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Inverse medium scattering from periodic structures with fixed-direction incoming waves

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Abstract

This paper is concerned with inverse time-harmonic acoustic and electromagnetic scattering from an infinite biperiodic medium (diffraction grating) in three dimensions. In the acoustic case, we prove that the near-field data of fixed-direction plane waves incited at multiple frequencies uniquely determine a refractive index function which depends on two variables. An analogous uniqueness result holds for time-harmonic Maxwell's system if the inhomogeneity is periodic in one direction and remains invariant along the other two directions. Uniqueness for recovering (non-periodic) compactly supported contrast functions are also presented.

Keywords: uniqueness, inverse scattering, multi-frequency data, periodic media, acoustic waves, electromagnetic waves

(Some figures may appear in colour only in the online journal)

1. Introduction

Scattering theory in periodic structures (diffraction gratings) has attracted a lot of attention since Lord Rayleigh's original work [27]. It has many applications in micro-optics, radar imaging, nondestructive testing and so on. We refer to [5, 25, 30] for historical remarks and details of these applications.

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Assume a time-harmonic acoustic or electromagnetic plane wave is incident onto a bi-periodic inhomogeneous medium with finite height in $\mathbb{R}^3$. The media above and below this bi-periodic layer are assumed to be homogeneous and isotropic with the same physical properties. The direct (forward) scattering problem is, given the incident field and the bi-periodic refractive index, to determine the scattered field distribution, whereas the inverse scattering problem is to recover the refractive index from knowledge of the incident waves and their corresponding measured scattered fields. In this paper, we suppose the incident direction is fixed and the scattered fields are generated at multiple frequencies.

Imaging a general periodic interface has been well studied by using fixed-frequency incoming waves with the same quasiperiodic parameter. Such incident waves include both plane wave modes with different directions and evanescent surface wave modes. We refer to [3, 20, 21, 23, 28, 32] for uniqueness results and an inversion algorithm (e.g. sampling-type approaches or optimization-based iterative schemes) in acoustics and to [19, 23, 33] for the corresponding references in electromagnetism. However, the inverse medium scattering problem of recovering refractive indices have not been fully analyzed yet, due to its severe ill-posedness and high nonlinearity; see [6]. A limited number of papers are devoted to recovering periodic inhomogenities, in most of which scattering interfaces are given by constant functions. In two dimensions, it was shown in [22, 29] that a periodic potential function depending on one direction can be uniquely determined by all quasiperiodic acoustic incident waves at the fixed frequency. An analogous uniqueness result applies to Maxwell equations if the unknown refractive index changes periodically in one-direction only [18, 33]. A continuation technique with multiple wave numbers is implemented in [34], motivated by the linear recursive method for reconstructing a compactly supported contrast function at a fixed frequency [8] and at multiple frequencies in [9]. We also refer to [12, 15, 31] where inverse coefficient problems for gratings and waveguides were treated using the data of Dirichlet-to-Neumann map or outgoing propagating modes incited by all quasi-periodic incoming wave modes with the same parameter for all frequencies. To the best of our knowledge, uniqueness with the fixed-direction multi-frequency near-field data still remains open, and this is the concern of the present paper.

Compared with recent efforts made for (linear) inverse source problems (see e.g. [7]), the inverse medium scattering problems are typically non-linear and thus challenging. To overcome the nonlinearity, we first derive an orthogonal identity (see (2.12) and (3.11)) from the coincidence of near-field data for a band of frequencies and by making use of the analyticity of the wave fields with respect to frequencies. Then we analyze the limit of this orthogonal relation as the wavenumber tends to zero. The leading term in the asymptotics shows that the contrast function is orthogonal to any quasiperiodic harmonic function. This allows us to identify a refractive index which depends on two variables in acoustics. In the Maxwell case, the potential function is required to depend on one variable only. Such an idea has been used in [26] for determining the inhomogeneous media around an unknown bounded sound-soft obstacle in two dimensions from the far-field data of multiple incident directions and frequencies.

The remaining part of this paper is organized as follows. In the next section 2, we shall formulate direct and inverse acoustic scattering from penetrable periodic structures and review the solvability results for forward scattering. Uniqueness to inverse medium scattering will be presented in section 2.3. These results will be carried over to Maxwell equations in section 3. As a by-product, we also show uniqueness in recovering a compactly supported contrast function under proper *a priori* assumptions.

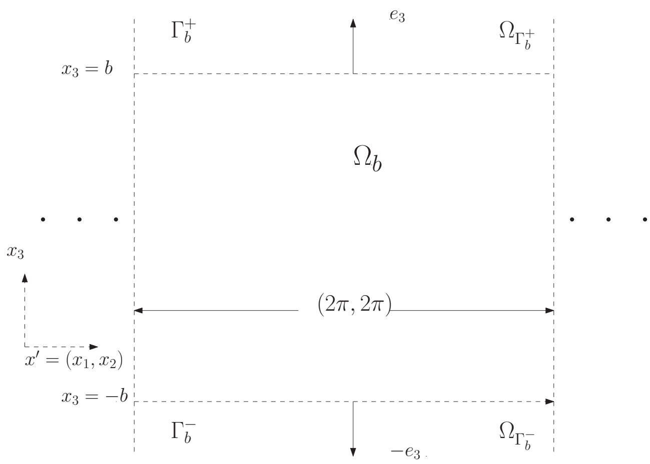

Figure 1. Problem geometry. $\Omega_{\Gamma_b^+} := \{x : x_3 > b\}$ and $\Omega_{\Gamma_b^-} := \{x : x_3 < -b\}$.

