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Inverse source problems in elastodynamics

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Abstract

We are concerned with time-dependent inverse source problems in elastodynamics. The source term is supposed to be the product of a spatial function and a temporal function with compact support. We present frequency-domain and time-domain approaches to show uniqueness in determining the spatial function from wave fields on a large sphere over a finite time interval. The stability estimate of the temporal function from the data of one receiver and the uniqueness result using partial boundary data are proved. Our arguments rely heavily on the use of the Fourier transform, which motivates inversion schemes that can be easily implemented. A Landweber iterative algorithm for recovering the spatial function and a non-iterative inversion scheme based on the uniqueness proof for recovering the temporal function are proposed. Numerical examples are demonstrated in both two and three dimensions.

Keywords: inverse source problems, Lamé system, uniqueness, Landweber iteration, Fourier transform

(Some figures may appear in colour only in the online journal)

1. Introduction

Consider the radiation of elastic (seismic) waves from a time-varying source term $F(x,t)$, $x \in \mathbb{R}^3$, embedded in an infinite and homogeneous elastic medium. The real-valued radiated field is governed by the inhomogeneous Lamé system:

$$\rho\partial_{tt}U(x,t) = \nabla\cdot\sigma(x,t) + F(x,t), \quad x = (x_1,x_2,x_3) \in \mathbb{R}^3,\ t > 0 \tag{1.1}$$

together with the initial conditions

$$U(x,0)=\partial_t U(x,0)=0,\quad x\in\mathbb{R}^3. \tag{1.2}$$

Here, $\rho>0$ denotes the density, $U=(u_1,u_2,u_3)^\top$ is the displacement vector, $\sigma=\sigma(U)$ is the stress tensor and F is the source term that causes the elastic vibration in $\mathbb{R}^3$. By Hooke's law, the stress tensor relates to the stiffness tensor $\tilde{C}=(C_{ijkl})_{i,j,k,l=1}^3$ via the identity $\sigma(U):=\tilde{C}:\nabla U$, where the action of $\tilde{C}$ on a matrix $A=(a_{ij})_{i,j=1}^3$ is defined as

$$\tilde{C}:A=(\tilde{C}:A)_{ij}=\sum_{k,l=1}^{3}C_{ijkl}\,a_{kl}.$$

In an isotropic and homogeneous elastic medium, the stiffness tensor is characterized by

$$C_{ijkl}(x)=\lambda\delta_{i,j}\delta_{k,l}+\mu(\delta_{i,k}\delta_{j,l}+\delta_{i,l}\delta_{j,k}), \tag{1.3}$$

where the Lamé constants satisfy $\mu>0, 3\lambda+2\mu>0$. Hence, the Lamé system (1.1) can be rewritten as

$$\rho\,\partial_{tt}\,U(x,t)=\mathcal{L}_{\lambda,\mu}U(x,t)+F(x,t),\quad (x,t)\in\mathbb{R}^3\times\mathbb{R}^+,$$
$$\mathcal{L}_{\lambda,\mu}U:=-\mu\nabla\times\nabla\times U+(\lambda+2\mu)\nabla\nabla\cdot U=\mu\Delta U+(\lambda+\mu)\nabla\nabla\cdot U.$$

Note that the above equation has a more complex form than the scalar wave equation, because it accounts for both longitudinal and transverse motions. Throughout this paper it is supposed that ρ,λ,μ are given as a prior data and that the dependence of the source term on time and space variables are separated, that is,

$$F(x,t)=f(x)\,g(t). \tag{1.4}$$

In other words, the source term is a product of the spatial function f and the temporal function g. The source term (1.4) can be regarded as an approximation of the elastic pulse and is commonly used in modeling vibration phenomena in seismology. In the standard model for teleseismic earthquake estimation, f is a vector multipole (partial differential operator acting on a spatial delta function) representing the complex force exerted by the earthquake and g is the seismic source–time function. In many applications, f can be expressed by the exponentially decaying function

$$f(x)=\exp(-a|x-x_0|^2)$$

where $x_0\in\mathbb{R}^3$ denotes the epicentral location and $a>0$ is some factor. The temporal function is usually given by the Ricker wavelet

$$g(t)=[1-2(\pi f_{\mathrm{p}}(t-t_0))^2]\mathrm{e}^{-(\pi f_{\mathrm{p}}(t-t_0))^2}$$

with the the center frequency f_{p} and the delay time t_0. The central issue of teleseismic inversion is to recover the source terms f and g. We refer to [2, 28] for an introduction to seismology and earthquake prediction. On the other hand, in biomedical imaging, source terms can be temporally localized, taking the form of $d\delta(t)/\mathrm{d}tf(x)$. This type of source has been considered in numerous applications in biomedical imaging; see [3, 4] and the references therein. In this paper, we suppose that f is compactly supported in the space region $B_{R_0}:=\{x:|x|<R_0\}$ and the source radiates only over a finite time period $[0,T_0]$ for some $T_0>0$. This implies that $g(t)=0$ for $t\geqslant T_0$ and $t\leqslant 0$.

Inverse hyperbolic problems have attracted considerable attention over the last few years. Most of the existing works treated scalar acoustic wave equations. We refer to Bukhgaim and

Klibanov [12], Klibanov [24], Yamamoto [30, 31], Khaĭdarov [23], Isakov [19, 20], Imanuvilov and Yamamoto [17, 18], Choulli and Yamamoto [14], Kian *et al* [22] and the recent work by Jiang *et al* [21] for uniqueness and stability of inverse source problems using Carleman estimates, and refer also to Fujishiro and Kian [16] for results of recovery of a time-dependent source. In addition, it is worth mentioning the work of Rakesh and Symes [26], dealing with coefficient determination problems based on the construction of appropriate geometric optic solutions. There are also rich references on inverse problems arising in the context of linear elasticity. Many investigations are devoted to mathematical and numerical techniques for the identification of elastic coefficients and buried objects of a geometrical nature (such as cracks, cavities and inclusions) in the time-harmonic regime; see e.g. the review article [11] by Bonnet and Constantinescu, the monograph [5] by Ammari *et al* and references therein. Due to our limited knowledge, we have found only a few mathematical works on inverse source problems for the time-dependent Lamé system. In [4], a time-reversal imaging algorithm based on a weighted Helmholtz decomposition was proposed for reconstructing f in a homogeneous isotropic medium, where the temporal function takes the special form $g(t) = \mathrm{d}\delta(t)/\mathrm{d}t$.

This paper concerns the uniqueness and numerical reconstructions of f or g from radiated elastodynamic fields over a finite time interval. This kind of inverse problem has many significant applications in biomedical engineering (see, e.g. [5]) and geophysics (see, e.g. [2, 28]). The uniqueness issue is important in inverse scattering theory, while it provides insight into whether the measurement data are sufficient for recovering the unknowns and ensuring the uniqueness of global minimizers in iterative schemes. Note that one cannot expect to uniquely recover source terms of the form $F(x, t)$ in general. It seems that the source given in (1.4) is the best we can expect; see remark 4.5. Being different from previously mentioned existing works, our uniqueness proofs rely heavily on the use of the Fourier transform, which motivates novel inversion schemes that can be easily implemented. Our arguments carry over the scalar wave equations without any additional difficulties. We believe that the Fourier-transform-based approach explored in this paper will also lead to stability estimates of our inverse problems, which deserves to be further investigated in future. We shall address the following inverse issues.

(i) Uniqueness in recovering f from emitted waves on a closed surface surrounding the source. We present frequency-domain and time-domain approaches for recovering the spatial function. The frequency-domain approach is of independent interest, and it reduces the time-dependent inverse problem to an inverse scattering problem in the Fourier domain with multi-frequency data. Our arguments are motivated by recent studies on inverse source problems for the time-harmonic Helmholtz equation with multi-frequency data (see e.g. [3, 8–10, 15]). The time-domain approach is inspired by the Lipschitz stability estimate of source terms for the scalar acoustic wave equation with additional *a priori* assumptions; see e.g. [17, 18, 30, 31]. A Landweber iterative algorithm is proposed for recovering f in 2D and numerical tests are presented to show the validity and effectiveness of the proposed inversion scheme; see section 5.1.

(ii) Stability estimate of g from measured data of one receiver. Under the assumption that the spatial function does not vanish at the position of the receiver, we estimate a vector-valued temporal function in section 4. The stability estimate relies on an explicit expression of the solution in terms of f and g. Such an idea seems well known in the case of scalar acoustic wave equations, but to the best of our knowledge not available for time-dependent Lamé systems.

(iii) Unique determination of g from partial boundary measurement data. If the spatial function f is known not to be a non-radiating source (see definition 4.3), we prove that the temporal function g can be uniquely determined by the time-domain data on any sub-boundary of a large sphere; see theorem 4.4. The uniqueness proof is based on the Fourier transform and yields a non-iterative inversion scheme in section 5.2. Numerical examples are demonstrated to verify our theory.

The remainder of this paper is organized as follows. In section 1, preliminary studies of the time-dependent Lamé system are carried out. Unique determination of spatial and temporal functions will be presented in sections 3 and 4, respectively. In particular, as a bi-product of the Fourier-domain approach presented in section 3.1, we show uniqueness in recovering a source term of the time-dependent Schrödinger equation. Finally, numerical tests are reported in section 5 and proofs of several lemmas are postponed to the appendix.

2. Preliminaries

For all $r > 0$, we denote by B_r the open ball of $\mathbb{R}^3$ defined by $B_r := \{x \in \mathbb{R}^3 : \ |x| < r\}$. By Helmholtz decomposition, the function $f \in (L^2(\mathbb{R}^3))^3$ supported in B_{R_0} admits a unique decomposition of the form (see lemma A.1 in the appendix)

$$f(x) = \nabla f_{\rm p}(x) + \nabla \times f_{\rm s}(x), \quad \nabla \cdot f_{\rm s} \equiv 0, \tag{2.1}$$

where $f_{\rm p} \in H^1(B_{R_0})$, $f_{\rm s} \in H_{\rm curl}(B_{R_0}) := \{u : u \in (L^2(B_{R_0}))^3, {\rm curl}\, u \in (L^2(B_{R_0}))^3\}$ also have compact support in B_{R_0}. We choose also $g \in \mathcal{C}(\mathbb{R})$ supported in $[0,T_0]$. By the completeness theorem (see [1, theorem 3.3] or [2, chapter 4.1.1]), there exist vector-valued functions $U_{\rm p}(x,t)$ and $U_{\rm s}(x,t)$ such that $U(x,t)$ can be expressed as

$$U = U_{\rm p} + U_{\rm s}, \quad U_{\rm p} = \nabla u_{\rm p}, \quad U_{\rm s} = \nabla \times u_{\rm s}, \quad \nabla \cdot u_{\rm s} = 0. \tag{2.2}$$

Moreover, the scalar function $u_{\rm p}$ and the vector function $u_{\rm s}$ satisfy the inhomogeneous wave equations

$$\frac{1}{c_\alpha^2}\partial_{tt} u_\alpha - \Delta u_\alpha = \frac{1}{\gamma_\alpha} f_\alpha(x) g(t) \quad \text{in} \quad \mathbb{R}^3 \times (0, +\infty), \qquad \alpha = p, s, \tag{2.3}$$

together with the initial conditions

$$u_\alpha|_{t=0} = \partial_t u_\alpha|_{t=0} = 0 \quad \text{in} \quad \mathbb{R}^3.$$

Note that

$$c_{\rm p} := \sqrt{(\lambda + 2\mu)/\rho}, \quad c_{\rm s} := \sqrt{\mu/\rho}, \quad \gamma_{\rm p} := \lambda + 2\mu,\ \gamma_{\rm s} := \mu, \tag{2.4}$$

and that $\lambda + 2\mu > 0$ since $\mu > 0$, $3\lambda + 2\mu > 0$. This implies that $U_{\rm p}$ and $U_{\rm s}$ propagate at different wave speeds, which will be referred as compressional waves (or simply P-waves) and shear waves (or simply S-waves), respectively.

It is well-known that the electrodynamic Green's tensor $G(x,t) = (G_{ij}(x,t))_{i,j=1}^3 \in \mathbb{C}^{3\times 3}$, which satisfies

$$\rho\partial_{tt} G(x,t) e_j - \nabla \cdot \sigma(x,t) = -\delta(x)\delta(t) e_j, \quad j = 1,2,3,$$
$$G(x,0) = \partial_t G(x,0) = 0, \quad x \neq y,$$

is given by (see e.g. [13])

$$\begin{aligned} G_{i,j}(x,t) = &\frac{1}{4\pi\rho|x|^3}\left\{t^2\left(\frac{x_j x_k}{|x|^2}\delta(t - |x|/c_{\rm p}) + (\delta_{jk} - \frac{x_j x_k}{|x|^2})\delta(t - |x|/c_{\rm s})\right)\right\} \\ &+ \frac{1}{4\pi\rho|x|^3}\left\{t\left(3\frac{x_j x_k}{|x|^2} - \delta_{jk}\right)(\Theta(t - |x|/c_{\rm p}) - \Theta(t - |x|/c_{\rm s}))\right\}. \end{aligned} \tag{2.5}$$

Here, δ_{ij} is the Kronecker symbol, δ is the Dirac distribution, Θ is the Heaviside function and e_j ($j = 1, 2, 3$) are the unit vectors in $\mathbb{R}^3$. Physically, the Green's tensor $G(x, t)$ is the response of the Lamé system to a point body force at the origin that emits an impulse at time $t = 0$. Using the above Green's tensor, the solution U to the inhomogeneous Lamé system (1.1) can be represented as

$$U(x,t) = \int_0^\infty \int_{\mathbb{R}^3} G(x-y, t-s) f(y) g(s) \, \mathrm{d}x\mathrm{d}s, \quad x \in \mathbb{R}^3, \ t \in \mathbb{R}. \tag{2.6}$$

Note that, since $\mathrm{supp}(g) \subset [0, +\infty)$, for every $t \in (-\infty, 0]$, and $x \in \mathbb{R}^3$, we have $U(x, t) = 0$. Throughout the paper we define

$$T_\mathrm{p} := T_0 + (R + R_0)/c_\mathrm{p}, \quad T_\mathrm{s} := T_0 + (R + R_0)/c_\mathrm{s}, \tag{2.7}$$

for some $R > R_0$. Obviously, it holds that $T_\mathrm{s} > T_\mathrm{p}$, since $c_\mathrm{p} > c_\mathrm{s}$ by (2.4). The following lemma states that the wave fields over B_R must vanish after a finite time that depends on R and the support of f and g.

Lemma 2.1. *We have* $U(x, t) \equiv 0$ *for all* $x \in B_R$ *and* $t > T_\mathrm{s}$.

