R_{2,\lambda}(\tau,x)\overline{v(t,x)}dxd\tau\right| \\ &\leq C\left(\int_0^t \|\partial_t v(\tau,\cdot)\|_{L^2(\Omega)}\|q_2(\tau,\cdot)\|_{L^{p_2}(\Omega)}d\tau\right)\|v(t,\cdot)\|_{H^1(\Omega)} \\ &\leq C\left(\int_0^t E(\tau)^{1/2}\|q_2(\tau,\cdot)\|_{L^{p_2}(\Omega)}d\tau\right)E(t)^{1/2} \\ &\leq C^2\left(\int_0^t E(\tau)^{1/2}\|q_2(\tau,\cdot)\|_{L^{p_2}(\Omega)}d\tau\right)^2+\frac{E(t)}{4} \\ &\leq C^2\left(\|q_2\|_{L^{p_1}(0,T;L^{p_2}(\Omega))}\left(\int_0^t E(\tau)^{\frac{p_1}{2(p_1-1)}}d\tau\right)^{\frac{p_1-1}{p_1}}\right)^2+\frac{E(t)}{4} \\ &\leq C^2\|q_2\|^2_{L^{p_1}(0,T;L^{p_2}(\Omega))}\left(\int_0^t E(\tau)^{\frac{p_1}{2(p_1-1)}}d\tau\right)^{\frac{2(p_1-1)}{p_1}}+\frac{E(t)}{4}, \end{aligned}$$

with $C$ depending only on $\Omega$, $p_2$, $n$. Then, applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned} &\left|\int_0^t\int_\Omega q_2(\tau,x)R_{2,\lambda}(\tau,x)\overline{v(t,x)}dxd\tau\right| \\ &\leq C^2\|q_2\|^2_{L^{p_1}(0,T;L^{p_2}(\Omega))}\left(\left(\int_0^t E(\tau)^{\frac{p_1}{2(p_1-1)}}d\tau\right)^2\right)^{\frac{p_1-1}{p_1}}+\frac{E(t)}{4} \\ &\leq C^2\|q_2\|^2_{L^{p_1}(0,T;L^{p_2}(\Omega))}\left(t\left(\int_0^t E(\tau)^{\frac{p_1}{(p_1-1)}}d\tau\right)\right)^{\frac{p_1-1}{p_1}}+\frac{E(t)}{4} \end{aligned} \tag{3.15}$$

$$\leq C^2 \|q_2\|^2_{L^{p_1}(0,T;L^{p_2}(\Omega))} T^{\frac{(p_1-1)}{p_1}} \left(\int_0^t E(\tau)^{\frac{p_1}{(p_1-1)}} d\tau\right)^{\frac{(p_1-1)}{p_1}} + \frac{E(t)}{4}.$$

In the same way, we obtain

$$\begin{aligned} &\left|\int_0^t \int_\Omega q_2(\tau,x) R_{2,\lambda}(\tau,x)\overline{v(\tau,x)}dxd\tau\right| \\ &\leq C\int_0^t E(\tau)\|q_2(\tau,\cdot)\|_{L^{p_2}(\Omega)} d\tau \\ &\leq C\left(\int_0^t E(\tau)^{\frac{p_1}{p_1-1}} d\tau\right)^{\frac{p_1-1}{p_1}} \|q_2\|_{L^{p_1}(0,T;L^{p_2}(\Omega))}. \end{aligned} \tag{3.16}$$

Finally, fixing

$$\beta_\lambda(t,x) := \int_0^t H_\lambda(\tau,x)d\tau,$$

we find

$$\begin{aligned} &\left|\int_0^t \int_\Omega \left(\int_0^s H_\lambda(\tau,x)d\tau\right)\overline{\partial_t v(s,x)}dxds\right| \\ &\leq \|\beta_\lambda\|_{L^{p_1}(0,T;L^2(\Omega))} \left(\int_0^t E(\tau)^{\frac{p_1}{2(p_1-1)}} d\tau\right)^{\frac{p_1-1}{p_1}} \\ &\leq \|\beta_\lambda\|^2_{L^{p_1}(0,T;L^2(\Omega))} + \left(\int_0^t E(\tau)^{\frac{p_1}{2(p_1-1)}} d\tau\right)^{\frac{2(p_1-1)}{p_1}} \\ &\leq \|\beta_\lambda\|^2_{L^{p_1}(0,T;L^2(\Omega))} + T^{\frac{p_1-1}{p_1}} \left(\int_0^t E(\tau)^{\frac{p_1}{p_1-1}} d\tau\right)^{\frac{p_1-1}{p_1}}. \end{aligned} \tag{3.17}$$

Combining (3.14)-(3.17), we deduce that

$$E(t) \leq \frac{E(t)}{4} + C(\|q_2\|_{L^{p_1}(0,T;L^{p_2}(\Omega))}+1)^2 \left(\int_0^t E(\tau)^{\frac{p_1}{p_1-1}} d\tau\right)^{\frac{p_1-1}{p_1}} + \|\beta_\lambda\|^2_{L^{p_1}(0,T;L^2(\Omega))}$$

and we get

$$E(t) \leq C(\|q_2\|_{L^{p_1}(0,T;L^{p_2}(\Omega))} + 1)^2 \left(\int_0^t E(\tau)^{\frac{p_1}{p_1-1}} d\tau\right)^{\frac{p_1-1}{p_1}} + \frac{4\|\beta_\lambda\|^2_{L^{p_1}(0,T;L^2(\Omega))}}{3},$$

with $C$ depending only on $\Omega$, $T$, $p_1$, $p_2$, $n$. Now taking the power $\frac{p_1}{p_1-1}$ on both side of this inequality, we get

$$\begin{aligned} E(t)^{\frac{p_1}{p_1-1}} &\leq 2^{\frac{p_1}{p_1-1}} C^{\frac{p_1}{p_1-1}} (\|q_2\|_{L^{p_1}(0,T;L^{p_2}(\Omega))} + 1)^{\frac{2p_1}{p_1-1}} \int_0^t E(\tau)^{\frac{p_1}{p_1-1}} d\tau \\ &\quad + 2^{\frac{p_1}{p_1-1}} \left(\frac{4\|\beta_\lambda\|^2_{L^{p_1}(0,T;L^2(\Omega))}}{3}\right)^{\frac{p_1}{p_1-1}} \end{aligned}$$

and applying the Gronwall inequality, we obtain

$$E(t)^{\frac{p_1}{p_1-1}} \leq C_1 \|\beta_\lambda\|^{\frac{2p_1}{p_1-1}}_{L^{p_1}(0,T;L^2(\Omega))} e^{C_2 t} \leq C_1 \|\beta_\lambda\|^{\frac{2p_1}{p_1-1}}_{L^{p_1}(0,T;L^2(\Omega))} e^{C_2 T},$$

where $C_1$ depends only on $p_1$ and $C_2$ on $\|q_2\|_{L^{p_1}(0,T;L^{p_2}(\Omega))}$, $p_1$, $p_2$, $n$, $\Omega$ and $T$. According to this estimate, the proof of the lemma will be completed if we prove that

$$\lim_{\lambda\to+\infty}\|\beta_\lambda\|_{L^\infty(0,T;L^2(\Omega))}=0. \tag{3.18}$$

This last property follows from some arguments similar to the end of the proof of [38, Lemma 2.2]. This completes the proof the lemma. $\square$

3.2. **Proof of Theorem 1.1 with restriction at $t=0$ or $t=T$.** This subsection is devoted to the first part of Theorem 1.1, which consists of showing that (1.2) or (1.3) implies $q_1=q_2$. We start by assuming that (1.2) is fulfilled and we fix $q=q_2-q_1$ on $Q$ extended by 0 on $\mathbb{R}^{1+n}\setminus Q$. We set $\lambda>1$, $\omega\in\mathbb{S}^{n-1}$ and $\xi\in\mathbb{R}^{1+n}$ satisfying $(1,-\omega)\cdot\xi=0$. Then, in view of Lemma 3.1, we can consider $u_j\in\mathcal{K}(Q)$, $j=1,2$, solving (3.1), of the form (3.3)-(3.4), with $a_{1,1}(t,x)=(2\pi)^{-\frac{n+1}{2}}e^{-i(t,x)\cdot\xi}$, $a_{1,2}=0$, $a_{2,1}=1$, $a_{2,2}=-1$ and with condition (3.8)-(3.9) fulfilled, that is,

$$\begin{aligned}&u_1(t,x)=(2\pi)^{-\frac{n+1}{2}}e^{-i(t,x)\cdot\xi}e^{-i\lambda(t+x\cdot w)}+R_{1,\lambda}(t,x),\\&u_2(t,x)=e^{i\lambda(t+x\cdot w)}-e^{i\lambda(-t+x\cdot w)}+R_{2,\lambda}(t,x).\end{aligned} \tag{3.19}$$