2. Inverse acoustic scattering

Let us first specify the geometric settings for both acoustic and electromagnetic scattering problems in $\mathbb{R}^3$. Write $x = (x', x_3) \in \mathbb{R}^3$, where $x' = (x_1, x_2) \in \mathbb{R}^2$. Denote by $S_b = \{(x', x_3) \in \mathbb{R}^3 : -b < x_3 < b\} = \mathbb{R}^2 \times (-b, b)$ an infinitely long slab with the height $2b$, where $b > 0$ is a constant. We suppose that the region S_b is filled with a biperiodic medium which can be characterized by the refractive index function $q(x)$, and that an incoming plane wave is incident onto S_b from above. For simplicity, we assume that q is 2π-periodic in both x_1- and x_2-directions. Due to the periodicity of the structure and quasi-periodicity of the incident wave, our scattering problems can be restricted to a single period cell $\Omega_b = (0, 2\pi) \times (0, 2\pi) \times (-b, b)$; see figure 1 and the problem formulation in sections 2.1 and 3.1 for more details. Let $\mathbb{R}^+ = \{s \in \mathbb{R} : s \geqslant 0\}$ and $\mathbb{C}^+ = \{z_1 + \mathrm{i}z_2 : z_1, z_2 \in \mathbb{R}, z_2 \geqslant 0\}$, where $i = \sqrt{-1}$. Throughout the paper we denote the two-dimensional flat surfaces

$$\Gamma_b^\pm := \{x : x' \in (0, 2\pi) \times (0, 2\pi), x_3 = \pm b\}, \quad b > 0.$$

2.1. Problem formulation for acoustic waves

Let $u^{\mathrm{inc}}(x) = \mathrm{e}^{\mathrm{i}\kappa x \cdot d}$ be a plane wave incident onto S_b, where

$$d := d(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, -\cos\theta)$$

is the incident direction, $\kappa > 0$ is the wave number, and $0 \leqslant \theta < \pi/2,\ 0 \leqslant \varphi < 2\pi$ are the latitudinal and longitudinal incident angles, respectively. Evidently, the incident field is governed by the Helmholtz equation

$$\triangle u^{\mathrm{inc}}(x) + \kappa^2 u^{\mathrm{inc}}(x) = 0 \quad \text{in } \mathbb{R}^3. \tag{2.1}$$

Set $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ with $\alpha_1 := \kappa\sin\theta\cos\varphi, \alpha_2 := \kappa\sin\theta\sin\varphi$. Then the incident wave u^{inc} is α-quasiperiodic in the following sense.

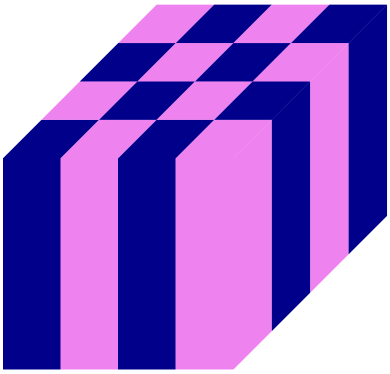

Figure 2. A bi-periodic medium in $x' = (x_1, x_2)$ which remains invariant in $x_3 \in (-b, b)$.

Definition 2.1. A function $u(x)$ is called α-quasiperiodic in x' if the function $\mathrm{e}^{-\mathrm{i}\alpha\cdot x'}u(x', x_3)$ is 2π-biperiodic with respect to the x' variable. Equivalently, it holds that $u(x' + 2\pi n, x_3) = \mathrm{e}^{\mathrm{i}2\pi\alpha\cdot n}u(x', x_3)$ for all $n = (n_1, n_2) \in \mathbb{Z}^2$ if u is α-quasiperiodic.

In this section, the refractive index function of an acoustically periodic medium is supposed to be $q(x) \in L^2_{loc}(\mathbb{R}^3)$. With some normalization we suppose that q takes the form

$$q(x) = \begin{cases} 1 & \text{if} \quad x_3 > b, \\ q(x') & \text{if} \quad x \in S_b, \\ 1 & \text{if} \quad x_3 < -b, \end{cases} \tag{2.2}$$

which implies that the media above and below S_b are the same. We want to emphasize that the medium inside S_b does not depend on x_3 and is 2π-biperiodic in x' (that is, $q(x') = q(x' + 2\pi n)$ for all $n \in \mathbb{Z}^2$); see figure 2 for an illustration of such bi-periodic media.

The total wave u is assumed to be governed by the Helmholtz equation

$$\triangle u(x) + \kappa^2 q(x)u(x) = 0 \quad \text{in } \mathbb{R}^3. \tag{2.3}$$

Thanks to the periodicity of q, the scattered field $u^{\mathrm{sc}} := u - u^{\mathrm{inc}}$ in $x_3 > b$ and the transmitted wave field u in $x_3 < -b$ are also α-quasiperiodic. In this paper, they are called bounded outgoing waves as $x_3 \to \pm\infty$, and they can be expanded into the Rayleigh series

$$\begin{aligned} u^{\mathrm{sc}}(x) &= \sum_{n\in\mathbb{Z}^2} A_n^+ \mathrm{e}^{\mathrm{i}\alpha_n\cdot x' + \mathrm{i}\beta_n x_3}, \quad x_3 > b, \\ u(x) &= \sum_{n\in\mathbb{Z}^2} A_n^- \mathrm{e}^{\mathrm{i}\alpha_n\cdot x' - \mathrm{i}\beta_n x_3}, \quad x_3 < -b, \end{aligned} \tag{2.4}$$

where $A_n^\pm := A_n^\pm(k) \in \mathbb{C}$, $\alpha_n := n + \alpha$ and $\beta_n^2 = \kappa^2 - |\alpha_n|^2$ with $\mathrm{Im}\beta_n \geqslant 0$ if $|\alpha_n|^2 > \kappa^2$.

Given u^{inc} and q, the direct (forward) scattering is to determine the scattered field in $\Omega_{\Gamma_b^+}$ and the transmitted field in $\Omega_{\Gamma_b^-}$. The inverse medium scattering problem considered in this paper is to recover $q|_{\Omega_b}$ from the multi-frequency near-field data in a single periodic cell

$$\{u(x; k) : x \in \Gamma_h^+ \cup \Gamma_h^-, k \in I\}, \quad \text{for some} \quad h > b,$$

incited by one direction and an interval of frequencies $k \in I \subset \mathbb{R}^+$.