Proof. For $x = (x_1, x_2, x_3)^\top$, $y = (y_1, y_2, y_3)^\top \in \mathbb{R}^3$, write $x \otimes y = xy^\top \in \mathbb{R}^{3\times 3}$ and $\hat{x} = x/|x|$ for simplicity. We introduce

$$V(x,t) = \Theta(t - |x|/c_\mathrm{p}) - \Theta(t - |x|/c_\mathrm{s}).$$

Combining (2.6) and (2.5), we have

$$\begin{aligned}
U(x,t) = & \int_0^\infty \int_{\mathbb{R}^3} \frac{(t-s)^2 (\hat{x}-\hat{y}) \otimes (\hat{x}-\hat{y})}{4\pi\rho |x-y|^3} \, \delta(t - s - \frac{|x-y|}{c_\mathrm{p}}) f(y) g(s) \, \mathrm{d}y\mathrm{d}s \\
& + \int_0^\infty \int_{\mathbb{R}^3} \frac{(t-s)^2 [\mathbf{I} - (\hat{x}-\hat{y}) \otimes (\hat{x}-\hat{y})]}{4\pi\rho |x-y|^3} \delta(t - s - \frac{|x-y|}{c_\mathrm{s}}) f(y) g(s) \, \mathrm{d}y\mathrm{d}s \\
& + \int_0^\infty \int_{\mathbb{R}^3} \frac{(t-s)[3(\hat{x}-\hat{y}) \otimes (\hat{x}-\hat{y}) - \mathbf{I}]}{4\pi\rho |x-y|^3} V(x-y, t-s) f(y) g(s) \, \mathrm{d}y\mathrm{d}s \\
= & \int_{|y-x| \leqslant c_\mathrm{p}(t+T_0)} \frac{(\hat{x}-\hat{y}) \otimes (\hat{x}-\hat{y})}{4\pi(\lambda+2\mu)|x-y|} \, g(t - \frac{|x-y|}{c_\mathrm{p}}) f(y) \, \mathrm{d}y\mathrm{d}s \\
& + \int_{|y-x| < c_\mathrm{s}(t+T_0)} \frac{\mathbf{I} - (\hat{x}-\hat{y}) \otimes (\hat{x}-\hat{y})}{4\pi\mu |x-y|} g(t - \frac{|x-y|}{c_\mathrm{s}}) f(y) \, \mathrm{d}y\mathrm{d}s \\
& + \int_0^{T_0} \int_{B_R} \frac{(t-s)[3(\hat{x}-\hat{y}) \otimes (\hat{x}-\hat{y}) - \mathbf{I}]}{4\pi\rho |x-y|^3} \, V(x-y, t-s) f(y) g(s) \, \mathrm{d}y\mathrm{d}s,
\end{aligned}$$

where $\mathbf{I}$ denotes the 3-by-3 unit matrix. For $t > T_0 + (R + R_0)/c_\mathrm{s}$, one can readily observe that

$$g(t - \frac{|x-y|}{c_\mathrm{s}}) = g(t - \frac{|x-y|}{c_\mathrm{p}}) = 0, \quad V(x-y, t-s) = 0$$

uniformly in all $x \in B_R, y \in B_{R_0}$ and $s \in (0, T_0)$, which implies the desired result. □

Denote by $\hat{f}$ the Fourier transform of f with respect to $t \in \mathbb{R}$, that is,

$$\hat{f}(\omega) = \mathcal{F}_{t\to\omega}[f] := \int_{\mathbb{R}} f(t)\exp(\mathrm{i}\omega t)\,\mathrm{d}t, \quad \omega \in \mathbb{R}.$$

Denote by $\hat{G} = \hat{G}(x,\omega)$ the Fourier transform of $G(x,t)$ with respect to t, and define the compressional and shear waves numbers k_p and k_s in the Fourier domain as

$$k_\mathrm{p} := \omega\sqrt{\rho/(\lambda+2\mu)}, \quad k_\mathrm{s} := \omega\sqrt{\rho/\mu}.$$

Then we find that

$$\mu\Delta\hat{G}(\cdot,\omega)e_j + (\lambda+\mu)\nabla(\nabla\cdot\hat{G}(\cdot,\omega)e_j) + \omega^2\rho\hat{G}(\cdot,\omega)e_j = -\delta(\cdot)e_j, \quad j=1,2,3$$

and

$$\hat{G}(x-y,\omega) = \frac{1}{\mu}\Phi_{k_\mathrm{s}}(x,y)\mathbf{I} + \frac{1}{\rho\omega^2}\,\mathrm{grad}_x\,\mathrm{grad}_x^\top\,\left[\Phi_{k_\mathrm{s}}(x,y) - \Phi_{k_\mathrm{p}}(x,y)\right], \quad x\neq y. \tag{2.8}$$

Here $\Phi_k(x,y) = \mathrm{e}^{\mathrm{i}k|x-y|}/(4\pi|x-y|)$ $(k = k_\mathrm{p}, k_\mathrm{s})$ is the fundamental solution to the Helmholtz equation $(\Delta + k^2)u = 0$ in $\mathbb{R}^3$. By lemma 2.1, we may take the Fourier transform of $U(x,t)$ with respect to t. Consequently, it holds in the frequency domain that

$$\mu\Delta\hat{U}(x,\omega) + (\lambda+\mu)\nabla(\nabla\cdot\hat{U}(x,\omega)) + \omega^2\rho\hat{U}(x,\omega) = -f(x)\hat{g}(\omega), \quad \omega\in\mathbb{R}. \tag{2.9}$$

Corresponding to the representation of $U(x,t)$ in the time domain, we have in the Fourier domain that

$$\hat{U}(x,\omega) = \int_{\mathbb{R}^3} \mathcal{F}[G(x-y,\cdot)*g(\cdot)]\,f(y)\mathrm{d}y = \hat{g}(\omega)\int_{\mathbb{R}^3}\hat{G}(x-y,\omega)f(y)\,\mathrm{d}y, \quad x\in\mathbb{R}^3, \quad \omega\in\mathbb{R}^+. \tag{2.10}$$

Here $*$ denotes the convolution product with respect to the time variable. Note that $\hat{U}(x,-\omega) = \overline{\hat{U}(x,\omega)}$, since $U(x,t)$ is real valued.

3. Unique determination of spatial functions

In this section we are interested in the inverse source problem of recovering f from the radiated wave field $\{U(x,t) : |x| = R, t > T\}$ for some $R > R_0$ and $T > T_0$ under the *a priori* assumption that g is given. We suppose that $f \in (L^2(\mathbb{R}^3))^3$, $\mathrm{supp}(f)\subset B_{R_0}$, $g\in\mathcal{C}_0([0,T_0])$. Since f and g have compact support, the initial boundary value problems (1.1), (1.2) and (1.4) admit a unique solution $U\in\mathcal{C}(\mathbb{R},H^1(B_R))^3\cap\mathcal{C}^1(\mathbb{R},L^2(B_R))^3$ for any $R>0$. Let f_α and U_α $(\alpha = p,s)$ be specified as in (2.1) and (2.2), respectively. Our uniqueness results are stated as follows.

Theorem 3.1. *(i) The data set $\{U(x,t) : |x| = R, t\in(0,T_\mathrm{s})\}$ uniquely determines the spatial function f. (ii) The data set of pure P- and S-waves, $\{U_\alpha(x,t) : |x| = R, t\in(0,T_\alpha)\}$, uniquely determines f_α $(\alpha = p,s)$.*

We remark that, since the measurement surface is spherical, the compressional and shear components $U_\alpha(x,t)$ $(\alpha = p,s)$ can be decoupled from the whole wave fields $U(x,t)$ on $|x| = R$. In fact, in the Fourier domain, $\hat{U}_\alpha(x,\omega)$ can be decoupled from $\hat{U}(x,\omega)$ on $|x| = R$ for every fixed $\omega\in\mathbb{R}^+$; see e.g. [7] or section 5.1 in the 2D case. Hence the decoupling in the time domain can be achieved via Fourier transform. Below we present a frequency-domain approach and a time-domain approach to the proof of theorem 3.1.

3.1. Frequency-domain approach

Proof of theorem 3.1.

(i) Assuming that $U(x,t)$=0 for all $|x| = R$ and $t \in (0, T_s)$, we need to prove that $f \equiv 0$ in B_{R_0}. Recalling lemma 2.1, we have $U_\alpha(x,t)$=0 for all $|x| = R$, $t \in \mathbb{R}^+$. Combining this with the fact that $U(x,t) = 0$, $(x,t) \in \mathbb{R}^3 \times (-\infty, 0]$, we deduce that $U_\alpha(x,t) = 0$ for all $|x| = R$, $t \in \mathbb{R}$. Then, applying the Fourier transform in time to $U_\alpha(x,\cdot)$ gives

$$\hat{U}_\alpha(x,\omega) = \int_{\mathbb{R}} U_\alpha(x,t)e^{i\omega t}dt = 0, \quad \text{for all} \quad |x| = R,\ \omega \in \mathbb{R}^+.$$

We introduce the functions

$$v_p(x,\omega) := d\,e^{-ik_p d\cdot x}, \quad v_s(x,\omega) := d^\perp\,e^{-ik_s d\cdot x}, \quad d \in \mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\},$$

where $k_\alpha = k_\alpha(\omega)$ $(\alpha = p, s)$ are the compressional and shear wave numbers, respectively, and $d^\perp \in \mathbb{S}^2$ stands for a unit vector orthogonal to d. Physically, v_p and v_s denote the compressional and shear plane waves propagating along the direction d, respectively. They fulfill the time-harmonic Navier equation as follows

$$\mu\Delta v_\alpha + (\lambda + \mu)\nabla(\nabla \cdot v_\alpha) + \omega^2\rho v_\alpha = 0, \quad \alpha = p, s.$$

Multiplying v_α by (2.9) and applying Betti's formula to $\hat{U}$ and v_α in B_R, we obtain

$$-\hat{g}(\omega)\int_{B_R} f(x)\cdot v_\alpha(x,\omega)\,dx = \int_{|x|=R}\left[T_\nu\hat{U}(x,\omega)\cdot v_\alpha(x,\omega) - T_\nu v_\alpha(x,\omega)\cdot\hat{U}(x,\omega)\right]\,ds,$$

where $\nu = (\nu_1, \nu_2, \nu_3)^\top \in \mathbb{S}^2$ is the normal direction on $|x| = R$ pointing into $|x| > R$ and $T_\nu = T_\nu^{(\lambda,\mu)}$ is the traction operator defined by

$$T_\nu\hat{U} := 2\mu\,\partial_\nu\hat{U} + \lambda\,\nu\,\text{div}\,\hat{U} + \mu\nu\times\text{curl}\,\hat{U}. \tag{3.1}$$

It follows from (2.10) that $\hat{U}(x,\omega)$ satisfies the Kupradze radiation when $|x| \to \infty$. By the well-posedness of the Dirichlet boundary value problem for the time-harmonic Navier system in $|x| > R$, we obtain $T_\nu\hat{U}(x,\omega) \equiv 0$ for all $|x| = R$, $\omega \in \mathbb{R}^+$. This also follows from the well-defined Dirichlet-to-Neumann operator applied to $\hat{U}|_{|x|=R}$ for fixed $\omega \in \mathbb{R}^+$; see e.g. [7]. Hence,

$$\hat{g}(\omega)\int_{B_R} f(x)\cdot v_\alpha(x,\omega)\,dx = 0, \quad \text{for all} \quad \omega \in \mathbb{R}^+. \tag{3.2}$$

On the other hand, applying integration by parts we get

$$\int_{\mathbb{R}^3}\nabla\times f_s(x)\cdot de^{-ik_p x\cdot d}dx = -\int_{\mathbb{R}^3} f_s(x)\cdot\nabla\times(de^{-ik_p x\cdot d})dx = 0,$$

in which the boundary integral over ∂B_R vanish due to the compact support of f in $B_{R_0} \subset B_R$. It then follows that

$$\begin{aligned}
&\int_{B_R} f(x)\cdot v_{\rm p}(x,\omega)\,{\rm d}x\\
&=\int_{\mathbb{R}^3}\nabla f_{\rm p}(x)\cdot d\,{\rm e}^{-{\rm i}k_{\rm s}x\cdot d}\,{\rm d}x+\int_{\mathbb{R}^3}\nabla\times f_{\rm s}(x)\cdot d\,{\rm e}^{-{\rm i}k_{\rm s}x\cdot d}\,{\rm d}x\\
&={\rm i}k_{\rm s}(2\pi)^{\frac{3}{2}}\,\hat{f}_{\rm p}(k_{\rm p}d).
\end{aligned}$$

Note that here $\hat{f}_{\rm p}$ refers to the Fourier transform of $f_{\rm p}$ with respect to spatial variables, given by

$$\hat{f}_{\rm p}(\xi):=\int_{\mathbb{R}^3} f(x){\rm e}^{{\rm i}x\cdot\xi}{\rm d}x,\quad \xi\in\mathbb{R}^3.$$

In the same way, we have

$$\int_{\mathbb{R}^3}\nabla f_{\rm p}(x)\cdot d^{\perp}\,{\rm e}^{-{\rm i}k_{\rm s}x\cdot d}\,{\rm d}x=-\int_{\mathbb{R}^3} f_{\rm p}(x)\,\nabla\cdot(d^{\perp}\,{\rm e}^{-{\rm i}k_{\rm s}x\cdot d})\,{\rm d}x={\rm i}k_{\rm s}(d\cdot d^{\perp})\int_{\mathbb{R}^3} f_{\rm p}(x)\,{\rm e}^{-{\rm i}k_{\rm s}x\cdot d}\,{\rm d}x=0$$

and we find

$$\int_{B_R} f(x)\cdot v_{\rm s}(x,\omega)\,{\rm d}x=\int_{\mathbb{R}^3}\nabla\times f_{\rm s}(x)\cdot d^{\perp}\,{\rm e}^{-{\rm i}k_{\rm s}x\cdot d}\,{\rm d}x={\rm i}k_{\rm s}(2\pi)^{\frac{3}{2}}\hat{f}_{\rm s}(k_{\rm s}d)\cdot(d\times d^{\perp}).$$

Therefore, it follows from (3.2) that

$$\hat{f}_{\rm p}(k_{\rm p}d)=\hat{f}_{\rm s}(k_{\rm s}d)\cdot(d\times d^{\perp})=0$$

for all $d\in\mathbb{S}^2$ and for all $\omega\in\{\omega\in\mathbb{R}^+:\hat{g}(\omega)\neq 0\}$. Since $g\neq 0$, one can always find an interval $(a,b)\subset\mathbb{R}^+$ such that $\hat{g}(\omega)\neq 0$ for $\omega\in(a,b)$. By the analyticity of $\hat{f}_\alpha$ $(\alpha=p,s)$ and the arbitrariness of $d\in\mathbb{S}^2$, we finally obtain $\hat{f}_\alpha\equiv 0$. Applying an inverse Fourier transform we get $f_\alpha\equiv 0$, implying that $f\equiv 0$.