Obviously, we have $u_2(0,x)=0$, since $R_{2,\lambda}(0,x)=0$ by (3.7). In view of Proposition 1, there exists a unique weak solution $v\in\mathcal{K}(Q)$ to the IBVP:

$$\begin{aligned}&\partial_t^2v-\Delta v+q_1v=0\quad\text{in}\quad Q,\\&v|_{t=0}=u_2|_{t=0}=0,\ \partial_tv|_{t=0}=\partial_tu_2|_{t=0},\quad v|_\Sigma=u_2|_\Sigma.\end{aligned}$$

Setting $u:=v-u_2$, we see

$$\begin{aligned}&\partial_t^2u-\Delta u+q_1u=(q_2-q_1)u_2\quad\text{in}\quad Q,\\&u|_{t=0}=0,\ \partial_tu|_{t=0}=0,\quad u|_\Sigma=0.\end{aligned} \tag{3.20}$$

Note that the inhomogeneous term $(q_2-q_1)u_2\in L^{p_1}(0,T;L^2(\Omega))$, due to the fact that $q_2-q_1\in L^{p_1}(0,T;L^{p_2}(\Omega))$ and $u_2\in L^\infty(0,T;H^1(\Omega))$. Hence, using again Proposition 1 gives that $u\in\mathcal{C}([0,T];H_0^1(\Omega))\cap\mathcal{C}^1([0,T];L^2(\Omega))$. Moreover, $u$ solves

$$\begin{cases}\partial_t^2u-\Delta u=Z\quad\text{in}\quad Q,\\u|_{t=0}=0,\ \partial_tu|_{t=0}=0,\quad u|_\Sigma=0.\end{cases}$$

with $Z=-q_1u+(q_2-q_1)u_2\in L^1(0,T;L^2(\Omega))$. Therefore, in light of [5, Theorem A.1], we have $\partial_\nu u\in L^2(\Sigma)$. In the same way, we can prove that $u_1\in\mathcal{K}(Q)$ with $\partial_\nu u_1\in L^2(\Sigma)$. Thus, we can multiply $u_1$ to the equation in (3.20) and apply Green formula to get

$$\begin{aligned}\int_Q(q_2-q_1)u_2u_1\,dxdt=&\int_\Omega\partial_tu(T,x)u_1(T,x)dx-\int_\Omega u(T,x)\partial_tu_1(T,x)dx\\&-\int_\Sigma\partial_\nu u(t,x)u_1(t,x)d\sigma(x)dt.\end{aligned} \tag{3.21}$$

Since $C_{q_1}(0)=C_{q_2}(0)$ and $v|_{t=0}=u_2|_{t=0}=0$, we see $\partial_\nu u|_\Sigma=u|_{t=T}=\partial_tu|_{t=T}=0$, in addition to the boundary conditions of $u$ in (3.20). Consequently, it follows from (3.21) that

$$\int_Q(q_2-q_1)u_2u_1\,dxdt=0.$$

Inserting the expressions of $u_j$ ($j=1,2$) given by (3.19) to the previous identity gives the relation

$$0=(2\pi)^{-(n+1)/2}\int_Q q(t,x)e^{-i(t,x)\cdot\xi}\,dxdt+\tilde{R}_\lambda,$$

$$\tilde{R}_\lambda:=(2\pi)^{-(n+1)/2}\int_Q q(t,x)e^{-i(t,x)\cdot\xi}\left(-e^{-2i\lambda t}+e^{-i\lambda(t+x\cdot w)}\,R_{2,\lambda}(t,x)\right)\,dxdt$$
$$+\int_Q q(t,x)R_{1,\lambda}(t,x)\left(e^{i\lambda(t+x\cdot w)}-e^{i\lambda(-t+x\cdot w)}+R_{2,\lambda}(t,x)\right)\,dxdt$$

for all $\lambda>1$. Using the fact that $q\in L^{p_1}(0,T;L^2(\Omega))$ and applying the Riemann-Lebesgue lemma and (3.8), we deduce that

$$\left|\int_Q q(t,x)e^{-i(t,x)\cdot\xi}\left(-e^{-2i\lambda t}+e^{-i\lambda(t+x\cdot w)}\,R_{2,\lambda}(t,x)\right)\,dxdt\right|\to 0,$$
$$\left|\int_Q q(t,x)R_{1,\lambda}(t,x)\left(e^{i\lambda(t+x\cdot w)}-e^{i\lambda(-t+x\cdot w)}\right)\,dxdt\right|\to 0$$

as $\lambda\to\infty$. On the other hand, by Hölder inequality it holds that

$$\begin{aligned}&\left|\int_Q q(t,x)R_{1,\lambda}(t,x)R_{2,\lambda}(t,x)\,dxdt\right|\\ &\leq ||q\,R_{1,\lambda}||_{L^1(0,T;L^2(\Omega))}\,||R_{2,\lambda}||_{L^\infty(0,T;L^2(\Omega))}\\ &\leq C\,||q||_{L^{p_1}(0,T;L^{p_2}(\Omega))}\,||R_{1,\lambda}||_{L^\infty(0,T;H^1(\Omega))}\,||R_{2,\lambda}||_{L^\infty(0,T;L^2(\Omega))},\end{aligned}$$

which tends to zero as $\lambda\to\infty$ due to the decaying behavior of $R_{j,\lambda}$ (see (3.8)) and estimate (3.9). Therefore, $|\tilde{R}_\lambda|\to 0$ as $\lambda\to\infty$. It then follows that

$$\mathcal{F}q(\xi)=(2\pi)^{-\frac{n+1}{2}}\int_{\mathbb{R}^{1+n}}q(t,x)e^{-i(t,x)\cdot\xi}dxdt=0. \tag{3.22}$$

Since $\omega\in\mathbb{S}^{n-1}$ is arbitrary chosen, we deduce that for any $\omega\in\mathbb{S}^{n-1}$ and any $\xi$ lying in the hyperplane $\{\zeta\in\mathbb{R}^{1+n}:\ \zeta\cdot(1,-\omega)=0\}$ of $\mathbb{R}^{1+n}$, the Fourier transform $\mathcal{F}q$ is null at $\xi$. On the other hand, since $q\in L^1(\mathbb{R}^{1+n})$ is compactly supported in $\overline{Q}$, we know that $\mathcal{F}q$ is a complex valued real-analytic function and it follows that $\mathcal{F}q=0$. By inverse Fourier transform this yields the vanishing of $q$, which implies that $q_1=q_2$ in $Q$.

To prove that the relation (1.3) implies $q_1=q_2$, we shall consider $u_j\in\mathcal{K}(Q)$, $j=1,2$, solving (3.1), of the form (3.3)-(3.4), with $a_{1,1}=1$, $a_{1,2}=-1$, $a_{2,1}=(2\pi)^{-\frac{n+1}{2}}e^{-i(t,x)\cdot\xi}$, $a_{2,2}=0$ and with condition (3.8)-(3.9) fulfilled. Then, by using the fact that $u_1(T,x)=0$, $x\in\Omega$, and by repeating the above arguments, we deduce that $q_1=q_2$. For brevity we omit the details.

We have proved so far that either of the conditions (1.2) and (1.3) implies $q_1=q_2$. It remains to prove that for $T>\mathrm{Diam}(\Omega)$, the condition (1.4) implies $q_1=q_2$.

### 3.3. Proof of Theorem 1.1 with restriction at $t=0$ and $t=T$.

In this section, we assume that $T>\mathrm{Diam}(\Omega)$ is fulfilled and we will show that (1.4) implies $q_1=q_2$. For this purpose, we fix $\lambda>1$, $\omega\in\mathbb{S}^{n-1}$ and $\varepsilon=\frac{T-\mathrm{Diam}(\Omega)}{4}$. We set also $\chi\in\mathcal{C}_0^\infty(-\varepsilon,T+\mathrm{Diam}(\Omega)+\varepsilon)$ satisfying $\chi=1$ on $[0,T+\mathrm{Diam}(\Omega)]$ and $x_0\in\overline{\Omega}$ such that

$$x_0\cdot\omega=\inf_{x\in\overline{\Omega}}x\cdot\omega.$$

We introduce the solutions $u_j \in \mathcal{K}(Q)$, $j = 1, 2$, of (3.1), of the form (3.3)-(3.4), with

$$\begin{aligned}
&a_{1,1}(t,x) = (2\pi)^{-\frac{n+1}{2}}\chi(t + (x - x_0)\cdot\omega)e^{-i(t,x)\cdot\xi},\\
&a_{1,2}(t,x) = -(2\pi)^{-\frac{n+1}{2}}\chi((2T - t) + (x - x_0)\cdot\omega)e^{-i(2T-t,x)\cdot\xi},\\
&a_{2,1}(t,x) = \chi(t + (x - x_0)\cdot\omega),\\
&a_{2,2}(t,x) = -\chi(-t + (x - x_0)\cdot\omega)
\end{aligned}$$