2.2. Solvability of direct scattering

In this section, we review the variational approach to the direct scattering problem and discuss the analyticity of the solution with respect to the wave number. Following the lines of [1, 10, 14, 20], we introduce transparent boundary conditions on $\Gamma_b^\pm$ which are equivalent to the bounded outgoing radiation conditions in (2.4).

Quasiperiodic Sobolev spaces are introduced as follows. For an α-quasiperiodic function $v(x')$ on $\Gamma_b^\pm$ of the form

$$v(x') = \sum_{n\in\mathbb{Z}^2} v_n e^{i\alpha_n\cdot x'}, \quad v_n \in \mathbb{C},$$

we define the Sobolev spaces

$$H^s_\alpha(\Gamma_b^\pm) = \Big\{ v \in L^2(\Gamma_b^\pm) : \sum_{n\in\mathbb{Z}^2}(1+|\alpha_n|^2)^s|v_n|^2 < \infty \Big\}, \quad s \in \mathbb{R},$$

which are equipped with the norm

$$\|v\|_{s,\Gamma_b^\pm} = \Big(4\pi^2 \sum_{n\in\mathbb{Z}^2}(1+|\alpha_n|^2)^s|v_n|^2 \Big)^{1/2}.$$

We then define boundary operators

$$T_\pm v = \sum_{n\in\mathbb{Z}^2} i\beta_n v_n e^{i\alpha_n\cdot x'}, \quad v \in H^s_\alpha(\Gamma_b^\pm).$$

This implies that

$$T_+(u^{sc}(x',b)) = \partial_3 u^{sc}(x',b), \qquad T_-(u(x',-b)) = -\partial_3 u(x',-b)$$

for bounded outgoing waves u^{sc} and u given in (2.4). Here and in the following, we use the notation $\partial_j = \partial/\partial x_j$ for $j = 1, 2, 3$. One can prove that $T_\pm : H_\alpha^{1/2}(\Gamma_b^\pm) \to H_\alpha^{-1/2}(\Gamma_b^\pm)$ are continuous, satisfying

$$-\mathrm{Re}\, T_\pm \geqslant 0, \quad \mathrm{Im}\, T_\pm > 0.$$

With the help of above boundary operators, we impose transparent boundary conditions on $\Gamma_b^\pm$ as follows:

$$\partial_\nu u = T_+ u + f \quad \text{on } \Gamma_b^+, \quad f = -2i\beta e^{i(\alpha\cdot x' - \beta b)}, \tag{2.5}$$

$$\partial_\nu u = T_- u \quad \text{on } \Gamma_b^-, \tag{2.6}$$

where ν is the unit normal vector directed into the exterior of Ω_b.

The equivalent variational formulation will be posed over the α-quasiperiodic space

$$H^1_\alpha(\Omega_b) := \{u|_{\Omega_b} : u \in H^1(\Omega_b),\ u \text{ is } \alpha\text{-quasi-biperiodic in } S_b\}.$$

Multiplying the complex conjugate of a test function $v \in H^1_\alpha(\Omega_b)$, integrating over Ω_b, and using integration by parts, we arrive at the variational form: find $u \in H^1_\alpha(\Omega_b)$ such that

$$a(u, v) = \langle f, v\rangle_{\Gamma_b^+} \quad \text{for all } v \in H^1_\alpha(\Omega_b), \tag{2.7}$$

with the sesquilinear form

$$a(u,v) := \int_{\Omega_b} [(\nabla u \cdot \nabla \bar{v}) - \kappa^2 q u \bar{v}]\, \mathrm{d}x - \int_{\Gamma_b^+} (T_+ u)\bar{v}\, \mathrm{d}s - \int_{\Gamma_b^-} (T_- u)\bar{v}\, \mathrm{d}s,$$

and the linear functional

$$\langle f, v\rangle_{\Gamma_b^+} := \int_{\Gamma_b^+} f \bar{v} \mathrm{d}s.$$

To analyze the dependance of u on the wavenumber κ, we need an energy space independent of κ. Obviously, $u(x)\exp(-\mathrm{i}\alpha\cdot x'), v(x)\exp(-\mathrm{i}\alpha\cdot x')$ are 2π-periodic functions with respect to x_1 and x_2, and they belong to the periodic space $H_p^1(\Omega_b)$, which is defined as the space $H_\alpha^1(\Omega_b)$ with $\alpha = (0,0)$. Then, the variational formulation (2.7) can be rewritten as the operator equation (see e.g. [14, 25, 28, 29])

$$I + A^{(\kappa)} = F^{(\kappa)},$$

where I is the identify operator and $A^{(\kappa)}, F^{(\kappa)}$: $H_p^1(\Omega_b) \to H_p^1(\Omega_b)^*$ depend on $\kappa \in \mathbb{C}$ analytically. Here $H_p^1(\Omega_b)^*$ stands for the dual space of $H_p^1(\Omega_b)$. Moreover, the operators $A^{(\kappa)}$ are compact for all $\kappa > 0$. Since our original scattering problem admits a unique solution if $\operatorname{Im}\kappa^2 > 0$ (which means energy absorption in the background medium) or $\kappa^2 > 0$ is sufficiently small (see e.g. [10, 14, 28]), by analytic Fredholm theory [11], the operator $I + A^{(\kappa)}$ has a bounded inverse for all $\kappa \in \mathbb{C}\backslash\mathcal{D}$, where $\mathcal{D} \subset \mathbb{R}$ is a discrete subset with the only accumulating point at infinity. We write $u = u^{(\kappa)}$ to indicate the dependence of u on the wavenumber κ. The solvability results for forward scattering are summarized as follows.