(ii) By (2.3), the P- and S-waves fulfill the wave equations

$$\begin{aligned}
&\frac{1}{c_{\rm p}^2}\,\partial_{tt}\,U_p(x,t)-\Delta U_{\rm p}(x,t)=\frac{1}{\gamma_{\rm p}}\,\nabla f_{\rm p}(x)g(t),\\
&\frac{1}{c_{\rm s}^2}\,\partial_{tt}\,U_s(x,t)-\Delta U_{\rm s}(x,t)=\frac{1}{\gamma_{\rm s}}\,\nabla\times f_{\rm s}(x)\,g(t),
\end{aligned}$$

in $\mathbb{R}^3\times(0,+\infty)$ together with the zero initial conditions at $t=0$, where c_α and γ_α $(\alpha=p,s)$ are given in (2.4). Applying Duhalme's principle and Kirchhoff's formula for wave equations, we can represent these P- and S-waves as

$$\begin{aligned}
U_{\rm p}(x,t)&=\frac{1}{4\pi\gamma_{\rm p}}\int_{|y-x|\leqslant c_{\rm p}(t+T_0)}\frac{\nabla f_{\rm p}(y)g(t-|y-x|/c_{\rm p})}{|y-x|}{\rm d}y,\\
U_{\rm s}(x,t)&=\frac{1}{4\pi\gamma_{\rm s}}\int_{|y-x|\leqslant c_{\rm s}(t+T_0)}\frac{\nabla\times f_{\rm s}(y)g(t-|y-x|/c_{\rm s})}{|y-x|}{\rm d}y,
\end{aligned}$$

for $x\in\mathbb{R}^3$, $t>0$. As done for the Navier equation in the proof of lemma 2.1, one can show that $U_\alpha(x,t)=0$ for all $x\in B_R$, $t>T_\alpha$ $(\alpha=p,s)$. Hence, the relation $U_\alpha(x,t)=0$ for $x\in B_R$, $t\in(0,T_\alpha)$ would imply the vanishing of $U_\alpha(x,t)$ over B_R for all $t\in\mathbb{R}^+$. Now, repeating the argument in the proof of the first assertion we deduce that

$$\nabla f_{\mathrm{p}} = 0, \quad \nabla \times f_{\mathrm{s}} = 0, \quad \mathrm{div} f_{\mathrm{s}} = 0 \quad \text{in} \quad B_R.$$

This implies that $f_{\mathrm{p}} \equiv 0$, since $f_{\mathrm{p}} \in H^1(B_R)$ and $f_{\mathrm{p}} = 0$ in $B_R \backslash B_{R_0}$. To prove the vanishing of $f_{\mathrm{s}} \in (L^2(B_R))^3$, we apply the Helmholtz decomposition to f_{s}, i.e. $f_{\mathrm{s}} = \nabla h_{\mathrm{p}} + \mathrm{curl}\, h_{\mathrm{s}}$, where $h_{\mathrm{p}} \in H^1(B_R)$ and $h_{\mathrm{s}} \in H_{\mathrm{curl}}(B_R)$ are compactly supported in B_R. Then it follows that $h_{\mathrm{p}} = h_{\mathrm{s}} \equiv 0$ in B_R and thus, $f_{\mathrm{s}} \equiv 0$ in B_R; see the proof of lemma A.1 in the appendix. □

Remark 3.2.

(i) The above proof of theorem 3.1 by using Fourier transform is valid in odd dimensions only. The vanishing of the wavefields on $|x| = R$ for $t > T_{\mathrm{s}}$ can be physically interpreted by Huygens' principle, which however does not hold when the number of spatial dimensions is even. The frequency-domain approach applies to two dimensions if we know the time-domain data for all $0 < t < \infty$.

(ii) Theorem 3.1 remains valid if partial data $\{U_\alpha(x,t) : x \in \Gamma, t \in (0, T_{\mathrm{s}})\}$ $(\alpha = p, s, \emptyset)$ are available, where $\Gamma \subset \partial B_R$ is an arbitrary open subset. In fact, in the Fourier domain, the vanishing of $\hat{U}_\alpha(x,\omega)$ on Γ implies that $\hat{U}_\alpha(x,\omega) = 0$ on $|x| = R$ for each fixed $\omega \in \mathbb{R}^+$, due to the analyticity of the solution in a neighborhood of ∂B_R.

As a bi-product of the frequency-domain approach to the proof of theorem 3.1, we show uniqueness in recovering the source term of the time-dependent Schrödinger equation:

$$\begin{cases} \mathrm{i}\hbar\partial_t W(x,t) = [-\frac{\hbar^2}{2\mu}\Delta + q(x)]W(x,t) + f_0(x)g_0(t) & \text{in} \quad \mathbb{R}^3 \times (0, +\infty), \\ W(x,0) = 0 & \text{on} \quad \mathbb{R}^3, \end{cases} \tag{3.3}$$

where $\hbar$ is the reduced Planck constant, μ is the particle's reduced mass and q is the particle's potential energy, which is assumed to be time-independent. Similar to the Lamé system, we shall assume that $f_0 \in L^2(\mathbb{R}^3)$, $\mathrm{supp}(f_0) \subset B_{R_0}$, $g_0 \in H_0^1(0, T_0)$. The potential is supposed to be a real-valued non-negative function with compact support on $\overline{B_R}$ for some $R > R_0$. The number $\omega \in \mathbb{C}$ is called a Dirichlet eigenvalue of the operator $L_{\tilde{q}} := \Delta - \tilde{q}$ with $\tilde{q}(x) := 2\mu/\hbar^2 q(x)$ if there exists a non-trivial function $V \in H_0^1(B_R)$ such that

$$(L_{\tilde{q}} + \omega)V = 0 \quad \text{in} \quad B_R.$$

It can be easily proved that the set of Dirichlet eigenvalues is discrete, which we donote by $\{\omega_n\}_{n=1}^\infty$, and that each eigenvalue is positive. According to [25, theorem 10.1, chapter 3] and [25, remark 10.2, chapter 3], the initial problem (3.3) admits a unique solution $W \in \mathcal{C}([0,+\infty); H^1(\mathbb{R}^3)) \cap \mathcal{C}^1([0,+\infty); H^{-1}(\mathbb{R}^3))$. Therefore, we can introduce the data $\{W(x,t) : |x| = R, t \in \mathbb{R}^+\}$, for W the unique solution of (3.3). The following result extends the uniqueness proof of the inverse source problem for the Helmholtz equation [15] to the case of a time-dependent Lamé system with an inhomogeneous time-independent potential function.

Corollary 3.3. *Assume that $q \in \mathcal{C}_0(B_R)$ is known and that $\hat{g}_0(\omega_n') \neq 0$, $\omega_n' = \omega_n\hbar/(2\mu)$, for all $n = 1, 2, \ldots$. Then the data set $\{W(x,t) : |x| = R, t \in \mathbb{R}^+\}$ uniquely determines f_0.*

Proof. We assume that

$$W(x,t) = 0, \quad |x| = R, \ t \in [0, +\infty). \tag{3.4}$$

Since $g \in H_0^1(0,T)$, the extension of W by 0 on $\mathbb{R}^3 \times (-\infty, 0]$, is the unique solution of

$$\begin{cases} \mathrm{i}\hbar\partial_t W(x,t) = [-\frac{\hbar^2}{2\mu}\Delta + q(x)]W(x,t) + f_0(x)g_0(t) & \text{in} \quad \mathbb{R}^3 \times \mathbb{R} \\ W(x,0) = 0 & \text{on} \quad \mathbb{R}^3. \end{cases}$$

Thus, without lost of generality we can assume that the solution of (3.3) is the solution of the problem on $\mathbb{R}^3 \times \mathbb{R}$. Then, condition (3.4) implies that

$$W(x,t) = 0, \quad |x| = R, \ t \in \mathbb{R}. \tag{3.5}$$

According to the estimate (10.14) in the proof of [25, theorem 10.1, chapter 3] we have

$$\begin{aligned} \|W(\cdot,t)\|^2_{H^1(\mathbb{R}^3)} &\leqslant C \int_0^{+\infty} (|g_0(s)|^2 + |\mathrm{d}g_0(s)/\mathrm{d}s|^2)\, \|f_0\|^2_{L^2(\mathbb{R}^3)}\, \mathrm{d}s \\ &\leqslant C\|g_0\|^2_{H^1(0,T_0)}\|f_0\|^2_{L^2(\mathbb{R}^3)}, \end{aligned}$$

for $t \in [0,+\infty)$, where $C > 0$ is a constant independent of t. In particular, this estimate and the fact that $W(x,t) = 0$ for $(x,t) \in \mathbb{R}^3 \times (-\infty, 0]$, proves that $W \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^3)) \subset \mathcal{S}'(\mathbb{R}; H^1(\mathbb{R}^3))$. Therefore, we can apply the Fourier transform $\mathcal{F}_{t\to\omega}$ to W and deduce from (3.3) that $\hat{W} = \mathcal{F}_{t\to\omega} W \in \mathcal{S}'(\mathbb{R}; H^1(\mathbb{R}^3))$ satisfies

$$L_{\tilde{q}} \hat{W}(x,\omega) + \eta_1 \omega\, \hat{W}(x,\omega) = \eta_2 f_0(x)\, \hat{g}_0(\omega), \quad x \in \mathbb{R}^3, \ \omega \in \mathbb{R}^+, \tag{3.6}$$

with $\eta_1 = 2\mu/\hbar, \eta_2 = 2\mu/\hbar^2$. Note that the identity (3.6) is considered in the sense of distribution with respect to $(x,\omega) \in \mathbb{R}^3 \times \mathbb{R}^+$. In view of (3.6), we have $\Delta\hat{W} \in \mathcal{S}'(\mathbb{R}; L^2(\mathbb{R}^3))$, which implies that $\hat{W} \in \mathcal{S}'(\mathbb{R}; H^2(\mathbb{R}^3))$. The equation (3.6) can be rewritten as

$$\Delta\hat{W}(x,\omega) + k^2\, \hat{W}(x,\omega) = \eta_2 f_0(x)\, \hat{g}_0(\omega) + \tilde{q}(x)\, \hat{W}(x,\omega), \quad k := \sqrt{\eta_1\omega}.$$

Recalling Green's formula, for any $R_1 > R$ we may represent $\hat{W}$ as the integral equation

$$\begin{aligned} \hat{W}(x,\omega) = &\int_{\partial B_{R_1+1}} \left[\partial_\nu \hat{W}(y,\omega)\Phi_k(x-y) - \partial_\nu \Phi_k(x-y)\, \hat{W}(y,\omega)\right] \mathrm{d}s(y) \\ &- \int_{\mathbb{R}^3} \Phi_k(x-y)\, \tilde{q}(y)\, \hat{W}(y,\omega)\, \mathrm{d}y - \eta_2\, \hat{g}_0(\omega) \int_{\mathbb{R}^3} \Phi_k(x-y) f_0(y)\mathrm{d}y \end{aligned}$$

for $x \in B_{R_1}$, where Φ_k is the fundamental solution to the Helmholtz equation $(\Delta + k^2)u = 0$. On the other hand, we have

$$\begin{aligned} &\left|\int_{\partial B_{R_1+1}} \left[\partial_\nu \hat{W}(y,\omega)\Phi_k(x-y) - \partial_\nu \Phi_k(x-y)\, \hat{W}(y,\omega)\right] \mathrm{d}s\right| \\ &\leqslant C(\|\partial_\nu \hat{W}(y,\omega)\|_{L^2(\partial B_{R_1+1})} + \|\hat{W}(y,\omega)\|_{L^2(\partial B_{R_1+1})}) \left(\int_{\partial B_{R_1+1}} |x-y|^{-2}\mathrm{d}s(y)\right)^{\frac{1}{2}} \\ &\leqslant C(\|\partial_\nu \hat{W}(y,\omega)\|_{L^2(\partial B_{R_1+1})} + \|\hat{W}(y,\omega)\|_{L^2(\partial B_{R_1+1})}) \end{aligned}$$

and, since $\hat{W}(\cdot,\omega) \in H^2(\mathbb{R}^3)$, by density we deduce that

$$\lim_{R_1\to+\infty} (\|\partial_\nu \hat{W}(y,\omega)\|_{L^2(\partial B_{R_1+1})} + \|\hat{W}(y,\omega)\|_{L^2(\partial B_{R_1+1})}) = 0.$$

Therefore, sending $R_1 \to +\infty$, we get

$$\hat{W}(x,\omega) = -\int_{\mathbb{R}^3} \Phi_k(x-y)\, \tilde{q}(y)\, \hat{W}(y,\omega)\, \mathrm{d}y - \eta_2\, \hat{g}_0(\omega) \int_{\mathbb{R}^3} \Phi_k(x-y) f_0(y)\mathrm{d}y, \quad x \in \mathbb{R}^3.$$

This implies that $\hat{W}(\cdot,\omega)$ is the unique solution of (3.6) satisfying the Sommerfeld radiation condition when $|x| \to \infty$. Let $V_n \in (H_0^1(B_R))^2$ be an eigenfunction that corresponds to the Dirichlet eigenvalue ω_n. Using the fact that $\hat{W}(\cdot,\omega) \in \{S \in H^1(B_R):\ \Delta S \in L^2(B_R)\}$ and multiplying both sides of (3.6) by V_n, with $\omega = \omega_n'$, and applying integral by parts, we obtain

$$\begin{aligned}\eta_1\,\hat{g}_0(w_n')\int_{B_R} f_0(x)V_n(x)\mathrm{d}x &= \left\langle \partial_\nu \hat{W}_n(\cdot;\omega_n'), V_n\right\rangle_{H^{-\frac{1}{2}}(\partial B_R),H^{\frac{1}{2}}(\partial B_R)} - \int_{\partial B_R}\partial_\nu V_n(x)\,\hat{W}_n(x;\omega_n')\mathrm{d}s(x)\\ &= \left\langle T_n[\hat{W}(\cdot;\omega_n')], V_n\right\rangle_{H^{-\frac{1}{2}}(\partial B_R),H^{\frac{1}{2}}(\partial B_R)} - \int_{\partial B_R}\partial_\nu V_n(x)\,\hat{W}_n(x;\omega_n')\mathrm{d}s(x),\end{aligned}$$

where $T_n: H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ is the Dirichlet-to-Neumann map for radiating solutions to the Helmholtz equation $(\Delta + (\omega_n')^2)u = 0$, which satisfies the Sommerfeld radiation condition at infinity. In view of (3.5), the fact that $\hat{g}_0(w_n') \neq 0$ and the fact that T_n is a linear bounded map, for every $n = 1, 2, \ldots$, we deduce, from the previous identity that

$$\int_{B_R} f_0(x)V_n(x)\mathrm{d}x = 0, \quad n = 1, 2, \ldots.$$

Since the set of the Dirichlet eigenfunctions is complete over $(L^2(B_R))^2$, we conclude that the spatial function f_0 can be uniquely determined by the data. This finishes the uniqueness proof. □

3.2. Time-domain approach

In this subsection we present a time-domain proof of theorem 3.1. Note that this demonstration can be extended to dimension two provided that we replace the data $\{U(x,t):\ |x| = R,\ t \in (0,T_s)\}$ by $\{U(x,t):\ |x| = R,\ t \in (0,+\infty)\}$, since lemma 2.1 does not hold in dimension two.

Proof of theorem 3.1.