and with condition (3.8)-(3.9) fulfilled. Then, one can check that $u_1(T,x) = u_2(0,x) = 0$, $x \in \Omega$, and repeating the arguments of the previous subsection we deduce that condition (1.4) implies the orthogonality identity

$$\int_Q q(t,x)u_2(t,x)u_1(t,x)dxdt = 0. \tag{3.23}$$

It remains to prove that this implies $q = 0$. Note that

$$\begin{aligned}
&\int_Q q(t,x)u_2(t,x)u_1(t,x)dxdt\\
&= (2\pi)^{-\frac{n+1}{2}}\int_{\mathbb{R}^{1+n}} q(t,x)\chi^2(t + (x - x_0)\cdot\omega)e^{-i(t,x)\cdot\xi}dxdt\\
&+ \int_Q e^{-2i\lambda t}qa_{1,1}a_{2,2}dxdt + \int_Q e^{-2i\lambda(T-t)}qa_{1,2}a_{2,1}dxdt\\
&+ e^{-2i\lambda T}\int_Q qa_{1,2}a_{2,2}dxdt + \int_Q Z_\lambda(t,x)dxdt,
\end{aligned} \tag{3.24}$$

with

$$Z_\lambda = q(u_1 - R_{1,\lambda})R_{2,\lambda} + q(u_2 - R_{2,\lambda})R_{1,\lambda} + qR_{2,\lambda}R_{1,\lambda}.$$

In a similar way to the previous subsection, one can check that (3.8)-(3.9) imply that

$$\lim_{\lambda\to+\infty}\int_Q Z_\lambda dxdt = 0.$$

Moreover, the Riemann-Lebesgue lemma implies

$$\lim_{\lambda\to+\infty}\left(\int_Q e^{-2i\lambda t}qa_{1,1}a_{2,2}dxdt + \int_Q e^{-2i\lambda(T-t)}qa_{1,2}a_{2,1}dxdt\right) = 0.$$

In addition, using the fact that for $(t,x) \in Q$ we have

$$0 \le t + (x - x_0)\cdot\omega \le T + |x - x_0| \le T + \mathrm{Diam}(\Omega),$$

we deduce that

$$q(t,x)\chi^2(t + (x - x_0)\cdot\omega) = q(t,x), \quad (t,x) \in \mathbb{R}^{1+n}$$

and that

$$(2\pi)^{-\frac{n+1}{2}}\int_{\mathbb{R}^{1+n}} q(t,x)\chi^2(t + (x - x_0)\cdot\omega)e^{-i(t,x)\cdot\xi}dxdt = \mathcal{F}q(\xi).$$

Thus, repeating the arguments of the previous subsection we can deduce that $q_1 = q_2$ provided that

$$\int_Q qa_{1,2}a_{2,2}dxdt = 0. \tag{3.25}$$

Since $a_{2,2}(t,x) = -\chi(-t+(x-x_0)\cdot\omega)$ and $a_{1,2} = -(2\pi)^{-\frac{n+1}{2}}\chi((2T-t)+(x-x_0)\cdot\omega)e^{-i(t,x)\cdot\xi}$, we deduce that

$$\operatorname{supp}(a_{2,2}) \subset \{(t,x) \in \mathbb{R}^{1+n} :\ (x-x_0)\cdot\omega \geq t-\varepsilon\},$$

$$\operatorname{supp}(a_{1,2}) \subset \{(t,x) \in \mathbb{R}^{1+n} :\ 2T-t+(x-x_0)\cdot\omega \leq T+\operatorname{Diam}(\Omega)+\varepsilon\}.$$

But, for any $(t,x) \in \{(t,x) \in \mathbb{R}^{1+n} :\ (x-x_0)\cdot\omega \geq t-\varepsilon\}$, one can check that

$$2T-t+(x-x_0)\cdot\omega \geq 2T-\varepsilon = T+\operatorname{Diam}(\Omega)+3\varepsilon > T+\operatorname{Diam}(\Omega)+\varepsilon.$$

Therefore, we have

$$\begin{aligned}&\{(t,x) \in \mathbb{R}^{1+n} : (x-x_0)\cdot\omega \geq t-\varepsilon\}\\ &\cap \{(t,x) \in \mathbb{R}^{1+n} : 2T-t+(x-x_0)\cdot\omega \leq T+\operatorname{Diam}(\Omega)+\varepsilon\} = \emptyset\end{aligned}$$

and by the same way that $\operatorname{supp}(a_{2,2}) \cap \operatorname{supp}(a_{1,2}) = \emptyset$. This implies (3.25) and by the same way that $q_1 = q_2$. Thus, the proof of Theorem 1.1 is completed.

## 4. Proof of Theorem 1.2.

In the previous section we have seen that the oscillating geometric optics solutions (3.2) can be used for the recovery of some general singular time-dependent potentials. We have even proved that, by adding a second term, we can restrict the data on the bottom $t = 0$ and top $t = T$ of $Q$ while avoiding a "reflection". Nevertheless, as mentioned in the introduction, it is not clear how one can adapt this approach to restrict data on the lateral boundary $\Sigma$ without requiring additional smoothness or geometrical assumptions. In this section, we use a different strategy for restricting the data at $\Sigma$ which is not a flat part of the boundary $\partial Q$. Namely, we replace the oscillating GO solutions (3.2) by exponentially growing and decaying solutions in order to restrict the data on $\Sigma$ by means of a Carleman estimate. In this section, we assume that $q_1, q_2 \in L^p(Q)$, with $p > n+1$, and we will prove that (1.5) implies $q_1 = q_2$. For this purpose, we will start with the construction of some suitable solutions of (1.1). Then we will show Carleman estimates for unbounded potentials and we will complete the proof of Theorem 1.2.

### 4.1. Geometric optics solutions for Theorem 1.2.

Let $\omega \in \mathbb{S}^{n-1}$ and let $\xi \in \mathbb{R}^{1+n}$ be such that $\xi\cdot(1,-\omega) = 0$. This section is devoted to the construction of exponentially decaying solutions $u_1 \in H^1(Q)$ of the equation $(\partial_t^2 - \Delta_x + q_1)u_1 = 0$ in $Q$ taking the form

$$u_1(t,x) = e^{-\lambda(t+x\cdot\omega)}(e^{-i(t,x)\cdot\xi} + w_1(t,x)), \tag{4.1}$$

and exponentially growing solution $u_2 \in H^1(Q)$ of the equation $(\partial_t^2 - \Delta_x + q_2)u_2 = 0$ in $Q$ taking the form

$$u_2(t,x) = e^{\lambda(t+x\cdot\omega)}(1 + w_2(t,x)), \tag{4.2}$$

where $\lambda > 1$ and $w_j \in H^1(Q)$, $j = 1, 2$, fulfills the following decay property

$$\|w_j\|_{H^1(Q)} + \lambda\,\|w_j\|_{L^2(Q)} \leq C, \tag{4.3}$$

with $C$ independent of $\lambda$. The main results of this section can be stated as follows.

**Proposition 4.** *There exists $\lambda_1 > 1$ such that for $\lambda > \lambda_1$ we can find a solution $u_1 \in H^1(Q)$ of $\Box u_1 + q_1 u_1 = 0$ in $Q$ taking the form (4.1) with $w_1 \in H^1(Q)$ satisfying (4.3) for $j = 1$.*

**Proposition 5.** *There exists $\lambda_2 > \lambda_1$ such that for $\lambda > \lambda_2$ we can find a solution $u_2 \in H^1(Q)$ of $\Box u_2 + q_2 u_2 = 0$ in $Q$ taking the form (4.2) with $w_2 \in H^1(Q)$ satisfying (4.3) for $j = 2$.*

Since Proposition 4 and 5 are similar, we will only consider Proposition 4. To build solutions $u_1 \in H^1(Q)$ of the form (4.1), we first recall some preliminary tools and a suitable Carleman estimate in Sobolev space of negative order borrowed from [37]. For all $m \in \mathbb{R}$, we denote by $H^m_\lambda(\mathbb{R}^{1+n})$ the space

$$H^m_\lambda(\mathbb{R}^{1+n}) = \{u \in \mathcal{S}'(\mathbb{R}^{1+n}) :\ (|(\tau,\xi)|^2 + \lambda^2)^{\frac{m}{2}}\hat{u} \in L^2(\mathbb{R}^{1+n})\},$$

with the norm

$$\|u\|^2_{H^m_\lambda(\mathbb{R}^{1+n})} = \int_{\mathbb{R}}\int_{\mathbb{R}^n}(|(\tau,\xi)|^2 + \lambda^2)^m |\hat{u}(\tau,\xi)|^2 d\xi d\tau.$$