Theorem 2.2. *The forward scattering problem (2.2)–(2.4) admits a unique solution $u^{(\kappa)} \in H_\alpha^1(\Omega_h)$ for any $h > b$ and $\kappa \in \mathbb{C}\backslash\mathcal{D}$, where where $\mathcal{D} \subset \mathbb{R}$ is a discrete subset with the only accumulating point at infinity. Moreover, the solution $u^{(\kappa)}$ depends on $\kappa \in \mathbb{C}\backslash\mathcal{D}$ analytically.*

Remark 2.3. For real-valued refractive index functions, uniqueness to the direct scattering problem can be proved for any $\kappa^2 \in \mathbb{R}^+$, if q possesses certain monotone properties along the x_3-direction; see [14, 22]. In our case, since the media above and below S_b are identical, it is necessary to exclude a discrete set of resonance frequencies for which guided wave modes might exist in the slab S_b.

2.3. Uniqueness

In this subsection, we verify uniqueness of the inverse medium scattering problem. It is supposed that the value of b in the problem geometry is the prior information that we know in advance. Let the incident direction d be fixed. For notational convenience we write

$$u_h^{(\kappa)}(x') = u(x', x_3; \kappa)|_{x_3=h}, \quad u_{-h}^{(\kappa)}(x') = u(x', x_3; \kappa)|_{x_3=-h}.$$

The main result in the acoustic case is stated below.

Theorem 2.4. *Let $I \subset \mathbb{R}$ be an open interval and let $q \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ be given as in (2.2). Then the multiple-frequency data $\{u_{\pm h}^{(\kappa)},\ \kappa \in \mathrm{I}\}$ for some $h > b$ uniquely determine q.*

Proof. Suppose q and $\tilde{q}$ have corresponding solutions u, $\tilde{u}$ that give rise to the same data $u_{\pm h}^{(\kappa)} = \tilde{u}_{\pm h}^{(\kappa)}$ for all $\kappa \in \mathrm{I}$. Denote by $\tilde{q}$ the inhomogeneity function corresponding to $\tilde{u}$, which takes the same form as in (2.2). Since the solutions $u^{(\kappa)}$ and $\tilde{u}^{(\kappa)}$ are analytic functions with respect to the variable $\kappa \in \mathbb{C}\backslash\mathcal{D}$, we have that $u_{\pm h}^{(\kappa)} = \tilde{u}_{\pm h}^{(\kappa)}$ for all $\kappa \in \mathbb{R}\backslash\mathcal{D}$. Therefore, from the transparent boundary conditions (2.5) and (2.6) we have $T_{\pm}u_{\pm h}^{(\kappa)} = T_{\pm}\tilde{u}_{\pm h}^{(\kappa)}$. Hence, it holds that $\partial_3 u^{(\kappa)} = \partial_3 \tilde{u}^{(\kappa)}$ on $\Gamma_h^{\pm}$ for all $\kappa \in \mathbb{R}\backslash\mathcal{D}$.

We need to show that $q(x') = \tilde{q}(x')$. By rewriting (2.3) we obtain

$$\triangle u^{(\kappa)} + \kappa^2 u^{(\kappa)} = \kappa^2(1-q)u^{(\kappa)}, \tag{2.8}$$

$$\triangle \tilde{u}^{(\kappa)} + \kappa^2 \tilde{u}^{(\kappa)} = \kappa^2(1-\tilde{q})\tilde{u}^{(\kappa)}. \tag{2.9}$$

Setting $w^{(\kappa)} = u^{(\kappa)} - \tilde{u}^{(\kappa)}$ and subtracting (2.8) from (2.9) yield

$$\begin{cases} \triangle w^{(\kappa)} + \kappa^2 w^{(\kappa)} = \kappa^2((1-q)u^{(\kappa)} - (1-\tilde{q})\tilde{u}^{(\kappa)}) & \text{in } \mathbb{R}^3, \\ w^{(\kappa)} = \partial_3 w^{(\kappa)} = 0 & \text{on } \Gamma_h^{\pm}. \end{cases} \tag{2.10}$$

Let $v^{(\kappa)}$ be an α-quasi-periodic solution of

$$\triangle v^{(\kappa)} + \kappa^2 v^{(\kappa)} = 0 \quad \text{in } \Omega_h.$$

Explicitly, $v^{(\kappa)}$ has the form

$$v^{(\kappa)}(x) = \sum_{n\in\mathbb{Z}^2} A_n \mathrm{e}^{\mathrm{i}\alpha_n\cdot x' + \mathrm{i}\beta_n x_3} + B_n \mathrm{e}^{\mathrm{i}\alpha_n\cdot x' - \mathrm{i}\beta_n x_3}, \tag{2.11}$$

where A_n and B_n are complex numbers. Multiplying (2.10) by $\bar{v}^{(\kappa)}$ and using integration by parts yield

$$\kappa^2 \int_{\Omega_h} \Big((1-q)u^{(\kappa)} - (1-\tilde{q})\tilde{u}^{(\kappa)}\Big)\bar{v}^{(\kappa)}\mathrm{d}x = 0,$$

or equivalently,

$$\int_{\Omega_b} \Big((1-q)u^{(\kappa)} - (1-\tilde{q})\tilde{u}^{(\kappa)}\Big)\bar{v}^{(\kappa)}\mathrm{d}x = 0, \tag{2.12}$$

for all $\kappa \in \mathbb{R}\backslash\mathcal{D}$. This is the orthogonal identity derived from the coincidence of multl-frequency data measured on $\Gamma_h^{\pm}$. In particular, (2.12) remains valid for small wavenumbers, since the discrete set $\mathcal{D}$ keeps away from zero. The Lipmann–Schwinger integral equation in our case can be written as

$$u^{(\kappa)}(x) = \mathrm{e}^{\mathrm{i}\kappa x\cdot d} - \kappa^2 \int_{\Omega_b} \Phi_\alpha^{(\kappa)}(x,y)(1-q(y))u^{(\kappa)}(y)\,\mathrm{d}y, \tag{2.13}$$

where $\Phi_\alpha^{(\kappa)}(x,y)$ is the α-quasiperiodic Green's function to the Helmholtz equation, given by

$$\Phi_\alpha^{(\kappa)}(x,y) := \sum_{n\in\mathbb{Z}^2} \frac{1}{8\pi^2\beta_n}\mathrm{e}^{\mathrm{i}(\alpha_n\cdot(x'-y')+\beta_n|x_3-y_3|)}, \quad x \neq y + (2n\pi, 0).$$