(i) Again we assume that $U(x,t) = 0$ for all $|x| = R$, $t \in (0,T_s)$ and, in view of lemma 2.1, this implies that $U(x,t) = 0$ for all $|x| = R, t \in (0,+\infty)$. Our aim is to deduce that $f \equiv 0$. Since the temporal function g is known, we apply Duhalme's principle to U by setting

$$U(x,t) = \int_0^t V(t-s,x)g(s)\,\mathrm{d}s, \quad x \in \mathbb{R}^3, t > 0. \tag{3.7}$$

The function V then fulfills the homogeneous Lamé equation with non-zero initial conditions

$$\begin{aligned}&\partial_{tt}V(x,t) = -c_{\mathrm{p}}^2\,\nabla\times\nabla\times V(x,t) + c_{\mathrm{s}}^2\,\nabla(\nabla\cdot V(x,t)),\\ &V(x,0) = 0, \quad \partial_t V(x,0) = f(x).\end{aligned}$$

Further, we can continue V onto $\mathbb{R}^3 \times (-\infty, 0)$ preserving the Lamé equation and the initial conditions. Since $g(t) = 0$ for $t < 0$, the function $t \to U(t,x)$, given by (3.7), can be regarded as the convolution of $V(x,\cdot)\chi(\cdot)$ and $g(\cdot)\chi(\cdot)$, i.e.

$$U(x,t) = [V(x,t)\chi(t)] * [g(t)\chi(t)], \tag{3.8}$$

where χ is the characteristic function of $(0,\infty)$. By lemma 2.1, $U(x,t)=0$ for $|x|=R$ and $t\in\mathbb{R}$. Taking the Fourier transform to (3.8), we see

$$0=\mathcal{F}_{t\to\omega}[V(x,t)\chi(t)]\mathcal{F}_{t\to\omega}[g(t)\chi(t)]=\mathcal{F}_{t\to\omega}[V(x,t)\chi(t)]\,\hat{g}(\omega),\quad |x|=R.$$

Making use of the analyticity of $\mathcal{F}_{t\to\omega}[V(x,t)\chi(t)]$ with respect to ω and that the fact that g does not vanish identically, we deduce that $V(x,t)=0$ for $|x|=R$ and $t\in\mathbb{R}$.
We decouple V into the sum of the compressional part $V_{\rm p}$ and shear part $V_{\rm s}$:

$$V=V_{\rm p}+V_{\rm s},\quad V_{\rm p}=\nabla v_{\rm p}+\nabla\times v_{\rm s},\quad \nabla\cdot v_{\rm s}=0\quad\text{in}\quad\mathbb{R}^3,$$

where V_α ($\alpha=p,s$) fulfills the homogeneous wave equation

$$\partial_{tt}V_\alpha(x,t)=c_\alpha^2\,\Delta V_\alpha\qquad\text{in}\quad B_R$$

and the initial conditions

$$V_\alpha(x,0)=0,\quad \partial_t V_{\rm p}(x,0)=\nabla f_{\rm p}(x),\quad \partial_t V_{\rm s}(x,0)=\nabla\times f_{\rm s}(x).$$

Since $\mathrm{Supp}(f)\subset B_{R_0}\subset B_R$, $V(x,t)$ has zero initial conditions in the unbounded domain $|x|>R$. Consequently, we get $V\equiv 0$ for all $|x|>R$ and $t\in\mathbb{R}$, due to the unique solvability of the hyperbolic system in $|x|>R$ with the Dirichlet boundary condition at $|x|=R$ for all $t>0$. By uniqueness of the Helmholtz decomposition, it follows that $V_\alpha=0$ in $|x|>R$ for all $t\in\mathbb{R}$. In view of the unique continuation for the homogeneous wave equation (see e.g. [6, 27, 29]), it can be deduced that $V_\alpha(x,t)=0$ in $B_R\times\mathbb{R}$, implying that $V=0$ for $x\in B_R$ and $t\in\mathbb{R}$. In particular, $\partial_t V(x,0)=f(x)=0$ for $x\in B_R$.

(ii) If $U_\alpha(x,t)=0$ for $|x|=R$, $t\in(0,T_\alpha)$, then we have $V_\alpha=0$ on $\{|x|=R\}\times\mathbb{R}$. Repeating the arguments above, it follows that $V_\alpha(x,0)=0$ in $|x|>R$ for $t\in\mathbb{R}$. As a consequence of the unique continuation we get $V_\alpha(x,0)=0$ in $B_R\times\mathbb{R}$. Setting $t=0$ we obtain $f_\alpha=0$ for $\alpha=p,s$. □

Remark 3.4. We think that the frequency-domain and time-domain approaches presented above could also yield a stability estimate of the spatial function in terms of the time-domain data $\{U(x,t):|x|=R,0<t<T_{\rm s}\}$. The terminal time T_α ($\alpha=p,s$) in theorem 3.1 is optimal. Non-uniqueness examples can be readily reconstructed if the terminal time is less than T_α.

4. Unique determination of temporal functions

Given some $T>0$, we suppose that $g\in(L^2(0,T))^3$ is an unknown vector-valued temporal function and that the spatial function f is known to be compactly supported in B_{R_0} for some $R_0>0$. We consider the inverse problem of determining g from observations of the solution of

$$\begin{cases}\rho\partial_{tt}U(x,t)=\nabla\cdot\sigma(x,t)+f(x)g(t), & (x,t)\in\mathbb{R}^3\times(0,T),\\ U(x,0)=\partial_t U(x,0)=0, & x\in\mathbb{R}^3,\end{cases}\tag{4.1}$$

at one fixed point $x_0\in\mathrm{supp}(f)$ (i.e. interior observations) or at the subbounary $\Gamma\subset\partial B_R$ (i.e. partial boundary observations). In order to state rigorously our problem, we start by considering the regularity of this initial value problem (4.1).

Lemma 4.1. *Let $g\in(L^2(0,T))^3$ and let $f\in H^p(\mathbb{R}^3)$, with $p>5/2$, be supported on B_R for some $R>R_0$. Then problem (4.1) admits a unique solution $U\in\mathcal{C}([0,T];H^{p+1}(\mathbb{R}^3))^3\cap H^2((0,T);H^{p-1}(\mathbb{R}^3))^3$ satisfying*

$$\|U\|_{\mathcal{C}([0,T];H^{p+1}(\mathbb{R}^3))^3} + \|U\|_{H^2((0,T);H^{p-1}(\mathbb{R}^3))^3} \leqslant C\|g\|_{L^2(0,T)^3}\|f\|_{H^p(\mathbb{R}^3)}, \quad (4.2)$$

with $C>0$ depending on ρ, λ, μ, R.

Proof. Applying a Fourier transform to $U(\cdot,t)$ with respect to spatial variables, denoted by $\hat{U}$, we find

$$\begin{aligned} &\partial_{tt}\hat{U}(\xi,t) + A(\xi)\hat{U}(\xi,t) = \frac{g(t)\hat{f}(\xi)}{\rho} \quad \text{in} \quad \mathbb{R}^3\times(0,T),\\ &\hat{U}(\xi,0)=0, \quad \partial_t\hat{U}(\xi,0)=0, \quad \xi\in\mathbb{R}^3, \end{aligned} \quad (4.3)$$

where the matrix $A(\xi)\in\mathbb{R}^{3\times3}$ is defined by

$$A(\xi) := \frac{\mu}{\rho}|\xi|^2\,\mathbf{I} + \frac{(\lambda+\mu)}{\rho}\xi\otimes\xi, \quad \xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3.$$

Evidently, $A(\xi)$ is a real-valued symmetric matrix, with the eigenvalues given by $\frac{(\lambda+2\mu)|\xi|^2}{\rho}$, $\frac{\mu|\xi|^2}{\rho}$, $\frac{\mu|\xi|^2}{\rho}$; see lemma A.2 in the appendix. Denote by $A^{1/2}(\xi)$ the square roof of $A(\xi)$ and by $A^{-1/2}(\xi)$ the inverse of $A^{1/2}(\xi)$. Then the unique solution to (4.3) takes the form

$$\hat{U}^\top(\xi,t) = \int_0^t g^\top(s)\,A^{-1/2}(\xi)\,\sin\left(A^{1/2}(\xi)(t-s)\right)\frac{\hat{f}(\xi)}{\rho}\,\mathrm{d}s. \quad (4.4)$$

On the other hand, for all $t\in[0,T]$ and $s\in[0,t]$, fixing

$$H(t-s,\cdot) := \xi\mapsto A^{-1/2}(\xi)\,\sin\left(A^{1/2}(\xi)(t-s)\right)\frac{\hat{f}(\xi)}{\rho},$$

we have

$$\begin{aligned} \|H(t-s,\cdot)\|^2_{L^2(\mathbb{R}^3)^{3\times3}} &\leqslant \mu^{-1}\rho^{-1}\|\hat{f}\|^2_{L^\infty(\mathbb{R}^3)}\int_{B_1}|\xi|^{-2}\mathrm{d}\xi + 4\mu^{-1}\rho^{-1}\int_{\mathbb{R}^3\setminus B_1}(1+|\xi|^2)^{-1}|\hat{f}(\xi)|^2\mathrm{d}\xi\\ &\leqslant C\mu^{-1}\rho^{-1}|B_R|\|\hat{f}\|^2_{L^2(\mathbb{R}^3)} + 4\mu^{-1}\rho^{-1}\|f\|^2_{L^2(\mathbb{R}^3)}, \end{aligned} \quad (4.5)$$

with C a constant. Note that here we use the fact that $\xi\mapsto|\xi|^{-2}\in L^1(B_1)$, since $2<3$, and the fact that $\mathrm{supp}(f)\subset B_R$. Moreover, we apply the fact that $\lambda+\mu>0$ to deduce that $|A^{-1/2}(\xi)|\leqslant\rho^{\frac12}\mu^{-1/2}|\xi|^{-1}$. In the same way, we have

$$\||\xi|^{p+1}H(t-s,\cdot)\|^2_{L^2(\mathbb{R}^3)^{3\times3}} \leqslant \mu^{-1}\rho^{-1}\int_{\mathbb{R}^3}|\xi|^{2p}|\hat{f}(\xi)|^2\mathrm{d}\xi \leqslant \mu^{-1}\rho^{-1}\|f\|^2_{H^p(\mathbb{R}^3)}. \quad (4.6)$$

Combining estimates (4.5) and (4.6), one can easily deduce that $U\in\mathcal{C}([0,T];H^{p+1}(\mathbb{R}^3))^3$. In the same way, we have

$$\|(1+|\xi|^2)^{\frac{p-1}{2}}\partial_t H(t-s,\cdot)\|_{L^2(\mathbb{R}^3)^{3\times3}} + \|(1+|\xi|^2)^{\frac{p-1}{2}}\partial_t^2 H(t-s,\cdot)\|_{L^2(\mathbb{R}^3)^{3\times3}} \leqslant C\|f\|_{H^p(\mathbb{R}^3)}, \quad (4.7)$$

where C depends on ρ, λ, μ, R. Moreover, for almost every $\xi\in\mathbb{R}^3$, we have

$$\hat{U}^\top(\xi,\cdot): t \mapsto \hat{U}^\top(\xi,t) \in H^2(0,T),$$

with

$$\partial_t \hat{U}^\top(\xi,t) = \int_0^t g(s)^\top \partial_t H(t-s,\xi)\mathrm{d}s, \quad \partial_{tt}\hat{U}^\top(\xi,t) = \frac{g(t)^\top \hat{f}(\xi)}{\rho} + \int_0^t g(s)^\top \partial_{tt} H(t-s,\xi)\mathrm{d}s.$$

Combining this with (4.7), we deduce that $U \in H^2((0,T); H^{p-1}(\mathbb{R}^3))^3$ and we deduce (4.2) from the previous estimates. □

According to lemma 4.1 and the Sobolev embedding theorem we have $U \in \mathcal{C}([0,T]; \mathcal{C}^2(\mathbb{R}^3))^3 \cap H^2((0,T); \mathcal{C}(\mathbb{R}^3))^3$ and the trace $t \mapsto U(x_0,t)$, for some point $x_0 \in \mathbb{R}^3$, is well defined as an element of $H^2((0,T))^3$. Below we consider the inverse problem of determining the evolution function $g(t)$ from the interior observation of the wave fields $U(x_0,t)$ for $t \in (0,T)$ and some $x_0 \in \mathrm{supp}(f)$.

Theorem 4.2 (Uniqueness and stability with interior data). *Let $x_0 \in B_R$, $p > 5/2$ and consider $M, \delta > 0$ such that*

$$\mathcal{A}_{x_0,p,\delta,M} := \{h \in H^p(\mathbb{R}^3): \ \|h\|_{H^p(\mathbb{R}^3)} \leqslant M, \ |h(x_0)| \geqslant \delta, \ \mathrm{supp}(h) \subset B_R\} \neq \emptyset.$$

Then, for $f \in \mathcal{A}_{x_0,p,\delta,M}$, it holds that

$$\|g\|_{L^2(0,T)^3} \leqslant C\,\|\partial_{tt} U(x_0,\cdot)\|_{L^2(0,T)^3}$$

where C depends on λ, μ, ρ, p, x_0, M, R, δ and T. In particular, this estimate implies that the data $\{U(x_0,t) : t \in (0,T)\}$ determines uniquely the temporal function g.

Proof. According to (4.4), the solution U of (4.1) is given by

$$U(x,t)^\top = (2\pi)^{-3} \int_{\mathbb{R}^3} \left(\int_0^t g(s)^\top A^{-1/2}(\xi) \sin\left(A^{1/2}(\xi)(t-s)\right) \frac{\hat{f}(\xi)}{\rho} \right) \mathrm{e}^{\mathrm{i}\xi\cdot x}\, \mathrm{d}\xi, \quad (x,t) \in \mathbb{R}^3 \times [0,T]$$

and applying Fubini's theorem we find

$$U(x,t)^\top = (2\pi)^{-3} \int_0^t g(s)^\top \left(\int_{\mathbb{R}^3} A^{-1/2}(\xi) \sin\left(A^{1/2}(\xi)(t-s)\right) \frac{\hat{f}(\xi)}{\rho}\, \mathrm{e}^{\mathrm{i}\xi\cdot x}\, \mathrm{d}\xi \right) \mathrm{d}s, \quad (x,t) \in \mathbb{R}^3 \times [0,T].$$

In particular, in view of lemma 4.1, $U \in \mathcal{C}([0,T]; H^{p+1}(\mathbb{R}^3))^3 \cap H^2((0,T); H^{p-1}(\mathbb{R}^3))^3$ satisfies (4.2). Further, direct calculations show that

$$\mathcal{L}_{\lambda,\mu} U(x,t)^\top = -(2\pi)^{-3} \int_0^t g(s)^\top \left(\int_{\mathbb{R}^3} A^{1/2}(\xi) \sin\left(A^{1/2}(\xi)(t-s)\right) \frac{\hat{f}(\xi)}{\rho}\, \mathrm{e}^{\mathrm{i}\xi\cdot x}\, \mathrm{d}\xi \right) \mathrm{d}s.$$

Since $|A^{1/2}(\xi)\mathbf{a}| \leqslant \frac{\sqrt{\lambda+2\mu}}{\sqrt{\rho}} |\xi||\mathbf{a}|$ for all $\mathbf{a} \in \mathbb{C}^3$, the previous identity can be estimated by

$$
\begin{aligned}
|\mathcal{L}_{\lambda,\mu} U(x,t)| &\leqslant \frac{\sqrt{\lambda+2\mu}}{\sqrt{\rho}} \int_0^t |g(s)|\,\mathrm{d}s \int_{\mathbb{R}^3} |\hat{f}(\xi)|\,|\xi|\,\mathrm{d}\xi \\
&\leqslant \frac{\sqrt{\lambda+2\mu}}{\sqrt{\rho}} \int_0^t |g(s)|\,\mathrm{d}s\, \|\hat{f}(\xi)(1+|\xi|^2)^{p/2}\|_{L^2(\mathbb{R}^3)}\, \|(1+|\xi|^2)^{(1-p)/2}\|_{L^2(\mathbb{R}^3)} \\
&\leqslant M_0\, \frac{\sqrt{\lambda+2\mu}}{\sqrt{\rho}}\, \|f\|_{H^p(\mathbb{R}^3)} \int_0^t |g(s)|\,\mathrm{d}s
\end{aligned}
\tag{4.8}
$$

where $M_0 = \|(1+|\xi|^2)^{(1-p)/2}\|_{L^2(\mathbb{R}^3)} < \infty$. Since $|f(x_0)| \geqslant \delta$, we derive from the governing equation of U and (4.8) that

$$
\begin{aligned}
|g(t)| &= \frac{1}{|f(x_0)|}\, |\rho \partial_{tt} U(x_0,t) - \mathcal{L}_{\lambda,\mu} U(x_0,t)| \\
&\leqslant M_1\, |\partial_{tt} U(x_0,t)| + M_2 \int_0^t |g(s)|\,\mathrm{d}s
\end{aligned}
$$

for all $t \in (0,T)$, where $M_1 = \rho/\delta$, $M_2 = M_0 \frac{\sqrt{\lambda+2\mu}}{\sqrt{\rho}} M/\delta$. Applying the Grownwall inequality stated in lemma A.3, for almost every $t \in (0,T)$, we find

$$
\begin{aligned}
|g(t)| &\leqslant M_1 |\partial_{tt} U(x_0,t)| + M_1\, M_2 \int_0^t |\partial_{tt} U(x_0,s)|\, \mathrm{e}^{M_2(t-s)}\,\mathrm{d}s \\
&\leqslant M_1\, |\partial_{tt} U(x_0,t)| + M_1\, M_2\, T^{\frac{1}{2}}\, \mathrm{e}^{M_2 T}\, \|\partial_{tt} U(x_0,\cdot)\|_{L^2(0,T)^3}.
\end{aligned}
$$

Therefore, taking the norm $L^2(0,T)$ on both sides of the inequality, implies that

$$
\|g\|_{L^2(0,T)^3} \leqslant (M_1 + M_1\, M_2\, T\, \mathrm{e}^{M_2 T})\, \|\partial_{tt} U(x_0,\cdot)\|_{L^2(0,T)^3}.
$$

This completes the proof. □

To state uniqueness with partial boundary measurement data, we need the concept of non-radiating source.