Note that here we consider these spaces with $\lambda > 1$ and, for $\lambda = 1$, one can check that $H^m_\lambda(\mathbb{R}^{1+n}) = H^m(\mathbb{R}^{1+n})$. Here for all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^{1+n})$, we denote by $\hat{u}$ the Fourier transform of $u$. We fix the weighted operator

$$P_{\omega,\pm\lambda} := e^{\mp\lambda(t+x\cdot\omega)}\Box e^{\pm\lambda(t+x\cdot\omega)} = \Box \pm 2\lambda(\partial_t - \omega\cdot\nabla_x)$$

and we recall the following Carleman estimate

**Lemma 4.1.** (Lemma 5.1, [37]) *There exists $\lambda_1' > 1$ such that*

$$\|v\|_{L^2(\mathbb{R}^{1+n})} \leq C\,\|P_{\omega,\lambda}v\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})}, \quad v \in \mathcal{C}^\infty_0(Q), \quad \lambda > \lambda_1', \tag{4.4}$$

*with $C > 0$ independent of $v$ and $\lambda$.*

From this result we can deduce the Carleman estimate

**Lemma 4.2.** *Let $p_1 \in (n+1, +\infty)$, $p_2 \in (n, +\infty)$ and $q \in L^{p_1}(Q) \cup L^\infty(0,T;L^{p_2}(\Omega))$. Then, there exists $\lambda_1'' > \lambda_1'$ such that*

$$\|v\|_{L^2(\mathbb{R}^{1+n})} \leq C\,\|P_{\omega,\lambda}v + qv\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})}, \quad v \in \mathcal{C}^\infty_0(Q), \quad \lambda > \lambda_1'', \tag{4.5}$$

*with $C > 0$ independent of $v$ and $\lambda$.*

*Proof.* We start by considering the case $q \in L^{p_1}(Q)$. Note first that

$$\|P_{\omega,\lambda}v + qv\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})} \geq \|P_{\omega,\lambda}v\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})} - \|qv\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})}. \tag{4.6}$$

Fixing

$$r = \frac{n+1}{p_1}, \quad \frac{1}{p_3} = \frac{1}{2} - \frac{1}{p_1} = \frac{n+1-2r}{2(n+1)},$$

by the Sobolev embedding theorem we deduce that for any $w \in H^r(\mathbb{R}^{1+n})$ we have $w \in L^{p_3}(\mathbb{R}^{1+n})$ with

$$\|w\|_{L^{p_3}(\mathbb{R}^{1+n})} \leq C\,\|w\|_{H^r(\mathbb{R}^{1+n})}, \quad w \in H^r(\mathbb{R}^{1+n}), \tag{4.7}$$

with $C > 0$ depending only on $p_3$ and $n$. Applying (4.7) we deduce that for any $z \in L^{\frac{p_3}{p_3-1}}(\mathbb{R}^{1+n})$ we have

$$\left|\int_{\mathbb{R}^{1+n}} zw\,dx\,dt\right| \leq \|z\|_{L^{\frac{p_3}{p_3-1}}(\mathbb{R}^{1+n})}\|w\|_{L^{p_3}(\mathbb{R}^{1+n})} \leq C\,\|z\|_{L^{\frac{p_3}{p_3-1}}(\mathbb{R}^{1+n})}\|w\|_{H^r(\mathbb{R}^{1+n})}.$$

It follows that $z \in H^{-r}(\mathbb{R}^{n+1})$ and

$$\|z\|_{H^{-r}(\mathbb{R}^{n+1})} \leq C\,\|z\|_{L^{\frac{p_3}{p_3-1}}(\mathbb{R}^{1+n})}, \quad z \in L^{\frac{p_3}{p_3-1}}(\mathbb{R}^{1+n}). \tag{4.8}$$

Applying (4.8) to $qv$ and using the fact that $\mathrm{supp}(qv)\subset \overline{Q}$, we find

$$\begin{aligned}\|qv\|^2_{H^{-1}_\lambda(\mathbb{R}^{1+n})} &= \int_{\mathbb{R}^{1+n}} (|\zeta|^2+\lambda^2)^{-1}|\mathcal{F}(qv)(\zeta)|^2 d\zeta\\ &\leq \int_{\mathbb{R}^{1+n}} (\lambda^2)^{-(1-r)}(1+|\zeta|^2)^{-r}|\mathcal{F}(qv)(\zeta)|^2 d\zeta\\ &\leq \lambda^{2(r-1)}\|qv\|^2_{H^{-r}(\mathbb{R}^{n+1})}\\ &\leq C\lambda^{2(r-1)}\|qv\|^2_{L^{\frac{p_3}{p_3-1}}(\mathbb{R}^{n+1})} = C\lambda^{2(r-1)}\|qv\|^2_{L^{\frac{p_3}{p_3-1}}(Q)}.\end{aligned}$$

Combining this with the fact that

$$\frac{p_3-1}{p_3} = 1-\frac{1}{p_3} = \frac{1}{2}+\frac{1}{p_1},$$

we deduce from the Hölder inequality that

$$\|qv\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})} \leq C\lambda^{r-1}\|q\|_{L^{p_1}(Q)}\|v\|_{L^2(Q)}.$$

Thus, applying (4.4) and (4.6), we deduce (4.5) for $\lambda>1$ sufficiently large. Now let us consider the case $q\in L^\infty(0,T;L^{p_2}(\Omega))$. Note first that

$$\|qv\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})} \leq \|qv\|_{L^2(0,T;H^{-1}_\lambda(\mathbb{R}^n))}.$$

Therefore, by repeating the above arguments, we obtain

$$\|qv\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})} \leq C\lambda^{\frac{n}{p_2}-1}\|q\|_{L^\infty(0,T;L^{p_2}(\Omega))}\|v\|_{L^2(Q)}$$

which implies (4.5) for $\lambda>1$ sufficiently large. Combining these two results, one can find $\lambda_1''>\lambda_1'$ such that (4.5) is fulfilled. □

Using this new carleman estimate we are now in position to complete the proof of Proposition 4.

*Proof of Proposition 4.* Note first that

$$\begin{aligned}\Box(e^{-\lambda(t+x\cdot\omega)}e^{-i\xi\cdot(t,x)}) &= [2i\lambda(1,-\omega)\cdot\xi e^{-i\xi\cdot(t,x)}+\Box e^{-i\xi\cdot(t,x)}]e^{-\lambda(t+x\cdot\omega)}\\ &= [\Box e^{-i\xi\cdot(t,x)}]e^{-\lambda(t+x\cdot\omega)},\end{aligned}$$

$$\Box(e^{-\lambda(t+x\cdot\omega)}w_1) = e^{-\lambda(t+x\cdot\omega)}P_{\omega,-\lambda}w_1.$$

Therefore, we need to consider $w_1\in H^1(Q)$ a solution of

$$P_{\omega,-\lambda}w_1+q_1w_1 = -e^{\lambda(t+x\cdot\omega)}(\Box+q_1)(e^{-\lambda(t+x\cdot\omega)}e^{-i\xi\cdot(t,x)}) = -(\Box+q_1)e^{-i\xi\cdot(t,x)} = F \tag{4.9}$$

and satisfying (4.3) for $j=1$. Combining the Carleman estimate (4.5) with a classical application of the Hahn Banach theorem (e.g. [37, Lemma 5.3]), we prove that there exists $w_1\in H^1_\lambda(\mathbb{R}^{1+n})$ such that $P_{\omega,-\lambda}w_1+q_1w_1=F$ in $Q$ and $\|w_1\|_{H^1_\lambda(\mathbb{R}^{1+n})}\leq C\|F\|_{L^2(Q)}$. This proves that $w_1$ fulfills (4.3) which completes the proof of the proposition. □

4.2. **Carleman estimates for unbounded potentials.** This subsection is devoted to the proof of a Carleman estimate similar to [37, Theorem 3.1]. More precisely, we consider the following estimate.

**Theorem 4.3.** *Let $p_1 \in (n+1, +\infty)$, $p_2 \in (n, +\infty)$ and assume that $q \in L^{p_1}(Q)$ (resp $q \in L^\infty(0,T;L^{p_2}(\Omega))$) and $u \in \mathcal{C}^2(\overline{Q})$. If $u$ satisfies the condition*

$$u_{|\Sigma} = 0, \quad u_{|t=0} = \partial_t u_{|t=0} = 0, \tag{4.10}$$

*then there exists $\lambda_3 > \lambda_2$ depending only on $\Omega$, $T$ and $M \geq \|q\|_{L^{p_1}(Q)}$ (resp $M \geq \|q\|_{L^\infty(0,T;L^{p_2}(\Omega))}$) such that the estimate*