By Neumann's series, $||u^{(\kappa)}||_{H^1_\alpha(\Omega_b)}$ is uniformly bounded for small wavenumbers. Hence, letting $\kappa \to 0$ in (2.13) we obtain $u^{(0)}(x) := \lim_{\kappa\to 0} u^{(\kappa)}(x) = 1$ in $L^2_\alpha(\Omega_b)$. Similarly we also have $\tilde{u}^{(0)}(x) := \lim_{\kappa\to 0} \tilde{u}^{(\kappa)}(x) = 1$. In addition, by (2.11) the limiting function $v^{(0)}(x) := \lim_{\kappa\to 0} v^{(\kappa)}(x)$ is of the form

$$v^{(0)}(x) = \sum_{n\in\mathbb{Z}^2} A_n e^{in\cdot x' - |n|x_3} + B_n e^{in\cdot x' + |n|x_3}, \quad x \in \Omega_b, \tag{2.14}$$

and the convergence is understood to be pointwise over $\bar{\Omega}_b$. Obviously, $v^{(0)}(x)$ is 2π-biperiodic in the variable x', satisfying

$$\triangle v^{(0)} = 0.$$

Taking $\kappa \to 0$ in the orthogonal identify (2.12) we obtain

$$\int_{\Omega_b} (q - \tilde{q})(x)\, \bar{v}^{(0)}(x)\, dx = 0 \tag{2.15}$$

for all harmonic functions $v^{(0)}$ of the form (2.14). Since $q - \tilde{q}$ is 2π-biperiodic with respect to the variable x' in S_b, it admits the Fourier series expansion

$$q(x) - \tilde{q}(x) = \sum_{n\in\mathbb{Z}^2} a_n e^{in\cdot x'}, \quad x \in \Omega_b,$$

where $a_n \in \mathbb{C}$ are the Fourier coefficients. By choosing $v^{(0)}(x) = e^{im\cdot x' + |m|x_3}$ for $m = (m_1, m_2) \in \mathbb{Z}^2$ the identity (2.15) becomes

$$0 = \int_{-b}^{b} dx_3 \int_0^{2\pi}\int_0^{2\pi} \Big(\sum_{n\in\mathbb{Z}^2} a_n e^{in\cdot x'}\Big) e^{-im\cdot x' + |m|x_3} dx_1 dx_2 = 4\pi^2 a_m \int_{-b}^{b} e^{|m|x_3} dx_3$$

implying that $a_m = 0$. By the arbitrariness of $m \in \mathbb{Z}^2$, we get the vanishing of a_n for all $n \in \mathbb{Z}^2$, which proves $q = \tilde{q}$. □

2.4. Remark

Similar uniqueness result can be obtained for bounded penetrable scatterers in $\mathbb{R}^3$. Let $\Omega = \tilde{\Omega} \times (-b, b)$ with $\tilde{\Omega} \subset \mathbb{R}^2$, be a bounded penetrable scatterer. Suppose that $q|_\Omega = q(x') \in L^2(\Omega)$ for $x \in \Omega$ and $q(x) = 1$ for $x \in \mathbb{R}^3 \backslash \overline{\Omega}$. Moreover, the value of b is assumed to be given in advance. For bounded scatterers, the scattered field u^{sc} in $\mathbb{R}^3 \backslash \overline{\Omega}$ is required to satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} r(\partial_r u^{sc} - i\kappa u^{sc}) = 0, \quad r = |x|,$$

uniformly for all directions $x/|x|$. Hence, u satisfies the classical Lipmann–Schwinger integral equation

$$u(x) = u^{inc}(x) - \kappa^2 \int_\Omega G(x, y)(1 - q)(y)\, u(y) dy, \quad x \in \mathbb{R}^3,$$

where

$$G(x,y) = \frac{1}{4\pi}\frac{e^{i\kappa|x-y|}}{|x-y|}, \quad x \neq y,$$

is the free-space fundamental solution to the Helmholtz equation $(\Delta + k^2)u = 0$ in $\mathbb{R}^3$. Denote B_R an open ball in $\mathbb{R}^3$ with radius $R > 0$ such that $\Omega \subset B_R$. By using similar arguments as those of theorem 2.4, we can prove:

Corollary 2.5. *The multiple-frequency near-field data* $\{u^{(\kappa)}(x) : |x| = R, \kappa \in \mathrm{I}\}$ *uniquely determine* q*. Here* I *is an open interval of* $\mathbb{R}^+$.

3. Inverse electromagnetic scattering

3.1. Problem formulation for electromagnetic waves

In this section, we consider the time-harmonic Maxwell equations in an inhomogeneous medium:

$$\nabla \times E - i\omega\mu_0 H = 0, \quad \nabla \times H + i\omega\varepsilon_0 qE = 0 \quad \text{in } \mathbb{R}^3, \tag{3.1}$$

where $\omega > 0$ is the angular frequency, $\mu_0 > 0$ is the magnetic permeability and $\varepsilon_0 > 0$ is the electric permittivity. Eliminating the magnetic field from (3.1) we see

$$\nabla \times \nabla \times E - \kappa^2 qE = 0 \quad \text{in} \quad \mathbb{R}^3, \quad \kappa^2 := \omega^2\mu_0\varepsilon_0. \tag{3.2}$$

As seen in figure 3, we will assume in this section that the refractive index function is periodic in the x_1-direction only and remains invariant in $x_2 \in \mathbb{R}$ and $x_3 \in (-b, b)$, taking the form

$$q(x) = \begin{cases} 1 & \text{in} \quad x_3 > b, \\ q(x_1) & \text{in} \quad S_b, \\ 1 & \text{in} \quad x_3 < -b, \end{cases}$$

where $q(x_1) = q(x_1 + 2\pi n_1)$ for all $n_1 \in \mathbb{Z}$. Further we suppose that $q(x) \in H^1_{\mathrm{loc}}(S_b)$.