Definition 4.3. The compactly supported function f is called a non-radiating source at the frequency $\omega \in \mathbb{R}^+$ to the Lamé system if there exists a $P \in \mathbb{C}^3$ such that the unique radiating solution to the inhomogeneous Lamé system

$$
\mathcal{L}_{\lambda,\mu} u(x) + \omega^2 \rho u(x) = f(x)\, P, \quad j = 1,2,3,
\tag{4.9}
$$

vanishes identically in $\mathbb{R}^3 \backslash \overline{\mathrm{supp}(f)}$.

Obviously, f is not a non-radiating source at the frequency ω if the unique solution to (4.9) does not vanish in $\mathbb{R}^3 \backslash \overline{\mathrm{supp}(f)}$ for all $P \in \mathbb{C}^3$.

Theorem 4.4 (Uniqueness with partial boundary data). *Suppose that $f \in L^2(B_R)$ is known to be a compacted supported function over B_{R_0} for some $R_0 < R$ and that f is not a non-radiating source at any frequency $\omega \in \mathbb{R}^+$. Then the temporal function $g \in (\mathcal{C}_0([0,T_0]))^3$ can be uniquely determined by the partial boundary measurement data $\{U(x,t) : x \in \Gamma, t \in (0,T_s)\}$ where $\Gamma \subset \partial B_R$ is an arbitrary open subset and T_s is defined in (2.7).*

Proof. Let $w_j = w_j(x, \omega)$ $(j = 1, 2, 3)$ be the unique radiating solution to the inhomogeneous Lamé system

$$\mathcal{L}_{\lambda,\mu} w_j(x) + \omega^2 \rho w_j(x) = f(x)\, e_j, \quad j = 1, 2, 3,$$

which does not vanish identically in $|x| \geqslant R$ by our assumption. Set the matrix $W := (w_1, w_2, w_3) \in \mathbb{C}^{3\times 3}$. Then $W(\cdot, \omega)$ solves the matrix equation

$$\mathcal{L}_{\lambda,\mu} W(x, \omega) + \omega^2 \rho W(x, \omega) = f(x)\, \mathbf{I} \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty).$$

Note that here the action of the differential operator is understood column-wisely, and W can be represented as

$$W(x, \omega) = \int_{\mathbb{R}^3} \hat{G}(x - y, \omega) f(y)\, \mathrm{d}y, \quad x \in \mathbb{R}^3,$$

where $\hat{G}$ is the Green tensor to the time-harmonic Lamé system. In view of (2.10), the Fourier transform $\hat{U}(x, \omega)$ of $U(x, t)$ can be written as

$$\hat{U}(x, \omega) = W(x, \omega)\, \hat{g}(\omega) \quad \text{for all} \quad \omega \in \mathbb{R}^+,\ |x| = R. \tag{4.10}$$

We claim that for each $\omega_0 \in \mathbb{R}^+$, there always exists $x_0 \in \Gamma \subset \partial B_R$ such that $\mathrm{Det}(W(x_0, \omega_0)) \neq 0$. Suppose on the contrary that $\mathrm{Det}(W(x, \omega_0)) = 0$ for all $x \in \Gamma$. This implies that there exist $c_j \in \mathbb{C}^3$ such that

$$V(x) := c_1 w_1(x, \omega_0) + c_2 w_2(x, \omega_0) + c_3 w_3(x, \omega_0) = 0 \quad \text{on} \quad \Gamma. \tag{4.11}$$

By the analyticity of w_j in a neighborhood of $|x| = R$ and the analyticity of the surface $\Gamma \subset \partial B_R$, we conclude that (4.11) holds on $|x| = R$. By uniqueness of the exterior Dirichlet boundary value problem, we have $V(x) = 0$ in $|x| > R$, and by unique continuation it holds that $V(x) = 0$ for all x lying outside of the support of f. However, it is easy to observe that V satisfies the inhomogeneous equation (4.9) with $P = c_1 e_1 + c_2 e_2 + c_3 e_3$, which contradicts the fact that f is not a non-radiating source. Therefore, by (4.10) we get

$$\hat{g}(\omega_0) = [W(x_0, \omega_0)]^{-1}\, \hat{U}(x_0, \omega_0) \in \mathbb{C}^{3\times 1} \quad \text{for some} \quad x_0 \in \Gamma.$$

Note that ω_0 is arbitrary and the point x_0 depends on ω_0. Hence, if $U(x, t) = 0$ for all $x \in \Gamma$ and $t \in (0, T_s)$, then $\hat{U}(x, \omega) = 0$ for all $x \in \Gamma$ and $\omega \in \mathbb{R}^+$. This implies that $\hat{g}(\omega) = 0$ for all $\omega \in \mathbb{R}$ and thus $g \equiv 0$. □

Remark 4.5. We remark that from the mathematical point of view the recovery of source terms of the form $g(t) f(x)$ is the best we can expect.

(i) There is no hope of recovering general source terms of the form $F(x, t)$. Below is a counterexample. Let $\chi = (\chi_1, \chi_2, \chi_3)$ with $\chi_j \in C_0^\infty(B_{R_0} \times (0, T_0))$, $j = 1, 2, 3$. Now fix

$$F(x, t) := \rho \partial_{tt} \chi - \mathcal{L}_{\lambda,\mu} \chi$$

and consider the problem

$$\begin{cases} \rho \partial_{tt} U(x, t) = \mathcal{L}_{\lambda,\mu} U(x, t) + F(x, t), & (x, t) \in \mathbb{R}^3 \times (0, +\infty), \\ U(x, 0) = \partial_t U(x, 0) = 0, & x \in \mathbb{R}^3. \end{cases} \tag{4.12}$$

Clearly $U = \chi$ is the unique solution of (4.12). Assuming that $\chi \neq 0$, from the uniqueness of solutions of (4.12) one can check that $F \neq 0$. However since $\mathrm{supp}(\chi) \subset B_{R_0} \times (0, T_0)$, we have

$$U(x,t) = 0, \quad |x| = R, \ t \in (0, +\infty)$$

and supp $(F) \subset B_{R_0} \times (0, T_0)$, but $F \neq 0$. This proves that the data $\{U(x,t) : \ |x| = R, \ t > 0\}$ do not allow the recovery of general sources $F(x,t)$ satisfying $\mathrm{supp}(F) \subset B_{R_0} \times (0, T_0)$.

(ii) There is even an obstruction for the recovery of source terms of the form $g_1(t)f_1(x) + g_2(t)f_2(x)$. Let $\chi = (\chi_1, \chi_2, \chi_3)$ with $\chi_j \in \mathcal{C}_0^\infty(B_{R_0})$, $j = 1, 2, 3$ and choose $g \in \mathcal{C}_0^\infty((0, T_0))$. Fix

$$g_1(t) = g''(t), \quad f_1(x) = \rho\chi(x), \quad g_2(t) = g(t), \quad f_2(x) = -\mathcal{L}_{\lambda,\mu}\chi(x)$$

and consider the problem

$$\begin{cases} \rho\partial_{tt}U(x,t) = \mathcal{L}_{\lambda,\mu}U(x,t) + g_1(t)f_1(x) + g_2(t)f_2(x), & (x,t) \in \mathbb{R}^3 \times (0, +\infty), \\ U(x,0) = \partial_t U(x,0) = 0, & x \in \mathbb{R}^3. \end{cases} \tag{4.13}$$

Clearly $U(x,t) = g(t)\chi(x)$ is the unique solution of (4.13) and assuming that $\chi \neq 0$ and $g \neq 0$ from the uniqueness of solutions of (4.13) one can check that, for $F : (x,t) \mapsto g_1(t)f_1(x) + g_2(t)f_2(x)$, $F \neq 0$. However, since $\mathrm{supp}(\chi) \subset B_{R_0}$, we have

$$U(x,t) = 0, \quad |x| = R, \ t \in (0, +\infty)$$

and $\mathrm{supp}(F) \subset B_{R_0} \times (0, T_0)$, but $F \neq 0$. This proves that the data $\{U(x,t) : \ |x| = R, \ t > 0\}$ do not allow the recovery of sources of the form $F(x,t) = g_1(t)f_1(x) + g_2(t)f_2(x)$ satisfying $\mathrm{supp}(F) \subset B_{R_0} \times (0, T_0)$.

5. Numerical experiments

In this section, we propose a Landweber iterative method for reconstructing the spatial function f in 2D and a non-iterative inversion scheme based on the proof of theorem 4.4 for recovering the temporal function g in 3D. Several numerical examples will be illustrated to examine the effectiveness of the proposed methods.

5.1. Reconstruction of spatial functions

We consider the inverse source problem presented in section 3. Our aim is to reconstruct the spacial function in two dimensions, relying on the Landweber iterative method for solving linear algebraic equations. Assume that the time-dependent data $U(x,t), x \in \partial B_R$ $(R > R_0)$ are measured over the time interval $[0, T]$ where $T > 0$ is sufficiently large such that the integral

$$\int_0^T U(x,t) \exp(\mathrm{i}\omega t) \, \mathrm{d}t$$

can be used to approximate the Fourier transform $\hat{U}(x, \omega)$ for any $\omega \in \mathbb{R}^+$. In the time-harmonic regime, it is supposed that the multi-frequency data $\hat{U}(x, \omega_k), x \in \partial B_R$ for $k = 1, \ldots, K$ are available. Hence, the time-dependent inverse source problem can be transformed into a problem in the Fourier domain with near-field data of multi frequencies. In 2D, the Helmhotz decomposition of $\hat{U}$ takes the form $\hat{U} = \hat{U}_{\rm p} + \hat{U}_{\rm s}$, where the compressional part $\hat{U}_{\rm p}$ and shear part $\hat{U}_{\rm s}$ are given by

$$\hat{U}_{\rm p} = -\frac{1}{k_{\rm p}^2}\,\text{grad}\,\text{div}\,\hat{U}, \quad \hat{U}_{\rm s} = \frac{1}{k_{\rm s}^2}\,\overrightarrow{\text{curl}}\,\text{curl}\,\hat{U}. \tag{5.1}$$

Here the 2D operators curl and $\overrightarrow{\text{curl}}$ are defined respectively by

$$\text{curl}\, v = \partial_1 v_2 - \partial_2 v_1, \quad v = (v_1, v_2)^\top, \qquad \overrightarrow{\text{curl}}\, h := (\partial_2 h, -\partial_1 h)^\top.$$

Writing $\hat{u}_{\rm p} := -1/k_{\rm p}^2\,\text{div}\,\hat{U}$ and $\hat{u}_{\rm s} = 1/k_{\rm s}^2\,\text{curl}\,\hat{U}$, we have $\hat{U} = \text{grad}\,\hat{u}_{\rm p} + \overrightarrow{\text{curl}}\,\hat{u}_{\rm s}$ and the scalar functions $\hat{u}_\alpha$ $(\alpha = p, s)$ satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial \hat{u}_\alpha}{\partial r} - \mathrm{i}k_\alpha \hat{u}_\alpha\right) = 0, \quad r = |x|, \quad \alpha = p, s$$

uniformly with respect to all $\hat{x} = x/|x| \in \mathbb{S}^1$.

For $|x| \geqslant R$, the radiation solutions $\hat{u}_\alpha$ can be expressed in terms of Hankel functions of the first kind,

$$\hat{u}_\alpha(|x|, \theta) = \sum_{n\in\mathbb{Z}} \hat{u}_{\alpha,n} H_n^{(1)}(k_\alpha|x|)\exp(\mathrm{i}n\theta), \quad x = |x|(\cos\theta, \sin\theta),\ |x| \geqslant R. \tag{5.2}$$

For every fixed $\omega \in \mathbb{R}^+$, the coefficients $\hat{u}_{\alpha,n} \in \mathbb{C}$ are uniquely determined by $\hat{U}(x,\omega)|_{|x|=R}$ as follows (see e.g. [7])

$$\begin{pmatrix} \hat{u}_{p,n} \\ \hat{u}_{s,n} \end{pmatrix} = \frac{1}{2\pi R}[A_n(R)]^{-1} \int_0^{2\pi} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \hat{U}(R, \theta; \omega) d\theta, \tag{5.3}$$

where

$$A_n(R) = \begin{pmatrix} t_{\rm p} H_n^{(1)\prime}(t_{\rm p}) & \mathrm{i}nH_n^{(1)}(t_{\rm s}) \\ \mathrm{i}nH_n^{(1)}(t_{\rm p}) & -t_{\rm s} H_n^{(1)\prime}(t_{\rm s}) \end{pmatrix}, \quad t_\alpha = k_\alpha R,\ \alpha = p, s. \tag{5.4}$$

This means that, in the Fourier domain, the P and S-waves can be decoupled from the whole wave field $\hat{U}$ on $|x| = R$ for every fixed frequency ω.

Below we shall consider the inverse problems of reconstructing $f_{\rm p}, f_{\rm s}$ and f from the wave fields $\hat{u}_{\rm p}(x,\omega)|_{\partial B_R}$, $\hat{u}_{\rm s}(x,\omega)|_{\partial B_R}$ and $u(x,\omega)|_{\partial B_R}$ at a finite number of frequencies $\omega = \omega_k$, $k = 1, \ldots, K$, respectively. Recall from (2.10) that

$$\hat{U}(x,\omega)/\hat{g}(\omega) = \int_{B_R} \hat{G}(x-y) f(y)\mathrm{d}y, \quad |x| = R, \quad \hat{g}(\omega) \neq 0, \tag{5.5}$$

where $\hat{G}$ is the fundamental displacement tensor of the Navier equation of the form (2.8) with the fundamental solution of the 2D Helmholtz equation given by

$$\Phi_k(x,y) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|), \quad x \neq y, \quad x, y \in \mathbb{R}^2.$$

Analogously, the compressional and shear components of $\hat{U}$ can be represented by (see (2.3))

$$\hat{u}_\alpha(x,\omega)/\hat{g}(\omega) = \frac{1}{\gamma_\alpha} \int_{B_R} \Phi_{k_\alpha}(x,y) f_\alpha(y)\mathrm{d}y, \quad \alpha = p, s. \tag{5.6}$$

Table 1. Landweber iterative method for reconstructing spatial functions.