(4.11)

$$\begin{aligned}
&\lambda \int_\Omega e^{-2\lambda(T+\omega\cdot x)} \left|\partial_t u(T,x)\right|^2 dx + \lambda \int_{\Sigma_{+,\omega}} e^{-2\lambda(t+\omega\cdot x)} \left|\partial_\nu u\right|^2 \left|\omega\cdot\nu(x)\right| d\sigma(x)dt \\
&+ \int_Q e^{-2\lambda(t+\omega\cdot x)}[\lambda^2 \left|u\right|^2 + \left|\partial_t u\right|^2 + \left|\nabla_x u\right|^2]dxdt \\
&\leq C\left(\int_Q e^{-2\lambda(t+\omega\cdot x)} \left|(\partial_t^2 - \Delta_x + q)u\right|^2 dxdt + \lambda^3 \int_\Omega e^{-2\lambda(T+\omega\cdot x)} \left|u(T,x)\right|^2 dx\right) \\
&+C\left(\lambda \int_\Omega e^{-2\lambda(T+\omega\cdot x)} \left|\nabla_x u(T,x)\right|^2 dx + \lambda \int_{\Sigma_{-,\omega}} e^{-2\lambda(t+\omega\cdot x)} \left|\partial_\nu u\right|^2 \left|\omega\cdot\nu(x)\right| d\sigma(x)dt\right)
\end{aligned}$$

*holds true for $\lambda \geq \lambda_3$ with $C$ depending only on $\Omega$, $T$ and $M$.*

*Proof.* Since the proof of this result is similar for $q \in L^{p_1}(Q)$ or $q \in L^\infty(0,T;L^{p_2}(\Omega))$, we assume without loss of generality that $q \in L^{p_1}(Q)$. Note first that for $q = 0$, (4.11) follows from [37, Theorem 3.1]. On the other hand, we have

$$\begin{aligned}
&\left\|e^{-\lambda(t+\omega\cdot x)} \left|(\partial_t^2 - \Delta_x + q)u\right|\right\|_{L^2(Q)} \\
&\geq \left\|e^{-\lambda(t+\omega\cdot x)} \left|(\partial_t^2 - \Delta_x)u\right|\right\|_{L^2(Q)} - \left\|e^{-\lambda(t+\omega\cdot x)} qu\right\|_{L^2(Q)}
\end{aligned}$$

and by the Hölder inequality we deduce that

$$\begin{aligned}
&\left\|e^{-\lambda(t+\omega\cdot x)} \left|(\partial_t^2 - \Delta_x + q)u\right|\right\|_{L^2(Q)} \\
&\geq \left\|e^{-2\lambda(t+\omega\cdot x)} \left|(\partial_t^2 - \Delta_x)u\right|\right\|_{L^2(Q)} - \|q\|_{L^{p_1}(Q)} \left\|e^{-\lambda(t+\omega\cdot x)} u\right\|_{L^{p_3}(Q)}
\end{aligned}$$

with $p_3 = \frac{2p_1}{p_1-2}$. Now fix $s := \frac{n+1}{p_1} \in (0,1)$ and notice that

$$\frac{1}{p_3} = \frac{n+1-2s}{2(n+1)}.$$

Thus, by the Sobolev embedding theorem, we have

$$\left\|e^{-\lambda(t+\omega\cdot x)} u\right\|_{L^{p_3}(Q)} \leq C \left\|e^{-\lambda(t+\omega\cdot x)} u\right\|_{H^s(Q)}$$

and by interpolation we deduce that

$$\begin{aligned}
&\left\|e^{-\lambda(t+\omega\cdot x)} u\right\|_{L^{p_3}(Q)} \\
&\leq C \left\|e^{-\lambda(t+\omega\cdot x)} u\right\|_{H^s(Q)} \\
&\leq C \left\|e^{-\lambda(t+\omega\cdot x)} u\right\|_{H^1(Q)}^s \left\|e^{-\lambda(t+\omega\cdot x)} u\right\|_{L^2(Q)}^{1-s} \\
&\leq C \left(\int_Q e^{-2\lambda(t+\omega\cdot x)} (|u|^2 + |\partial_t u - \lambda u|^2 + |\nabla_x u - \lambda u\omega|^2 dxdt\right)^{\frac{s}{2}} \left\|e^{-\lambda(t+\omega\cdot x)} u\right\|_{L^2(Q)}^{1-s} \\
&\leq C \left(\int_Q e^{-2\lambda(t+\omega\cdot x)} (\lambda^2|u|^2 + |\partial_t u|^2 + |\nabla_x u|^2 dxdt\right)^{\frac{s}{2}} \left\|e^{-\lambda(t+\omega\cdot x)} u\right\|_{L^2(Q)}^{1-s} \\
&\leq C\lambda^{s-1} \left(\int_Q e^{-2\lambda(t+\omega\cdot x)} (\lambda^2|u|^2 + |\partial_t u|^2 + |\nabla_x u|^2 dxdt\right)^{\frac{1}{2}}.
\end{aligned}$$

On the other hand, in view of [37, Theorem 3.1], there exists $\lambda_3' > 1$ such that, for $\lambda > \lambda_3'$, we have

$$\begin{aligned}
&\int_Q e^{-2\lambda(t+\omega\cdot x)}(\lambda^2|u|^2+|\partial_t u|^2+|\nabla_x u|^2)dxdt\\
&\leq C\left(\int_Q e^{-2\lambda(t+\omega\cdot x)}\left|(\partial_t^2-\Delta_x)u\right|^2 dxdt+\lambda^3\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|u(T,x)\right|^2 dx\right)\\
&+C\lambda\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|\nabla_x u(T,x)\right|^2 dx\\
&+C\lambda\int_{\Sigma_{-,\omega}} e^{-2\lambda(t+\omega\cdot x)}\left|\partial_\nu u\right|^2\left|\omega\cdot\nu(x)\right| d\sigma(x)dt.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\int_Q e^{-2\lambda(t+\omega\cdot x)}\left|(\partial_t^2-\Delta_x+q)u\right|^2 dxdt+\lambda^3\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|u(T,x)\right|^2 dx\\
&+\lambda\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|\nabla_x u(T,x)\right|^2 dx+\lambda\int_{\Sigma_{-,\omega}} e^{-2\lambda(t+\omega\cdot x)}\left|\partial_\nu u\right|^2\left|\omega\cdot\nu(x)\right| d\sigma(x)dt\\
&\geq\frac{1}{2}\int_Q e^{-2\lambda(t+\omega\cdot x)}\left|(\partial_t^2-\Delta_x)u\right|^2 dxdt+\lambda^3\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|u(T,x)\right|^2 dx\\
&+\lambda\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|\nabla_x u(T,x)\right|^2 dx+\lambda\int_{\Sigma_{-,\omega}} e^{-2\lambda(t+\omega\cdot x)}\left|\partial_\nu u\right|^2\left|\omega\cdot\nu(x)\right| d\sigma(x)dt\\
&-C\left\|q\right\|_{L^{p_1}(Q)}^2\lambda^{2(s-1)}\big(\int_Q e^{-2\lambda(t+\omega\cdot x)}\left|(\partial_t^2-\Delta_x)u\right|^2 dxdt\\
&\quad+\lambda^3\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|u(T,x)\right|^2 dx\big)\\
&-C\left\|q\right\|_{L^{p_1}(Q)}^2\lambda^{2(s-1)}\big(\lambda\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|\nabla_x u(T,x)\right|^2 dx\\
&\quad+\lambda\int_{\Sigma_{-,\omega}} e^{-2\lambda(t+\omega\cdot x)}\left|\partial_\nu u\right|^2\left|\omega\cdot\nu(x)\right| d\sigma(x)dt\big).
\end{aligned}$$

Therefore, fixing $\lambda$ sufficiently large and applying [37, Theorem 3.1] with $a = q = 0$ we deduce (4.11). $\square$

**Remark 1.** Note that, by density, (4.11) remains true for $u \in \mathcal{C}^1([0,T]; L^2(\Omega)) \cap \mathcal{C}([0,T]; H^1(\Omega))$ satisfying (4.10), $(\partial_t^2 - \Delta_x)u \in L^2(Q)$ and $\partial_\nu u \in L^2(\Sigma)$.