Let $E^{\mathrm{inc}}(x) = pe^{i\kappa x\cdot d}$ be an incoming plane wave, where

$$d(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, -\cos\theta) =: (d_1, d_2, d_3)$$

is the unit propagation vector and $p = (p_1, p_2, p_3)$ is the unit polarization vector satisfying $p(\theta, \varphi) \cdot d(\theta, \varphi) = 0$. This implies that E^{inc} satisfies the Maxwell equation (3.2) in the background medium (i.e. $q = 1$). As done in the acoustic case, we set $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 := \kappa\sin\theta\cos\varphi, \alpha_2 := \kappa\sin\theta\sin\varphi$. In our studies, we choose the incident angles $0 \leqslant \theta < \pi/2,\ 0 \leqslant \varphi < 2\pi$ such that the second component of the polarization direction is non-vanishing, that is, $p_2 \neq 0$. Since E^{inc} is α-quasiperiodic, the perturbed scattered field E^{sc} in $x_3 > b$ and the transmitted field E in $x_3 < -b$ are also α-quasiperiodic, fulfilling the Rayleigh expansion conditions

$$\begin{aligned} E^{\mathrm{sc}}(x) &= \sum_{n\in\mathbb{Z}^2} E_n^+ e^{i\alpha_n\cdot x' + i\beta_n x_3}, \quad x_3 > b, \\ E(x) &= \sum_{n\in\mathbb{Z}^2} E_n^- e^{i\alpha_n\cdot x' - i\beta_n x_3}, \quad x_3 < -b, \end{aligned} \tag{3.3}$$

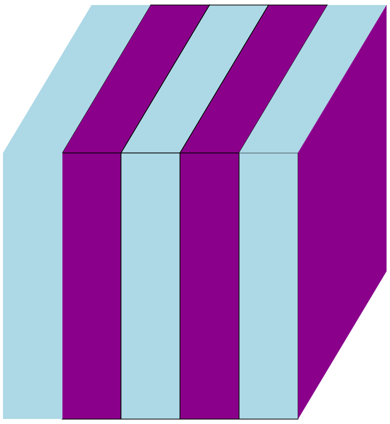

Figure 3. A periodic medium in $x_1 \in \mathbb{R}$ that remains invariant in $x_2 \in \mathbb{R}$ and $x_3 \in (-b, b)$.

where $\alpha_n = n + \alpha \in \mathbb{R}^2$, $E_n^\pm \in \mathbb{C}^3$ are complex vectors satisfying

$$E_n^\pm \cdot (\alpha_n, \pm\beta_n) = 0, \quad \beta_n := \beta_n(\omega) = \sqrt{\kappa^2 - |\alpha_n|^2} \quad \text{with } \mathrm{Im}\beta_n \geqslant 0.$$

3.2. Solvability of forward scattering

To reduce the electromagnetic scattering problem to a bounded periodic cell $\Omega_b := \{(x', x_3) : x' \in (0, 2\pi) \times (0, 2\pi), x_3 \in (-b, b)\}$, we shall review the transparent boundary operators $T^\pm$ on $\Gamma_b^\pm$ and an equivalent variational formulation, following the arguments presented in [1, 2, 4, 13]. Special attention will be paid to the analyticity of the solution with respect to incident frequencies.

We first introduce the quasi-periodic Sobolev space

$$H_\alpha(\mathrm{curl}, \Omega_b) = \{E|_{\Omega_b} : E \text{ is } \alpha\text{-quasiperiodic in } S_b, E \in L^2(\Omega_b)^3, \mathrm{curl}\, E \in L^2(\Omega_b)^3\},$$

equipped with the usual norm

$$||E||_{H_\alpha(\mathrm{curl}, \Omega_b)} = \left(||E||^2_{L^2(\Omega_b)^3} + ||\mathrm{curl}\, E||^2_{L^2(\Omega_b)^3} \right)^{1/2}.$$

For any smooth vector $E = (E_1, E_2, E_3)$ defined on $\Gamma_b^\pm$, denote by $\mathrm{div}_{\Gamma_b^\pm} E = \partial_1 E_1 + \partial_2 E_2$ and $\mathrm{curl}_{\Gamma_b^\pm} E = \partial_1 E_2 - \partial_2 E_1$ the surface divergence and the surface scalar curl, respectively, of the field E. We then introduce the following α-quasiperiodic vector trace spaces

$$H_{\mathrm{div}}^{-1/2}(\Gamma_b^\pm) = \{E \in H_\alpha^{-1/2}(\Gamma_b^\pm)^3 : E_3 = 0, \mathrm{div}_{\Gamma_b^\pm} E \in H_\alpha^{-1/2}(\Gamma_b^\pm)\},$$
$$H_{\mathrm{curl}}^{-1/2}(\Gamma_b^\pm) = \{E \in H_\alpha^{-1/2}(\Gamma_b^\pm)^3 : E_3 = 0, \mathrm{curl}_{\Gamma_b^\pm} E \in H_\alpha^{-1/2}(\Gamma_b^\pm)\}.$$