Step 1	Set an initial guess $S_{0,0}$
Step 2	Update the source function S by the iterative formula $S_{l,k} = S_{l-1,k} + \epsilon V_k^* (v_k - V_k S_{l-1,k}), \quad l = 1, \ldots, L,$ where ϵ and L are the step length and total number of iterations, respectively
Step 3	Set $S_{0,k+1} = S_{L,k}$ and repeat step 2 until the highest frequency ω_K is reached

Our numerical scheme relies on solvability of the ill-posed integral equations (5.5) and (5.6) for finding f and f_α. Since $f(x)$ is real-valued, it is more convenient to consider real-valued integral equations from a numerical point of view. Taking the real and imaginary parts of (5.5) gives

$$\mathrm{Re}\{\hat{U}(x,\omega)/\hat{g}(\omega)\} = \int_{B_R} \mathrm{Re}\{\hat{G}(x-y,\omega)\} f(y)\mathrm{d}y, \quad |x| = R, \tag{5.7}$$

$$\mathrm{Im}\{\hat{U}(x,\omega)/\hat{g}(\omega)\} = \int_{B_R} \mathrm{Im}\{\hat{G}(x-y,\omega)\} f(y)\mathrm{d}y, \quad |x| = R. \tag{5.8}$$

Furthermore, for the pressure part $\hat{u}_\mathrm{p}$ and shear part $\hat{u}_\mathrm{s}$, we have

$$\mathrm{Re}\{\hat{u}_\alpha(x,\omega)/\hat{g}(\omega)\} = \frac{1}{\gamma_\alpha} \int_{B_R} \mathrm{Re}\{\Phi_{k_\alpha}(x,y)\} f_\alpha(y)\mathrm{d}y, \quad |x| = R, \tag{5.9}$$

$$\mathrm{Im}\{u_\alpha(x,\omega)/\hat{g}(\omega)\} = \frac{1}{\gamma_\alpha} \int_{B_R} \mathrm{Im}\{\Phi_{k_\alpha}(x,y)\} f_\alpha(y)\mathrm{d}y, \quad |x| = R. \tag{5.10}$$

The equations (5.7)–(5.10) are Fredholm integral equations of the first kind. These equations are ill-posed, since the singular values of the matrix resulting from the discretized integral kernel are rapidly decaying. Now, we describe a Landweber iterative method to solve the ill-posed integral equations (5.7)–(5.10). Consider the linear operator equations

$$V_k(S) = v_k, \quad k = 1, \ldots, K, \quad S = f, f_\mathrm{p}, f_\mathrm{s}, \tag{5.11}$$

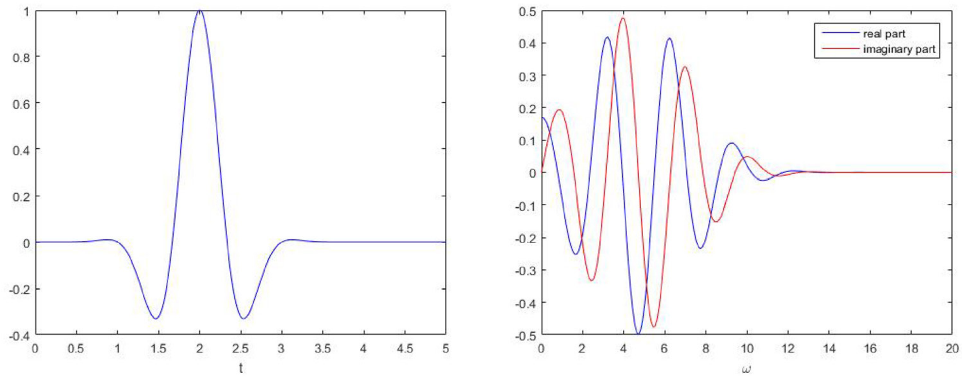
where $v_k = \hat{U}(x,\omega_k)$ or $v_k = \hat{u}_\alpha(x,\omega_k)$ denotes the measurement data at the frequency ω_k. We denote by $S_{l,k}$ the inverse solution obtained at the lth iteration step reconstructed from the data set at the frequency ω_k. Due to the linearity of (5.11), a straightforward Landweber iteration (see, e.g. [10]) can be applied as a regularization scheme for solving (5.11). For clarity we summarize the inversion process in table 1.

Below we present several numerical examples to demonstrate the validity and effectiveness of the proposed method. In the following we always choose

$$\hat{g}(\omega) = \int_0^T g(t)\exp(\mathrm{i}\omega t)\mathrm{d}t, \quad g(t) = \begin{cases} \cos(1.5\pi(t-t_0))\exp(-\pi(t-t_0)^2), & t \leqslant T, \\ 0, & t > T, \end{cases}$$

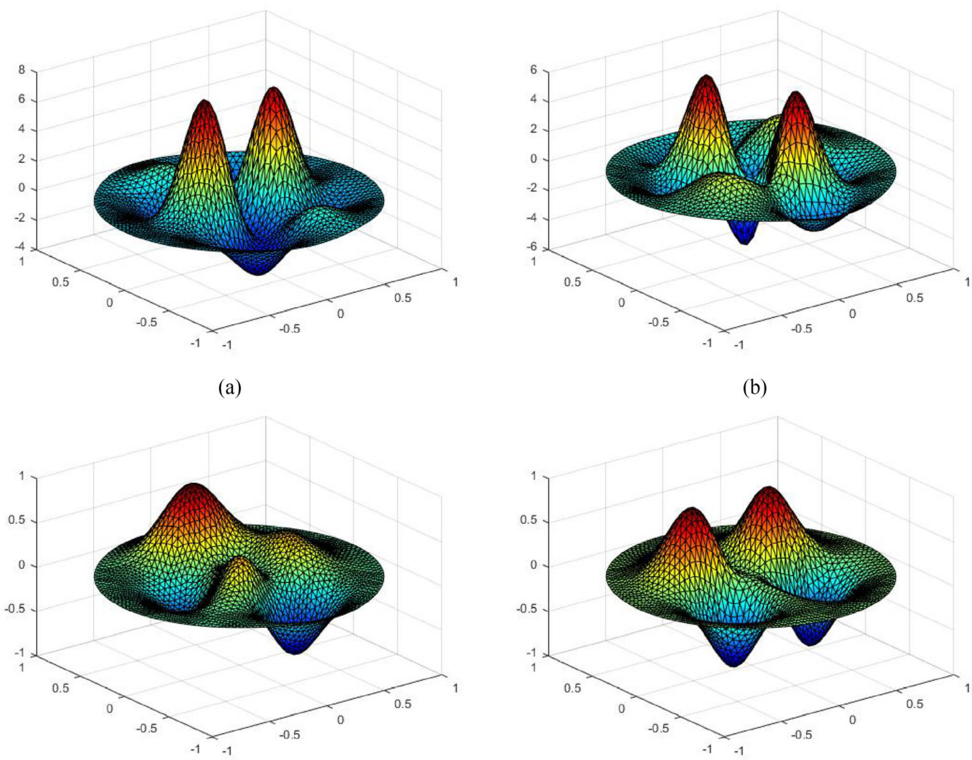
where $T = 5$, $t_0 = 2$. The functions $\hat{g}$ and g are plotted in figure 1, which shows that $\hat{g}$ is non-zero in $(0, 20)$. The source function f in B_R with $R = 1$ is defined by

$$f = (f_1, f_2)^\top = \nabla f_\mathrm{p} + \overrightarrow{\mathrm{curl}} f_\mathrm{s},$$



(a) (b)

Figure 1. The exact pulse function $g(t)$ and its Fourier transformation $\hat{g}(\omega)$. (a) $g(t)$ (b) $\hat{g}(\omega)$.



(a) (b) (c) (d)

Figure 2. The exact spatial source function $f = (f_1, f_2)$, and its compressional component f_p and shear component f_s. (a) f_1 (b) f_2 (c) f_p (d) f_s.

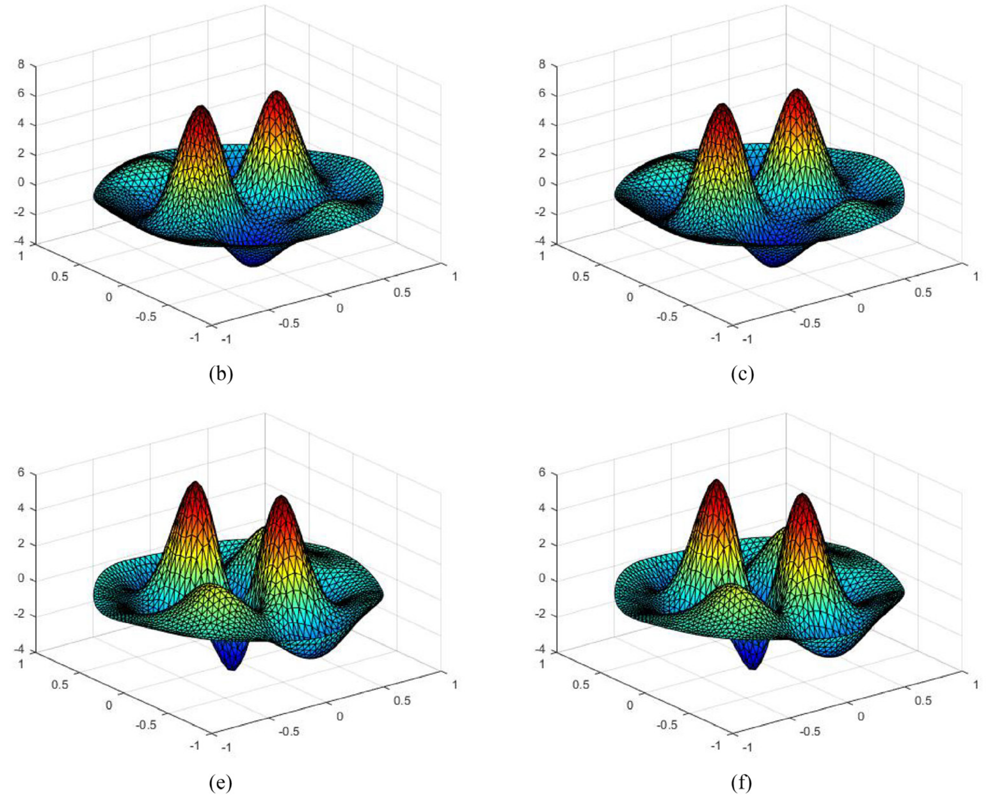


Figure 3. Reconstructions of $f = (f_1, f_2)$ from time-harmonic data at multi frequencies. Figures (b) and (e) are reconstructed from (5.7), whereas (c) and (f) are from (5.8). (b) Reconstructed f_1 (c) Reconstructed f_1 (e) Reconstructed f_2 (f) Reconstructed f_2.

where

$$\begin{aligned} f_{\rm p}(x) &= 0.3(1-3x_1)^2\exp(-9x_1^2-(3x_2+1)^2) - (0.6x_1 - 27x_1^3 - 3^5x_2^5)\exp(-9x_1^2-9x_2^2) \\ &\quad - 0.03\exp(-(3x_1+1)^2-9x_2^2), \\ f_{\rm s}(x) &= 135x_1^2x_2\exp(-9x_1^2-9x_2^2); \end{aligned}$$

see figure 2. We choose $\mu = 1$, $\lambda = 2$, $\rho = 1$ and $R = 2$. The scattering data is collected at 64 uniformly distributed points on the circle ∂B_R. The total number of iterations is set to be $L = 10$.

In the time-harmonic case, we simulate the data $\hat{U}(x, \omega)$ by solving the inhomogeneous time-harmonic Navier equation using a finite element method coupled with an exact transparent boundary condition. Then the compressional and shear parts, $\hat{u}_{\rm p}$ and $\hat{u}_{\rm s}$, are decoupled from $\hat{U}(x, \omega)$ via (5.2)–(5.4). The near-field data of twenty equally spaced frequencies from 1 to 20 are calculated. Figure 3 shows the reconstructed S_1 and S_2 from $\{\hat{U}(x, \omega_k) : |x| = R, k = 1, 2, \ldots, 20\}$, while figure 4 presents the reconstructed $f_{\rm p}$ and $f_{\rm s}$ from the counterpart of compressional and shear waves, respectively.

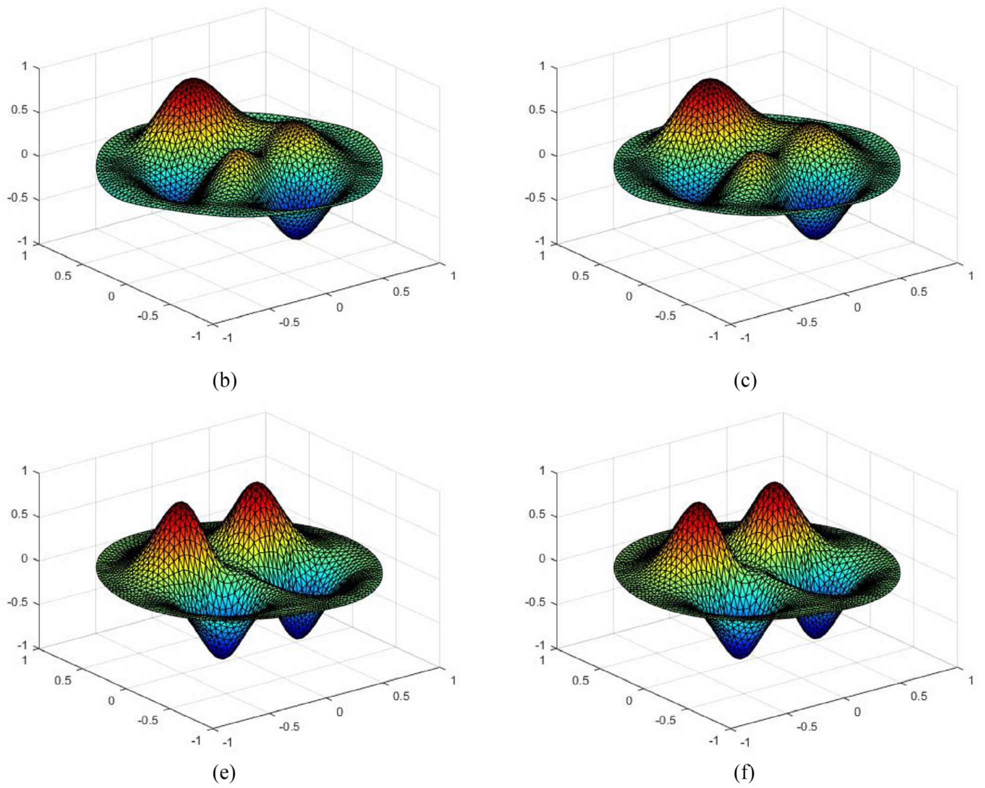


Figure 4. Reconstructions of the compressional and shear components of f. Figures (b) and (e) are reconstructed from (5.9), whereas (c) and (f) are from (5.10). (b) Reconstructed f_p (c) Reconstructed f_p (e) Reconstructed f_s (f) Reconstructed f_s.