4.3. **Completion of the proof of Theorem 1.2.** This subsection is devoted to the proof of Theorem 1.2. From now on, we set $q = q_2 - q_1$ on $Q$ and we assume that $q = 0$ on $\mathbb{R}^{1+n} \setminus Q$. For all $\theta \in \mathbb{S}^{n-1}$ and all $r > 0$, we set

$$\partial\Omega_{+,r,\theta} = \{x \in \partial\Omega:\ \nu(x)\cdot\theta > r\},\quad \partial\Omega_{-,r,\theta} = \{x \in \partial\Omega:\ \nu(x)\cdot\theta \leq r\}$$

and $\Sigma_{\pm,r,\theta} = (0,T)\times\partial\Omega_{\pm,r,\theta}$. We set $\varepsilon > 0$ such that for all $\omega \in \{\theta \in \mathbb{S}^{n-1} : |\theta-\omega_0| \leq \varepsilon\}$ we have $\partial\Omega_{-,\varepsilon,\omega} \subset V'$. Let $\lambda > \lambda_3$ and fix $\omega \in \{\theta \in \mathbb{S}^{n-1} : |\theta-\omega_0| \leq \varepsilon\}$. Applying Proposition 4, we define

$$u_1(t,x) = e^{-\lambda(t+x\cdot\omega)}(e^{-i(t,x)\cdot\xi} + w_1(t,x)),\ (t,x) \in Q,$$

where $u_1 \in H^1(Q)$ satisfies $\partial_t^2 u_1 - \Delta_x u_1 + q_1 u_1 = 0$, $\xi\cdot(1,-\omega) = 0$ and $w_1$ satisfies (4.3) for $j = 1$. Moreover, from Proposition 5, we fix $u_2 \in H^1(Q)$ a solution of

$\partial_t^2 u_2 - \Delta_x u_2 + q_2 u_2 = 0$, taking the form

$$u_2(t,x) = e^{\lambda(t+x\cdot\omega)}(1 + w_2(t,x)), \quad (t,x) \in Q,$$

where $w_2$ satisfies (4.3) for $j = 2$. In light of Proposition 2, the IBVP

$$\begin{cases} \partial_t^2 z_1 - \Delta_x z_1 + q_1 z_1 = 0 & \text{in } Q, \\ \tau_0 z_1 = \tau_0 u_2. \end{cases} \tag{4.12}$$

admits a unique weak solution $z_1 \in H_\square(Q)$. Choosing $u = z_1 - u_2$, we deduce that $u$ solves

$$\begin{cases} \partial_t^2 u - \Delta_x u + q_1 u = (q_2 - q_1)u_2 & \text{in } Q, \\ u(0,x) = \partial_t u(0,x) = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \Sigma. \end{cases} \tag{4.13}$$

Since $u_2 \in H^1(Q)$, by the Sobolev embedding theorem we have $(q_2 - q_1)u_2 \in L^2(Q)$. Hence, again using Proposition 1 gives that $u \in \mathcal{C}([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$. Therefore, we have $u \in \mathcal{K}(Q) \cap H_\square(Q)$. Combining this with the fact that $u_1 \in H^1(Q) \cap H_\square(Q)$, we obtain

$$(\partial_t u, -\nabla_x u), (\partial_t u_1, -\nabla_x u_1) \in H_{\mathrm{div}}(Q) = \{F \in L^2(Q; \mathbb{C}^{n+1}) : \ \mathrm{div}_{(t,x)} F \in L^2(Q)\}.$$

Now, according to [32, Lemma 2.2], we can multiply $u_1$ to the equation in (3.20) and apply Green formula to get

$$\begin{aligned} &\int_Q (q_2 - q_1) u_2 u_1 \, dxdt \\ &= \langle (\partial_t u, -\nabla_x u) \cdot \mathbf{k}, u_1 \rangle_{H^{-\frac{1}{2}}(\partial Q), H^{\frac{1}{2}}(\partial Q)} - \langle (\partial_t u_1, -\nabla_x u_1) \cdot \mathbf{k}, u \rangle_{H^{-\frac{1}{2}}(\partial Q), H^{\frac{1}{2}}(\partial Q)}, \end{aligned} \tag{4.14}$$

where $\mathbf{k}$ is the outward unite normal vector to $\partial Q$. On the other hand, we have $u_{|t=0} = \partial_t u_{|t=0} = u_{|\Sigma} = 0$ and condition (1.5) implies that $u_{|t=T} = \partial_\nu u_{|V} = 0$. In addition, in view of [41, Theorem 2.1], we have $\partial_\nu u \in L^2(\Sigma)$. Combining this with the fact that $u \in \mathcal{C}^1([0,T]; L^2(\Omega))$ and $u_1 \in H^1(Q) \subset H^1(0,T; L^2(\Omega))$, we obtain

$$\int_Q q u_2 u_1 dxdt = -\int_{\Sigma \setminus V} \partial_\nu u u_1 d\sigma(x) dt + \int_\Omega \partial_t u(T,x) u_1(T,x) dx. \tag{4.15}$$

Applying the decay (4.3), the Carleman estimate (4.11) and the fact that $u_{|t=T} = \partial_\nu u_{|\Sigma_{-,\omega}} = 0$, $\partial\Omega_{+,\varepsilon,\omega} \subset \partial\Omega_{+,\omega}$, we obtain

$$\begin{aligned} \left| \int_Q q u_2 u_1 dxdt \right|^2 &\le \varepsilon^{-1} C \lambda^{-1} \left( \int_Q \left| e^{-\lambda(t+\omega\cdot x)} q u_2 \right|^2 dxdt \right) \\ &\le \varepsilon^{-1} C \lambda^{-1} \left( \int_Q |q(1+w_2)|^2 \, dxdt \right), \end{aligned} \tag{4.16}$$

with $C > 0$ a constant independent of $\lambda$. On the other hand, in a similar way to Lemma 4.2, an application of the Hölder inequality and the Sobolev embedding theorem yields

$$\begin{aligned} \int_Q |q(1+w_2)|^2 \, dxdt &\le C \|q\|^2_{L^p(Q)} \|(1+w_2)\|^2_{H^{\frac{n+1}{p}}(Q)} \\ &\le C \|q\|^2_{L^p(Q)} (1 + \|w_2\|_{H^1(Q)})^2. \end{aligned}$$

Combining this with (4.3) and (4.16), we obtain

$$\left| \int_Q q u_2 u_1 dxdt \right| \le C \lambda^{-1/2}.$$

It follows

$$\lim_{\lambda\to+\infty}\int_Q qu_2u_1dxdt=0. \tag{4.17}$$

Moreover, (4.1)-(4.3) imply

$$\int_Q qu_2u_1dxdt=\int_{\mathbb{R}^{1+n}}q(t,x)e^{-i\xi\cdot(t,x)}dxdt+\int_Q W_\lambda(t,x)dxdt,$$

with

$$\int_Q |W_\lambda(t,x)|dxdt\leq C\lambda^{-1}.$$

From this estimate and (4.17) we deduce that, for all $\omega\in\{y\in\mathbb{S}^{n-1}:|y-\omega_0|\leq\varepsilon\}$ and all $\xi\in(1,-\omega)^\perp:=\{\zeta\in\mathbb{R}^{1+n}:\ \zeta\cdot(1,-\omega)=0\}$, we have $\mathcal{F}(q)(\xi)=0$. Combining this with the fact that $\mathcal{F}(q)$ is a complex valued real-analytic function, we deduce that $q=0$. Therefore, we have $q_1=q_2$ and the proof of Theorem 1.2 is completed.

5. **Proof of Theorem 1.3.** Let us first remark that, in contrast to Theorem 1.1, in Theorem 1.2 we do not restrict the data to solutions of (1.1) satisfying $u_{|t=0}=0$. In this section we will show Theorem 1.3 by combining the restriction on the bottom $t=0$, the top $t=T$ of $Q$ stated in Theorem 1.1 with the restriction on the lateral boundary $\Sigma$ stated in Theorem 1.2. From now on, we fix $q_1,q_2\in L^\infty(0,T;L^p(\Omega))$, $p>n$, and we will show that condition (1.6) implies $q_1=q_2$. For this purpose we still consider exponentially growing and decaying GO solutions close to those of the previous subsection, but this time we need to take into account the constraint $u_2(0,x)=0$ required in Theorem 1.3. For this purpose, we will consider a different construction compared to the one of the previous section which will follow from a Carleman estimate in negative order Sobolev space only with respect to the space variable.

5.1. **Carleman estimate in negative Sobolev space for Theorem 1.3.** In this subsection we will derive a Carleman estimate in negative order Sobolev space which will be one of the main tools for the construction of exponentially growing solutions $u_2$ of (3.1) taking the form

$$u_2(t,x)=e^{\lambda(t+x\cdot\omega)}(1+w_2(t,x))-e^{\lambda(-t+x\cdot\omega)}, \tag{5.1}$$

with the restriction $\tau_{0,2}u_2=0$ (recall that for $v\in\mathcal{C}^\infty(\overline{Q})$, $\tau_{0,2}v=v_{|t=0}$). In a similar way to the previous section, for all $m\in\mathbb{R}$, we introduce the space $H^m_\lambda(\mathbb{R}^n)$ given by

$$H^m_\lambda(\mathbb{R}^n):=\{u\in\mathcal{S}'(\mathbb{R}^n):\ (|\xi|^2+\lambda^2)^{\frac{m}{2}}\hat{u}\in L^2(\mathbb{R}^n)\},$$

with the norm

$$\|u\|^2_{H^m_\lambda(\mathbb{R}^n)}:=\int_{\mathbb{R}^n}(|\xi|^2+\lambda^2)^m|\hat{u}(\xi)|^2d\xi.$$

In order to construct solutions $u_2$ of the form (5.1) and satisfying $\tau_{0,2}u_2=0$ we consider the following.