Using the Fourier expansion

$$E_j(x)|_{\Gamma_b^\pm} = \sum_{n=(n_1,n_2)\in\mathbb{Z}^2} E_{jn} \exp(\mathrm{i}\alpha_n \cdot x'), \quad j = 1, 2,$$

the norms on the spaces $H_{\rm div}^{-1/2}(\Gamma_b^\pm)$ and $H_{\rm curl}^{-1/2}(\Gamma_b^\pm)$ can be characterized by

$$\|E\|^2_{H_{\rm div}^{-1/2}(\Gamma_b^\pm)} = \sum_{n\in\mathbb{Z}^2}(1+|\alpha_n|^2)^{-1/2}(|(E_{1n}|^2+|E_{2n}|^2+|n_1E_{1n}+n_2E_{2n}|^2),$$
$$\|E\|^2_{H_{\rm curl}^{-1/2}(\Gamma_b^\pm)} = \sum_{n\in\mathbb{Z}^2}(1+|\alpha_n|^2)^{-1/2}(|(E_{1n}|^2+|E_{2n}|^2+|n_1E_{2n}-n_2E_{1n}|^2).$$

For any vector field $E=(E_1,E_2,E_3)$, denote its tangential components on $\Gamma_b^\pm$ by

$$E_T=\nu\times(E\times\nu)|_{\Gamma_b^\pm}=(E_1(x',\pm b),E_2(x',\pm b),0).$$

If $E\in H(\mathrm{curl},\Omega_b)$, then

$$\nu\times E|_{\Gamma_b^\pm}\in H_{\rm div}^{-1/2}(\Gamma_b^\pm),\quad E_T|_{\Gamma_b^\pm}\in H_{\rm curl}^{-1/2}(\Gamma_b^\pm).$$

For any tangential field $V(x')=\sum_{n\in\mathbb{Z}^2}V_n\exp(\mathrm{i}\alpha_n\cdot x')\in H_{\rm div}^{-1/2}(\Gamma_b^+)$, the transparent operator $T^+:H_{\rm div}^{-1/2}(\Gamma_b^+)\to H_{\rm curl}^{-1/2}(\Gamma_b^+)$ is defined as $T^+V:=(\mathrm{curl}E)_T$ on Γ_b^+, where E solves the quasiperiodic boundary value problem

$$\mathrm{curl}\ \mathrm{curl}E-\kappa^2E=0\quad\text{in}\quad x_3>b,\qquad \nu\times E=V\quad\text{on}\quad\Gamma_b^+,$$

and fulfills the upward Rayleigh expansion (3.3) with the Rayleigh coefficients E_n^+. The transparent operator $T^-:H_{\rm div}^{-1/2}(\Gamma_b^-)\to H_{\rm curl}^{-1/2}(\Gamma_b^-)$ can be defined analogously, which maps $V\in H_{\rm div}^{-1/2}(\Gamma_b^-)$ to the tangential components of $\mathrm{curl}\,E$ where E satisfies the downward Rayleigh expansion (3.3) and the boundary value condition $\nu\times E=V$ on Γ_b^-. It was shown in [1] that $T^\pm$ takes the explicit representation

$$(T^\pm V)(x')=\sum_{n\in\mathbb{Z}^2}\frac{1}{\mathrm{i}\beta_n}\left\{\kappa^2V_n-(\alpha_n\cdot V_n)\,\alpha_n\right\}\exp(\mathrm{i}\alpha_n\cdot x').$$

By definition,

$$T^+(\nu\times E^{\rm sc}(x',b))=(\mathrm{curl}\,E^{\rm sc}(x',b))_T,\qquad T^-(\nu\times E(x',-b))=(\mathrm{curl}\,E(x',-b))_T,$$

where $E^{\rm sc}$ and E are given in (3.3), ν is the unit normal vector at $\Gamma_b^\pm$ directed into the exterior of S_b. Multiplying the complex conjugate of a test function $F\in H_\alpha(\mathrm{curl},\Omega_b)$, integrating over Ω_b, and using integration by parts, we arrive at the variational form for our scattering problem: find $E\in H_\alpha(\mathrm{curl},\Omega_b)$ such that

$$a(E,F)=\langle f,F\rangle_{\Gamma_b^+}\quad\text{for all } F\in H_\alpha(\mathrm{curl},\Omega),\tag{3.4}$$

with the sesquilinear form

$$a(E,F):=\int_{\Omega_b}(\nabla\times E)\cdot(\nabla\times\bar F)\mathrm{d}x-\kappa^2\int_{\Omega_b}q\,E\cdot\bar F\,\mathrm{d}x$$
$$-\int_{\Gamma_b^+}T^+(\nu\times E)\cdot(\nu\times\bar F)\,\mathrm{d}s-\int_{\Gamma_b^-}T^-(\nu\times E)\cdot(\nu\times\bar F)\,\mathrm{d}s$$

and the linear functional

$$\langle f,F\rangle_{\Gamma_b^+}=\int_{\Gamma_b^+}f\cdot(e_3\times\bar F)\,\mathrm{d}s,\quad f:=T^+(e_3\times E^{\rm in}|_{\Gamma_b^+})-(E^{\rm in}|_{\Gamma_b^+})_T\in H_{\rm curl}^{-1/2}(\Gamma_b^+).$$

Using mapping properties of the transparent operator $T^{\pm}$ (see [1]) together with a Hodge decomposition of the $H_\alpha(\mathrm{curl}, \Omega_b)$, the variational formulation (3.4) can be equivalently written as the operator equation $A^{(\omega)} + B^{(\omega)} = G^{(\omega)}$ over the periodic Sobolev space $H_p(\mathrm{curl}, \Omega_b)$, that is, the space $H_\alpha(\mathrm{curl}, \Omega_b)$ with the quasi-periodic parameter $\alpha = (0, 0)$. Here, $A^{(\omega)} : H_p(\mathrm{curl}, \Omega_b) \to H_p(\mathrm{curl}, \Omega_b)^*$ is coercive and $B^{(\omega)} : H_p(\mathrm{curl}, \Omega_b) \to H_p(\mathrm{curl}, \Omega_b)^*$ is compact. Moreover, the operators $A^{(\omega)}, B^{(\omega)}$ and $G^{(\omega)}$ all depend on $\omega \in \mathbb{C}\backslash \mathrm{I}_0$ analytically, where

$$\mathrm{I}_0 = \{\omega \in \mathbb{R}^+ : \omega^2 \mu_0 \varepsilon_0 = |\alpha_n|^2 \text{ for some } n \in \mathbb{Z}^2\}$$

denotes the set of resonance frequencies. Like the acoustic case, uniqueness to our scattering problem can be proved if $\omega \in \mathbb{R}^+$ is sufficiently small or $\mathrm{Im}\,(\omega^2) > 0$; see [2, 4, 13, 16, 17]. Hence, by analytic Fredholm theory we obtain