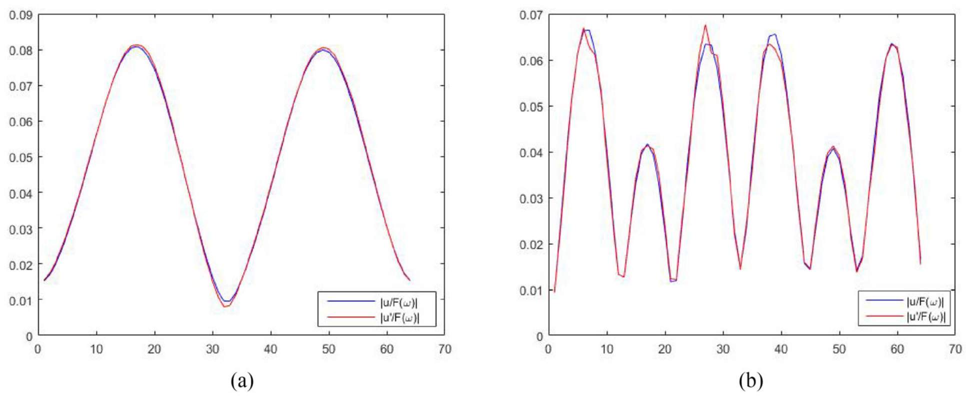


Figure 5. Comparison of the scattering data $\hat{u}(x,\omega)/\hat{g}(\omega)$ and $\hat{u}'(x,\omega)/\hat{g}(\omega)$ at $\omega = 3, 10$ obtained, respectively, by solving the time-harmonic Navier equation (blue) and by applying Fourier transform to the time-domain data (red). (a) $\omega = 3$ (b) $\omega = 10$.

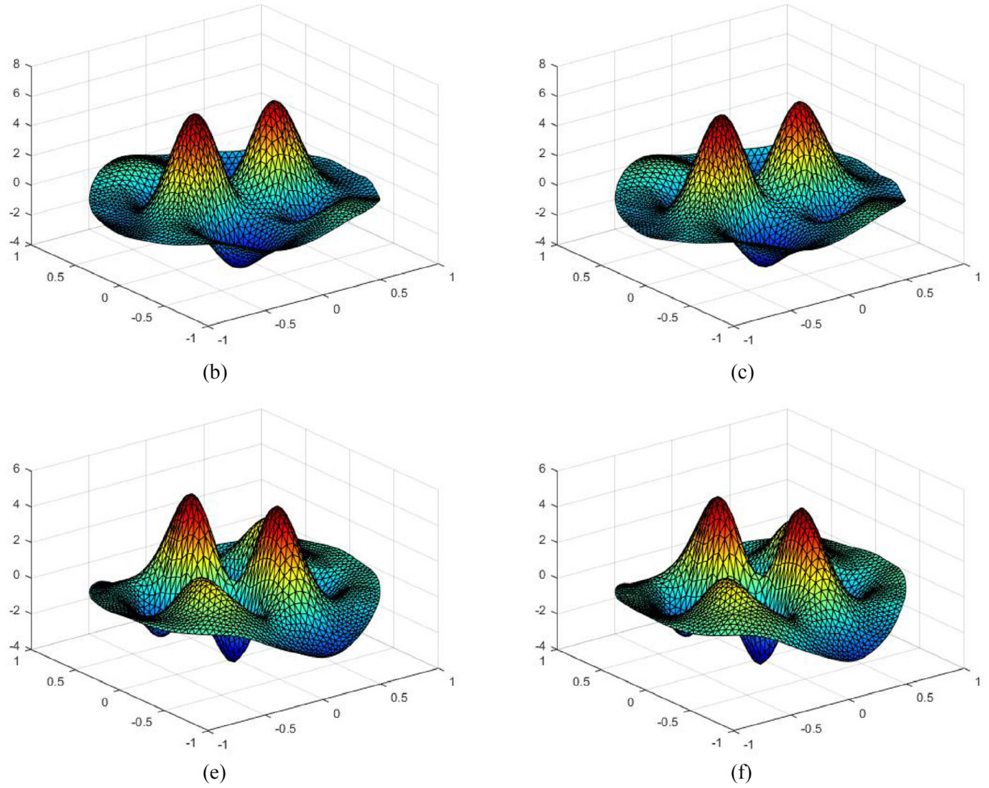


Figure 6. Reconstructions of $f = (f_1, f_2)$ from Fourier-transformed time-domain scattering data. Figures (b) and (e) are reconstructed from (5.7), whereas (c) and (f) are from (5.8). (b) Reconstructed f_1 (c) Reconstructed f_1 (e) Reconstructed f_2 (f) Reconstructed f_2.

In the time-dependent case, we first consider the numerical solution of the acoustic wave equation

$$\frac{1}{c_\alpha^2}\partial_{tt}\, u_\alpha(x,t) - \Delta u_\alpha(x,t) = g(t) f_\alpha(x), \text{ in } \quad \mathbb{R}^2 \times \mathbb{R}^+, \tag{5.12}$$

$$u_\alpha|_{t=0} = \partial_t u_\alpha|_{t=0} = 0 \quad \text{in} \quad \mathbb{R}^2, \quad \alpha = p, s. \tag{5.13}$$

To reduce the unbounded solution domain to a bounded computational domain, we use the local absorbing boundary condition

$$\partial_\nu u_\alpha + \frac{1}{c_\alpha}\partial_t u_\alpha + \frac{1}{2R} u_\alpha = 0 \quad \text{on} \quad \partial B_R.$$

Then the solutions to the acoustic scattering problem (5.12) and (5.13) are computed over B_R by using the interior penalty discontinuous Galerking method in space and the Newmark method in time. Consequently, the data $U(x,t)$ of the Lamé system are obtained through

$$U(x,t) = \frac{1}{\gamma_{\rm p}} \mathrm{grad}\, u_{\rm p}(x,t) + \frac{1}{\gamma_{\rm s}} \overrightarrow{\mathrm{curl}}\, u_{\rm s}(x,t).$$

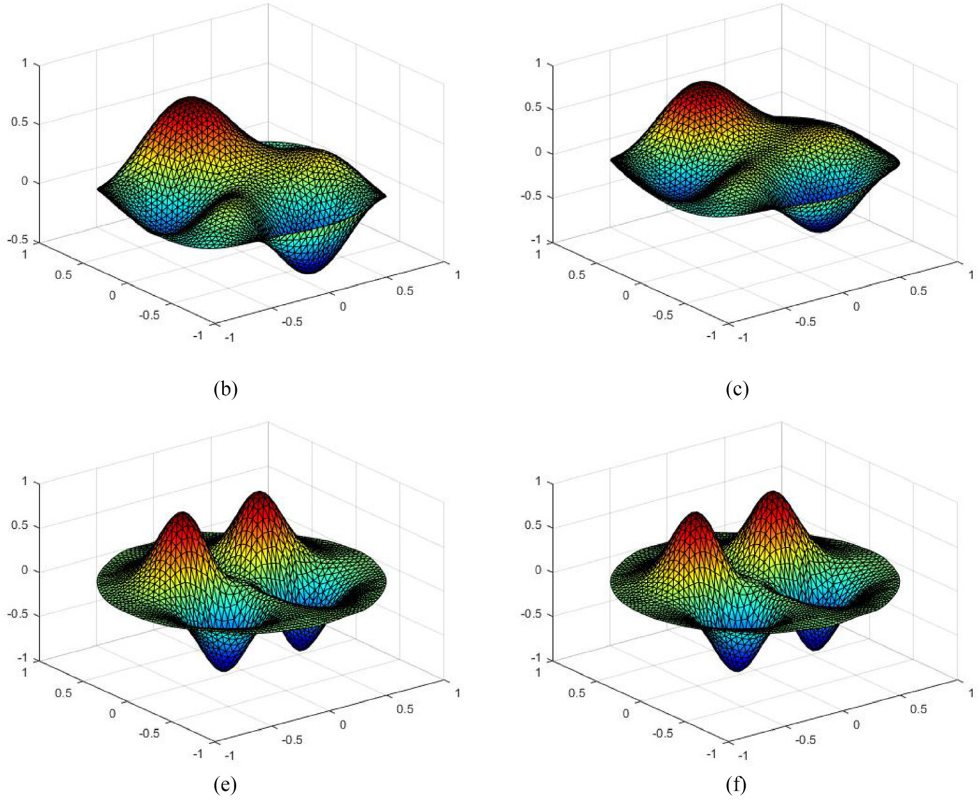
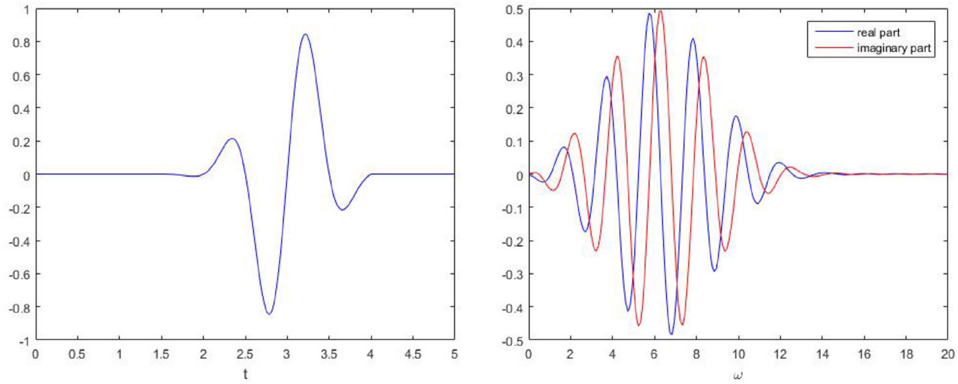


Figure 7. Reconstructions of the compressional and shear components of f from Fourier-transformed time-domain scattering data. Figures (b) and (e) are reconstructed from (5.9), whereas (c) and (f) are from (5.10). (b) Reconstructed f_p (c) Reconstructed f_p (e) Reconstructed f_s (f) Reconstructed f_s.

In our numerical examples, we collect the scattering data $U(x,t)|_{\partial B_{R_1}}$ for $t \in [0,T]$ with $T = 20 > T_0 + (R + R_1)/c_{\mathrm{s}} = 8$. In figure 5, we compare the scattering data $\hat{u}(x,\omega)|_{\partial B_R}$ at frequencies $\omega = 3$ and $\omega = 10$ obtained by solving the time-harmonic Lamé system and then by applying a Fourier transform (denoted by $\hat{u}'(x,\omega)|_{\partial B_R}$) to the time-dependent data $U(x,t)|_{\partial B_R}$. It can be seen that the data set via the Fourier transformation slightly differs from the time-harmonic data, possibly due to numerical errors in the Fourier transform and in the numerical scheme for solving time-dependent Lamé systems as well. To Fourier transform the time domain data, we use fifteen equally spaced frequencies from 1 to 15. Numerical solutions for reconstructing f and $f_\alpha, \alpha = p, s$ are presented in figures 6 and 7, respectively. We conclude from figures 3–7 that satisfactory reconstructions are obtained through the proposed Landweber iterative algorithm.

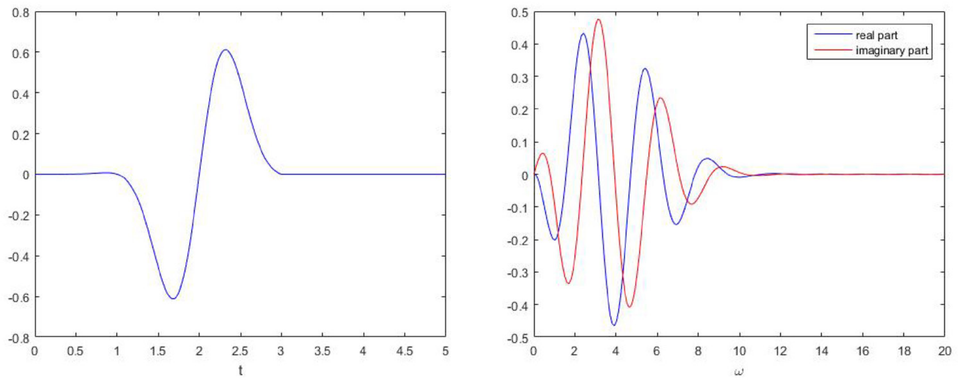
5.2. Reconstruction of temporal functions

We consider the inverse problem of reconstructing g from the wave fields $\{U(x,t) : x \in \Gamma \subset \partial B_R, t \in (0,T)$ for some $T > 0$ in three dimensions. For simplicity we choose the scalar spatial function to be the delta function, i.e. $f(x) = \delta(x)$. Then the function W (see the proof of theorem 4.4) takes the form



(a) (b)

Figure 8. The exact pulse function $g_2(t)$ and its Fourier transformation $\hat{g}_2(\omega)$. (a) $g_2(t)$ (b) $\hat{g}_2(\omega)$.



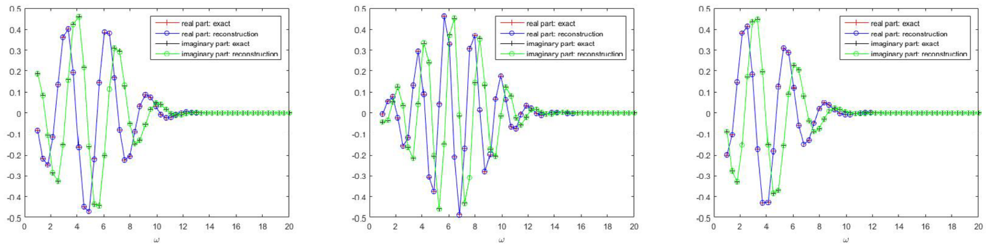
(a) (b)

Figure 9. The exact pulse function $g_3(t)$ and its Fourier transformation $\hat{g}_3(\omega)$. (a) $g_3(t)$ (b) $\hat{g}_3(\omega)$.

$$
\begin{aligned}
W(x,\omega) &= \int_{\mathbb{R}^3} \hat{G}(x-y,\omega) f(y)\,\mathrm{d}y \\
&= \hat{G}(x,\omega) \\
&= \frac{1}{\mu}\Phi_{k_s}(x)\mathbf{I} + \frac{1}{\rho\omega^2}\operatorname{grad}_x \operatorname{grad}_x^\top \left[\Phi_{k_s}(x) - \Phi_{k_p}(x)\right],
\end{aligned}
$$

where $\Phi_k(x) = \mathrm{e}^{\mathrm{i}k|x|}/(4\pi|x|)(k = k_\mathrm{p}, k_\mathrm{s})$. Hence, f is indeed not a non-radiation source for all $\omega \in \mathbb{R}^+$. In our example, we set the vector temporal function $g(t)$ to be

$$
\begin{aligned}
& g(t) = (g_1, g_2, g_3)^\top, \\
& g_1(t) = \begin{cases} \cos(1.5\pi(t-t_1))\exp(-\pi(t-t_1)^2), & t \leqslant T_1, \\ 0, & t > T_1, \end{cases} \\
& g_2(t) = \begin{cases} \sin(2\pi(t-t_2))\exp(-\pi(t-t_2)^2), & t \leqslant T_2, \\ 0, & t > T_2, \end{cases} \\
& g_3(t) = \begin{cases} \sin(\pi(t-t_3))\exp(-\pi(t-t_3)^2), & t \leqslant T_3, \\ 0, & t > T_3, \end{cases}
\end{aligned}
$$



(a) (a) (a)