**Theorem 5.1.** *There exists $\lambda_2'>0$ such that for $\lambda>\lambda_2'$ and $v\in\mathcal{C}^2([0,T];\mathcal{C}^\infty_0(\Omega))$ satisfying*

$$v(T,x)=\partial_tv(T,x)=v(0,x)=0,\quad x\in\mathbb{R}^n, \tag{5.2}$$

*we have*

$$\|v\|_{L^2((0,T)\times\mathbb{R}^n)} \le C\,\|P_{\omega,-\lambda}v + q_2 v\|_{L^2(0,T;H^{-1}_\lambda(\mathbb{R}^n))}, \tag{5.3}$$

*with* $C>0$ *independent of* $v$ *and* $\lambda$.

In order to prove this theorem, we start by recalling the following intermediate tools. For $m\in\mathbb{R}$ and $\xi\in\mathbb{R}^n$, we fix

$$\langle \xi,\lambda\rangle = (|\xi|^2+\lambda^2)^{\frac12}$$

and we denote by $\langle D_x,\lambda\rangle^m u$ the operator

$$\langle D_x,\lambda\rangle^m u = \mathcal{F}^{-1}(\langle \xi,\lambda\rangle^m \mathcal{F}u).$$

We recall also the class of symbols of order $m\in\mathbb{R}$ given by

$$S^m_\lambda := \{c_\lambda\in\mathcal{C}^\infty(\mathbb{R}^n\times\mathbb{R}^n):\ |\partial_x^\alpha\partial_\xi^\beta c_\lambda(x,\xi)|\le C_{\alpha,\beta}\langle\xi,\lambda\rangle^{m-|\beta|},\ \alpha,\beta\in\mathbb{N}^n\}.$$

In view [27, Theorem 18.1.6], for any $m\in\mathbb{R}$ and $c_\lambda\in S^m_\lambda$, we fix $c_\lambda(x,D_x)$, with $D_x=-i\nabla_x$, defined by

$$c_\lambda(x,D_x)u(x) := (2\pi)^{-\frac n2}\int_{\mathbb{R}^n} c_\lambda(x,\xi)\hat u(\xi)e^{ix\cdot\xi}d\xi.$$

For any $m\in\mathbb{R}$, we introduce $OpS^m_\lambda := \{c_\lambda(x,D_x):\ c_\lambda\in S^m_\lambda\}$ and for $m=-\infty$ we set

$$OpS^{-\infty}_\lambda := \bigcap_{m\in\mathbb{R}} OpS^m_\lambda.$$

Now let us consider the following intermediate result.

**Lemma 5.2.** *There exists* $\lambda_2''>0$ *such that for* $\lambda>\lambda_2''$ *and for all* $v\in\mathcal{C}^2([0,T];\mathcal{C}^\infty_0(\Omega))$ *satisfying* (5.2)*, we have*

$$\|v\|_{L^2(0,T;H^1_\lambda(\mathbb{R}^n))} \le C\,\|P_{\omega,-\lambda}v\|_{L^2((0,T)\times\mathbb{R}^n)}, \tag{5.4}$$

*with* $C>0$ *independent of* $v$ *and* $\lambda$.

*Proof.* Consider $w(t,x)=v(T-t,x)$ and note that according to (5.2), we have $w\in\mathcal{C}^2([0,T];\mathcal{C}^\infty_0(\Omega))$ and

$$w(0,x)=\partial_t w(0,x)=w(T,x)=0,\quad x\in\mathbb{R}^n.$$

Therefore, in a similar way to the proof of [36, Lemma 4.1], one can check that

$$\int_Q |P_{-\omega,\lambda}w|^2dxdt \ge \int_Q |\Box w|^2dxdt + c\lambda^2\int_Q |w|^2dxdt, \tag{5.5}$$

with $c>0$ independent of $w$ and $\lambda$. Now, recalling that $w$ solves

$$\begin{cases} \partial_t^2 w-\Delta_x w=\Box w, & (t,x)\in Q,\\ w(0,x)=0,\quad \partial_t w(0,x)=0, & x\in\Omega\\ w(t,x)=0 & (t,x)\in\Sigma,\end{cases}$$

we deduce that

$$\int_Q |\nabla_x w|^2dxdt \le C\int_Q |\Box w|^2dxdt,$$

where $C$ depends only on $T$ and $\Omega$. Combining this with (5.5), we obtain

$$\|w\|^2_{L^2(0,T;H^1_\lambda(\mathbb{R}^n))}\le C\int_Q |P_{-\omega,\lambda}w|^2dxdt.$$

Using the fact that $P_{-\omega,\lambda}w(t,x)=P_{\omega,-\lambda}v(T-t,x)$, we deduce (5.4). $\square$

Armed with this Carleman estimate, we are now in position of completing the proof of Theorem 5.1.

*Proof of Theorem 5.1.* Let $v \in \mathcal{C}^2([0,T];\mathcal{C}_0^\infty(\Omega))$ satisfy (5.2), consider $\Omega_j$, $j=1,2$, two bounded open smooth domains of $\mathbb{R}^n$ such that $\overline{\Omega} \subset \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$ and let $\psi \in \mathcal{C}_0^\infty(\Omega_2)$ be such that $\psi = 1$ on a neighborhood of $\overline{\Omega_1}$. We consider $w \in \mathcal{C}^2([0,T];\mathcal{C}_0^\infty(\overline{\Omega}))$ given by

$$w(t,\cdot) = \psi \left\langle D_x, \lambda\right\rangle^{-1} v(t,\cdot)$$

and we remark that $w$ satisfies

$$w(T,x) = \partial_t w(T,x) = w(0,x) = 0, \quad x \in \mathbb{R}^n. \tag{5.6}$$

Now let us consider the quantity $\left\langle D_x, \lambda\right\rangle^{-1} P_{\omega,-\lambda} \left\langle D_x, \lambda\right\rangle w$. Note first that

$$\left\| P_{\omega,-\lambda} \left\langle D_x, \lambda\right\rangle w\right\|_{L^2(0,T;H_\lambda^{-1}(\mathbb{R}^n))} = \left\| \left\langle D_x, \lambda\right\rangle^{-1} P_{\omega,-\lambda} \left\langle D_x, \lambda\right\rangle w\right\|_{L^2((0,T)\times\mathbb{R}^n)}.$$

Moreover, it is clear that

$$\left\langle D_x, \lambda\right\rangle^{-1} P_{\omega,-\lambda} \left\langle D_x, \lambda\right\rangle = P_{\omega,-\lambda}.$$

Therefore, we have

$$\left\| P_{\omega,-\lambda} \left\langle D_x, \lambda\right\rangle w\right\|_{L^2(0,T;H_\lambda^{-1}(\mathbb{R}^n))} = \left\| P_{\omega,-\lambda} w\right\|_{L^2((0,T)\times\mathbb{R}^n)}$$

and, since $w$ satisfies (5.6), combining this with (5.4) we deduce that

$$\|w\|_{L^2(0,T;H_\lambda^1(\mathbb{R}^n))} \leq C \left\| P_{\omega,-\lambda} \left\langle D_x, \lambda\right\rangle w\right\|_{L^2(0,T;H_\lambda^{-1}(\mathbb{R}^n))}. \tag{5.7}$$

On the other hand, fixing $\psi_1 \in \mathcal{C}_0^\infty(\Omega_1)$ satisfying $\psi_1 = 1$ on $\overline{\Omega}$, we get

$$w(t,\cdot) = \left\langle D_x, \lambda\right\rangle^{-1} v(t,\cdot) + (\psi - 1)\left\langle D_x, \lambda\right\rangle^{-1} \psi_1 v(t,\cdot)$$

and, combining this with (5.7), we deduce that

$$\begin{aligned}
&\|v\|_{L^2((0,T)\times\mathbb{R}^n)} \\
&= \left\| \left\langle D_x, \lambda\right\rangle^{-1} v\right\|_{L^2(0,T;H_\lambda^1(\mathbb{R}^n))} \\
&\leq \|w\|_{L^2(0,T;H_\lambda^1(\mathbb{R}^n))} + \left\| (\psi-1)\left\langle D_x, \lambda\right\rangle^{-1} \psi_1 v\right\|_{L^2(0,T;H_\lambda^1(\mathbb{R}^n))} \\
&\leq C\left\| P_{\omega,-\lambda} \left\langle D_x, \lambda\right\rangle w\right\|_{L^2(0,T;H_\lambda^{-1}(\mathbb{R}^n))} + \left\| (\psi-1)\left\langle D_x, \lambda\right\rangle^{-1} \psi_1 v\right\|_{L^2(0,T;H_\lambda^1(\mathbb{R}^n))} \\
&\leq C\left\| P_{\omega,-\lambda} v\right\|_{L^2(0,T;H_\lambda^{-1}(\mathbb{R}^n))} \\
&\quad + C\left\| P_{\omega,-\lambda} \left\langle D_x, \lambda\right\rangle (\psi-1)\left\langle D_x, \lambda\right\rangle^{-1} \psi_1 v\right\|_{L^2(0,T;H_\lambda^{-1}(\mathbb{R}^n))} \\
&\quad + \left\| (\psi-1)\left\langle D_x, \lambda\right\rangle^{-1} \psi_1 v\right\|_{L^2(0,T;H_\lambda^1(\mathbb{R}^n))}.
\end{aligned} \tag{5.8}$$

Moreover, since $(\psi - 1) = 0$ on neighborhood of $\mathrm{supp}(\psi_1)$, in view of [27, Theorem 18.1.8], we have $(\psi-1)\left\langle D_x, \lambda\right\rangle^{-1} \psi_1 \in OpS_\lambda^{-\infty}$. In the same way, [27, Theorem 18.1.8] implies that

$$P_{\omega,-\lambda} \left\langle D_x, \lambda\right\rangle (\psi-1)\left\langle D_x, \lambda\right\rangle^{-1} \psi_1 \in OpS_\lambda^{-\infty}$$

and we deduce that

$$C\left\|P_{\omega,-\lambda}\left\langle D_x,\lambda\right\rangle(\psi-1)\left\langle D_x,\lambda\right\rangle^{-1}\psi_1 v\right\|_{L^2(0,T;H_\lambda^{-1}(\mathbb{R}^n))}$$
$$+\left\|(\psi-1)\left\langle D_x,\lambda\right\rangle^{-1}\psi_1 v\right\|_{L^2(0,T;H_\lambda^{-1}(\mathbb{R}^n))}$$
$$\leq \frac{\|v\|_{L^2((0,T)\times\mathbb{R}^n)}}{\lambda^2}.$$

Combining this estimate with (5.8) and choosing $\lambda$ sufficiently large, we find (5.3) for $q_2=0$. Then, we deduce (5.3) for $q_2\neq 0$ by applying arguments similar to Lemma 4.2. □

Applying the Carleman estimate (5.3), we can now build solutions $u_2$ of the form (5.1) and satisfying $\tau_{0,2}u_2=0$ in order to complete the proof of Theorem 1.3.

5.2. **Completion of the proof of Theorem 1.3.** We start by proving existence of a solution $u_2\in L^2(0,T;H^1(\Omega))$ of the form (5.1) with the term $w_2\in L^2(0,T;H^1(\Omega))\cap e^{-\lambda(t+x\cdot\omega)}H_{\Box,*}(Q)$, satisfying

$$\|w_2\|_{L^2(0,T;H^1(\Omega))}+\lambda\|w_2\|_{L^2(Q)}\leq C, \tag{5.9}$$

$$\tau_{0,2}w_2=0. \tag{5.10}$$

This result can be stated in the following way.

**Proposition 6.** *There exists $\lambda_2>\lambda_1$ such that for $\lambda>\lambda_2$ we can find a solution $u_2\in L^2(0,T;H^1(\Omega))$ of $\Box u_2+q_2u_2=0$ in $Q$ taking the form (5.1) with $w_2\in L^2(0,T;H^1(\Omega))\cap e^{-\lambda(t+x\cdot\omega)}H_{\Box,*}(Q)$ satisfying (5.9)-(5.10).*

*Proof.* We need to consider $w_2\in L^2(0,T;H^1(\Omega))$ a solution of

$$P_{\omega,\lambda}w_2+q_2w_2=-e^{-\lambda(t+x\cdot\omega)}(\Box+q_2)(e^{\lambda(t+x\cdot\omega)}-e^{\lambda(-t+x\cdot\omega)})=-q_2(1-e^{-2\lambda t}), \tag{5.11}$$

satisfying (5.9)-(5.10). Note that here, we use (5.11) and the fact that $P_{\omega,\lambda}w_2=e^{-\lambda(t+x\cdot\omega)}\Box e^{\lambda(t+x\cdot\omega)}w_2$ in order to prove that $w_2\in e^{-\lambda(t+x\cdot\omega)}H_{\Box,*}(Q)$ and we define $\tau_{0,2}w_2$ by $\tau_{0,2}w_2=e^{-\lambda x\cdot\omega}\tau_{0,2}e^{\lambda(t+x\cdot\omega)}w_2$. We will construct such a function $w_2$ by applying estimate (5.3). From now on, we fix $\lambda_2=\lambda_2''$. Applying the Carleman estimate (5.3), we consider the linear form $\mathcal{M}$ on

$$\mathcal{I}=\{P_{\omega,-\lambda}v+q_2v:\ v\in\mathcal{C}^2([0,T];\mathcal{C}_0^\infty(\Omega))\ \text{ satisfying (5.2)}\},$$

by

$$\mathcal{M}(P_{\omega,-\lambda}v+q_2v)=-\int_Q vq_2(1-e^{-2\lambda t})dxdt,\quad v\in\mathcal{I}.$$

In view of (5.3), we have

$$|\mathcal{M}(P_{\omega,-\lambda}v+q_1v)|\leq C\|q_2\|_{L^2(Q)}\|P_{\omega,-\lambda}v+q_1v\|_{L^2(0,T;H_\lambda^{-1}(\mathbb{R}^n)},\quad v\in\mathcal{I},$$

with $C>0$ independent of $\lambda$ and $v$. Applying the Hahn Banach theorem we can extend $\mathcal{M}$ to a continuous linear form on $L^2(0,T;H_\lambda^{-1}(\mathbb{R}^n))$ still denoted by $\mathcal{M}$ and satisfying $\|\mathcal{M}\|\leq C\|q_2\|_{L^2(Q)}$. Thus, we can find $w_2\in L^2(0,T;H_\lambda^1(\mathbb{R}^n))$ such that

$$\langle g,w_2\rangle_{L^2(0,T;H_\lambda^{-1}(\mathbb{R}^n)),L^2(0,T;H_\lambda^1(\mathbb{R}^n))}=\mathcal{M}(g),\quad g\in L^2(0,T;H_\lambda^{-1}(\mathbb{R}^n).$$

Choosing $g=P_{\omega,-\lambda}v+q_2v$ with $v\in\mathcal{C}_0^\infty(Q)$, we deduce that $w_2$ is a solution of $P_{\omega,\lambda}w_2+q_2w_1=-q_2(1-e^{-2\lambda t})$ in $Q$. In addition, taking $g=P_{\omega,-\lambda}v+q_1v$, with $v\in\mathcal{I}$ and $\partial_t v_{|t=0}$ arbitrary, proves that (5.10) is fulfilled. Finally, using the fact that $\|w_2\|_{L^2(0,T;H^1_\lambda(\mathbb{R}^n))} \leq \|\mathcal{M}\| \leq C\|q\|_{L^2(Q)}$ proves that $w_2$ fulfills (5.9) which completes the proof of the proposition. $\square$

Using this proposition, we are now in position to complete the proof of Theorem 1.3.

*Proof of Theorem 1.3.* Let us remark that since Lemma 4.2 and Theorem 4.3 are valid when $q \in L^\infty(0,T;L^p(\Omega))$ one can easily extend Proposition 4 to the case $q_1 \in L^\infty(0,T;L^p(\Omega))$. Therefore, in the context of this section, Proposition 4 holds true. Combining Proposition 4 with Proposition 6, we deduce existence of a solution $u_1 \in H^1(Q)$ of $\Box u_1 + q_1u_1 = 0$ in $Q$ taking the form (4.1), with $w_1 \in H^1(Q)$ satisfying (4.3) for $j = 1$, as well as the existence of a solution $u_2 \in L^2(0,T;H^1(\Omega))$ of $\Box u_2 + q_2u_2 = 0$ in $Q$, $\tau_{0,2}u_2 = 0$, taking the form (5.1) with the term $w_2 \in L^2(0,T;H^1(\Omega))$ satisfying (5.9). Repeating the arguments of the end of the proof of Theorem 1.2 (see Subsection 4.4), we can deduce the following orthogonality identity

$$\lim_{\lambda\to+\infty}\int_Q qu_1u_2dxdt = 0. \tag{5.12}$$

Moreover, one can check that

$$\int_Q qu_1u_2dxdt = \int_{\mathbb{R}^{1+n}} q(t,x)e^{-i\xi\cdot(t,x)}dxdt + \int_Q Y_\lambda(t,x)dxdt,$$

with

$$Y_\lambda(t,x) = q[e^{-i(t,x)\cdot\xi}w_2 + w_1 + w_1w_2 - e^{-2\lambda t}e^{-i(t,x)\cdot\xi} - e^{-2\lambda t}w_1].$$

Combining (4.3), (5.9) with the fact that

$$\int_Q |q(t,x)|\left|e^{-2\lambda t}e^{-i(t,x)\cdot\xi}\right|dxdt \leq \|q\|_{L^2(Q)}|\Omega|^{\frac{1}{2}}\left(\int_0^{+\infty}e^{-4\lambda t}dt\right)^{\frac{1}{2}} \leq C\lambda^{-\frac{1}{2}},$$

we deduce that

$$\lim_{\lambda\to+\infty}\int_Q Y_\lambda(t,x)dxdt = 0.$$

Combining this asymptotic property with (5.12), we can conclude in a similar way to Theorem 1.2 that $q_1 = q_2$. $\square$

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