Figure 10. Reconstruction of temporal functions from I_1 without noise. (a) $\hat{g}_1$ (a) $\hat{g}_2$ (a) $\hat{g}_3$

where $T_1 = 5$, $T_2 = 4$, $T_3 = 3$, $t_1 = 2$, $t_2 = 3$ and $t_3 = 2$. The function pairs $(g_1, \hat{g}_1)$, $(g_2, \hat{g}_2)$ and $(g_3, \hat{g}_3)$ are plotted in figures 1, 8 and 9, respectively. Moreover, we set $g(t) = 0$ for $t < 0$. With the choice of f and g, the forward time-domain scattering data can be expressed as $U = (u_1, u_2, u_3)$, where

$$\begin{aligned} u_i(x,t) &= \sum_{j=1}^{3} \int_0^\infty \int_{\mathbb{R}^3} G_{i,j}(x-y, t-s) f(y) g_j(s)\, \mathrm{d}x\mathrm{d}s \\ &= \sum_{j=1}^{3} \int_0^\infty G_{i,j}(x, t-s) g_j(s)\, \mathrm{d}s \\ &= \frac{1}{4\pi\rho|x|^3} \sum_{j=1}^{3} \left(\frac{x_i x_j}{c_\mathrm{p}^2} g_j(t - |x|/c_\mathrm{p}) + \frac{1}{c_\mathrm{s}^2} (\delta_{ij}|x|^2 - x_i x_j) g_j(t - |x|/c_\mathrm{s}) \right) \\ &\quad + \frac{1}{4\pi\rho|x|^3} \sum_{j=1}^{3} \left(3x_i x_j - \delta_{ij}|x|^2\right) \int_{1/c_\mathrm{p}}^{1/c_\mathrm{s}} s g_j(t-s)\, \mathrm{d}s. \end{aligned}$$

Taking the Fourier transform gives the data $\hat{U}(x, \omega_i)$ in the Fourier domain. The sampling frequencies are chosen as

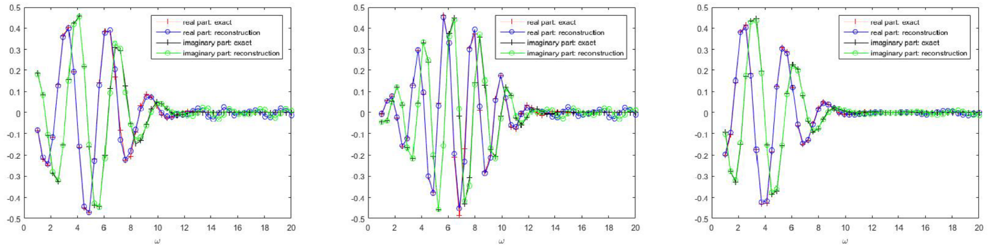
$$\omega_j = 1 + (j-1)h, \quad h = 19/49, \quad j = 1, \ldots, K, \quad K = 50.$$

Fixing $\omega_i \in \mathbb{R}^+$ $(i = 1, 2, \ldots, K)$, we can always find $x_{0,i} \in \Gamma$ such that $W(x_{0,i}, \omega_i)^{-1}$ exists and the value of the indicator

$$I_1(\omega_i) = [W(x_{0,i}, \omega_i)]^{-1} \hat{U}(x_{0,i}, \omega_i)$$

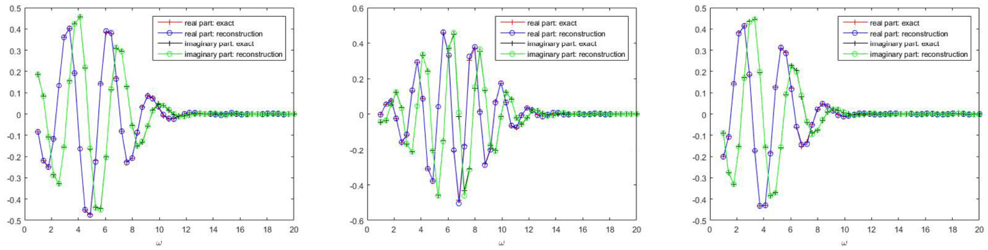
is identical to $\hat{g}(\omega_i)$. Taking the inverse Fourier transform of the indicator function $I_1(\omega)$ enables us to plot the function $t \to g_i(t)$ $(i = 1, 2, 3)$. In our tests we choose $x_{0,i} = (1, 1, 0)^\top$ uniformly in all $i = 1, 2, \ldots, K$. Numerical reconstructions of $\hat{g}_i, i = 1, 2, 3$ from the indicator I_1 are presented in figure 10.

One can readily observe that the choice of $x_{0,i}$ is not unique. Our numerics show that $\mathrm{Det}(W(x, \omega_i))$ does not vanish for almost all $x \in \partial B_R$. For $\omega_i \in \mathbb{R}^+$, we denote by $\{x_{j,i} : j = 1, 2, \ldots M\}$ a set of points lying on $|x| = R$ such that $W(x_{j,i}, \omega_i)$ is invertible for each j. To make our inversion scheme computationally stable, we can calculate $I_1(\omega_i)$ using each $x_{j,i}$ $(j = 1, 2, \ldots, M)$ and then take the average as the value of $\hat{g}(\omega_i)$. Hence, we propose another indicator function in the Fourier domain as follows



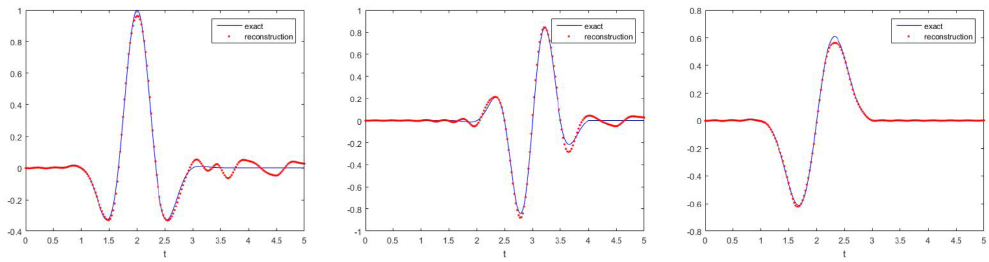
(a) (a) (a)

Figure 11. Reconstruction of temporal functions from I_1 with 30% noise. (a) $\hat{g}_1$ (a) $\hat{g}_2$ (a) $\hat{g}_3$



(a) (a) (a)

Figure 12. Reconstruction of temporal functions from I_2 with 30% noise. (a) $\hat{g}_1$ (a) $\hat{g}_2$ (a) $\hat{g}_3$



(a) (a) (a)

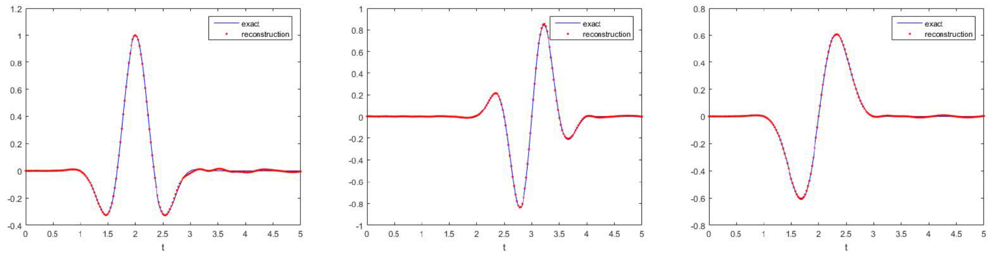
Figure 13. Reconstruction of temporal functions from $\tilde{I}_1$ with 30% noise. (a) g_1 (a) g_2 (a) g_3.

$$I_2(\omega_i) := \frac{1}{M}\sum_{j=1}^{M}[W(x_{j,i},\omega_i)]^{-1}\hat{U}(x_{j,i},\omega_i), \quad i = 1,2,\ldots,K,$$

where the time domain data $\{U(x_{j,i},t) : j = 1,2,\ldots,M, i = 1,2,\ldots,K\}$ are used. In our experiments, we make use of the boundary data equivalently distributed on $|x| = R$ and set

$$x_{j,i} = x_j = (\cos((j-1)d\theta), 1, \sin((j-1)h))^\top, \quad h = 2\pi/M, j = 1,2,\ldots,M,$$

uniformly in all $i = 1,2,\ldots,K$. Numerics show that such kind of boundary data is adequate for the choice of f and g. Next we consider reconstructions from the noised data



(a) (a) (a)

Figure 14. Reconstruction of temporal functions from $\tilde{I}_2$ with 30% noise. (a) g_1 (a) g_2 (a) g_3

$$U_\delta(x,t) = (1 + \delta\epsilon(x,t))U(x,t)$$

where $\epsilon(x,t)$ is a function whose value is random between -1 and 1, and the noise level δ is set to be 30%. We present the reconstructions of $\hat{g}_j$ $(j = 1, 2, 3)$ based on the indicators I_1 and I_2 in figures 11 and 12, respectively. Reconstructions from the inverse Fourier transform of I_j (that is, the temporal function $g(t)$) are illustrated in figures 13 and 14, where the time-domain data with 30% noise are again used. Comparing figures 11–14, one may conclude that the inversion scheme using I_2 is indeed more computationally stable than I_1.

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Appendix

Lemma A.1. *Suppose that $S \in (L^2(\mathbb{R}^3))^3$ has a compact support in B_R for some $R > 0$, then the Helmholtz decomposition of S is unique.*

Proof. Due to the Helmholtz decomposition, every $S \in (L^2(\mathbb{R}^3))^3$ admits a decomposition:

$$S = \nabla S_{\rm p} + \nabla \times S_{\rm s}, \quad \nabla \cdot S_{\rm s} = 0,$$

where

$$S_{\rm p} \in H^1(B_R), \quad S_{\rm s} \in H_{\rm curl}(B_R) := \{U : U \in L^2(B_R)^3, \mathrm{curl}\, U \in L^2(B_R)^3\}$$

also have compact support in B_R. Suppose that S admits another orthogonal decomposition $S = \nabla S'_{\rm p} + \nabla \times S'_{\rm s}, \nabla \cdot S'_{\rm s} = 0$. Then we have

$$\nabla(S_{\rm p} - S'_{\rm p}) + \nabla \times (S_{\rm s} - S'_{\rm s}) = 0. \tag{A.1}$$

Taking the divergence of both sides of (A.1) gives $\Delta(S_{\rm p}-S'_{\rm p})=0$ in B_R, i.e. $S_{\rm p}-S'_{\rm p}$ is harmonic over B_R. Note that $S_{\rm p}-S'_{\rm p}=0$ on ∂B_R. Applying the maximum principle for harmonic functions yields $S_{\rm p}=S'_{\rm p}$ in B_R. On the other hand, applying $\nabla\times$ to both sides of (A.1) we obtain

$$0=\nabla\times(\nabla\times\ (S_{\rm s}-S'_{\rm s}))=\nabla(\nabla\cdot\ (S_{\rm s}-S'_{\rm s}))-\Delta\,(S_{\rm s}-S'_{\rm s})=-\Delta\,(S_{\rm s}-S'_{\rm s}).$$

Then the relation $S_{\rm s}=S'_{\rm s}$ in B_R can be proved analogously. This completes the proof. □

In the following lemma, the notation $\mathbf{I}_{n\times n}$ denotes the unit matrix in $\mathbb{R}^{n\times n}$ for $n\geqslant 2$.

Lemma A.2. *Let $\xi=(\xi_1,\dots,\xi_n)^\top\in\mathbb{R}^{n\times 1}$ and $A(\xi)=\mu|\xi|^2\mathbf{I}_{n\times n}+(\lambda+\mu)\xi\otimes\xi\in\mathbb{R}^{n\times n}$. Then the eigenvalues τ_j ($j=1,2,\dots,n$) of $A(\xi)$ are given by*

$$\tau_1=(\lambda+2\mu)|\xi|^2,\quad \tau_2=\ldots=\tau_n=\mu\,|\xi|^2.$$

Proof. Set $\tilde{A}=A-\tau\mathbf{I}_{n\times n}$. We may rewrite $\tilde{A}$ in the form $\tilde{A}=B+VV^\top$, where

$$B=(\mu|\xi|^2-\tau)\,\mathbf{I}_{n\times n},\quad V=\sqrt{\lambda+\mu}\,\xi.$$

Straightforward calculations show that

$$\begin{aligned}\mathrm{Det}(\tilde{A})&=\mathrm{Det}(B+VV^\top)\\&=(1+V^\top B^{-1}V)\,\mathrm{Det}(B)\\&=\left(1+\frac{(\lambda+\mu)|\xi|^2}{\mu|\xi|^2-\tau}\right)(\mu|\xi|^2-\tau)^n\\&=[(\lambda+2\mu)|\xi|^2-\tau](\mu|\xi|^2-\tau)^{n-1},\end{aligned}$$

which implies the eigenvalues of A. □

Lemma A.3 (Grownwall-type inequality.). *Let $T>0$ and $u\in L^2(0,T)$ be non-negative and fulfill, for almost every $t\in(0,T)$, the inequality*

$$u(t)\leqslant a(t)+\int_0^t b(s)\,u(s)\,\mathrm{d}s,\tag{A.2}$$

where $a\in L^2(0,T)$ and $b\in\mathcal{C}([0,T])$ are two non-negative functions. Then, for almost every $t\in(0,T)$, we have

$$u(t)\leqslant a(t)+\int_0^t a(s)b(s)\mathrm{e}^{\int_s^t b(\tau)d\tau}\mathrm{d}s.\tag{A.3}$$

Proof. We consider Y defined, for almost every $t\in(0,T)$, by

$$Y(t):=\mathrm{e}^{-\int_0^t b(s)\mathrm{d}s}\int_0^t b(s)u(s)\mathrm{d}s$$

and we remark that $Y\in H^1(0,T)$ and satisfies $Y(0)=0$. Then, for almost every $t\in(0,T)$, we find

$$Y'(t) = b(t)u(t)\mathrm{e}^{-\int_0^t b(s)\mathrm{d}s} - b(t)\mathrm{e}^{-\int_0^t b(s)\mathrm{d}s} \int_0^t b(s)u(s)\mathrm{d}s.$$

On the other hand, in view of (A.2), for almost every $t \in (0, T)$, we get

$$\int_0^t b(s)u(s)\mathrm{d}s \geqslant u(t) - a(t)$$

and we deduce that

$$Y'(t) \leqslant a(t)b(t)\mathrm{e}^{-\int_0^t b(s)\mathrm{d}s}.$$

Integrating on both side of this inequality we get

$$\int_0^t Y'(s)\mathrm{d}s \leqslant \int_0^t a(s)b(s)\mathrm{e}^{-\int_0^s b(\tau)d\tau}\mathrm{d}s, \quad t \in (0, T).$$

On the other hand, since $Y \in H^1(0, T)$ and satisfies $Y(0) = 0$, by density one can check that $Y(t) = \int_0^t Y'(s)\mathrm{d}s$, which implies that

$$\mathrm{e}^{-\int_0^t b(s)\mathrm{d}s} \int_0^t b(s)u(s)\mathrm{d}s \leqslant \int_0^t a(s)b(s)\mathrm{e}^{-\int_0^s b(\tau)d\tau}\mathrm{d}s$$

and by the same way, for almost every $t \in (0, T)$, the following inequality

$$\int_0^t b(s)u(s)\mathrm{d}s \leqslant \mathrm{e}^{\int_0^t b(s)\mathrm{d}s}\left(\int_0^t a(s)b(s)\mathrm{e}^{-\int_0^s b(\tau)d\tau}\mathrm{d}s\right) = \int_0^t a(s)b(s)\mathrm{e}^{\int_s^t b(\tau)d\tau}\mathrm{d}s.$$

Finally, applying again (A.2), for almost every $t \in (0, T)$, we find

$$u(t) \leqslant a(t) + \int_0^t b(s)u(s)\mathrm{d}s \leqslant a(t) + \int_0^t a(s)b(s)\mathrm{e}^{\int_s^t b(\tau)d\tau}\mathrm{d}s.$$

This proves (A.3). □

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