Theorem 3.1. *The variational problem (3.4) admits a unique solution $E \in H_\alpha(\mathrm{curl}, \Omega_h)$ for all $\omega \in \mathbb{C}^+\backslash\mathcal{D}$ and $h > b$, where $\mathcal{D} \subset \mathbb{R}^+$ is a discrete set with the only accumulating point at infinity. Moreover, the solution $E = E^{(\omega)}$ depends on $\omega \in \mathbb{C}^+\backslash\mathcal{D}$ analytically.*

Remark 3.2. The transparent boundary operators $T^{\pm}$ are well-defined if ω is not a resonance frequency, i.e. $\omega \notin \mathrm{I}_0$. This implies that the resonance set I_0 is a subset of the discrete set $\mathcal{D}$ within this paper. If $\omega \in \mathrm{I}_0$ (which means that $\beta_n = 0$ for some $n \in \mathbb{Z}^2$), the definition of $T^{\pm}$ must be modified to incorporate the wave modes in the resonance case; we refer to [16, 17] for the derivation of an equivalent variational formulation based on the Fourier modes expansion. Uniqueness and existence holds for any frequency, if the refractive index function fulfills certain conditions; see e.g. [24].

3.3. Uniqueness

In this section, we prove uniqueness of the inverse electromagnetic medium scattering problem. In the following we use $E^{(\omega)}$ to indicate the dependence of E on the frequency ω. Moreover, it is assumed that the number b is known in advance.

Theorem 3.3. *The multiple-frequency data $\{e_3 \times E^{(\omega)}(x', \pm h) : x' \in (0, 2\pi) \times (0, 2\pi),\ \omega \in \mathrm{I}\}$ for some $h > b$ uniquely determine $q(x_1)$. Here $\mathrm{I} \subset \mathbb{R}^+$ is an open interval.*

Proof. Assume there are two refractive index functions q and $\tilde{q}$ with the corresponding measured data satisfying

$$\nu \times E^{(\omega)} = \nu \times \tilde{E}^{(\omega)} \quad \text{on} \quad \Gamma_h^{\pm} \quad \text{for all} \quad \omega \in \mathrm{I}. \tag{3.5}$$

By theorem 3.1, both the solutions $E^{(\omega)}$ and $\tilde{E}^{(\omega)}$ are analytic with respect to ω in $\mathbb{C}^+\backslash\mathcal{D}$. Therefore, the above relation holds for all ω in $\mathbb{C}^+\backslash\mathcal{D}$ and in particular for sufficiently small frequencies $\omega \in (0, \delta)$ where $\delta > 0$ is a small number. Note that both $\mathcal{D}$ and I_0 keep a positive distance from zero. By the boundedness of the transparent operators $T^{\pm}$, we obtain

$$(\nabla \times E^{(\omega)})_T = (\nabla \times \tilde{E}^{(\omega)})_T \quad \text{on } \Gamma_h^{\pm} \tag{3.6}$$

for $\omega \in \mathbb{R}^+\backslash\mathcal{D}$. The previous two relations (3.5) and (3.6) also hold on $\Gamma_b^{\pm}$, due to the analogue of Holmgren's theorem for Maxwell's equations.

By rewriting (3.2) we obtain

$$\nabla \times (\nabla \times E^{(\omega)}) - \omega^2 \mu_0 \varepsilon_0 E^{(\omega)} = \omega^2 \mu_0 \varepsilon_0 (q-1) E^{(\omega)}, \tag{3.7}$$

$$\nabla \times (\nabla \times \tilde{E}^{(\omega)}) - \omega^2 \mu_0 \varepsilon_0 \tilde{E}^{(\omega)} = \omega^2 \mu_0 \varepsilon_0 (\tilde{q}-1) \tilde{E}^{(\omega)}. \tag{3.8}$$

Here and in the following, the dependance of the equation on ω is written explicitly. Subtracting (3.7) by (3.8) and letting $F^{(\omega)} := E^{(\omega)} - \tilde{E}^{(\omega)}$ yield

$$\begin{cases} \nabla \times (\nabla \times F^{(\omega)}) - \omega^2 \mu_0 \varepsilon_0 F^{(\omega)} = \omega^2 \mu_0 \varepsilon_0 \Big((q-1) E^{(\omega)} - (\tilde{q}-1) \tilde{E}^{(\omega)} \Big) & \text{in } \Omega_b, \\ e_3 \times F^{(\omega)} = (\nabla \times F^{(\omega)})_T = 0 \quad \text{on } \Gamma_b^{\pm}, \\ F^{(\omega)}(x) \text{ is } \alpha\text{-quasiperiodic with respect to } x' \text{ in } S_b. \end{cases} \tag{3.9}$$

Let $Q^{(\omega)}$ be an α-quasi-biperiodic vector solution to the equation

$$\nabla \times (\nabla \times Q^{(\omega)}) - \omega^2 \mu_0 \varepsilon_0 Q^{(\omega)} = 0 \quad \text{in } \Omega_b.$$

Explicitly, $Q^{(\omega)}$ can be expanded into the series

$$Q^{(\omega)}(x) = \sum_{n \in \mathbb{Z}^2} C_n^+ \mathrm{e}^{\mathrm{i}\alpha_n \cdot x' + \mathrm{i}\beta_n x_3} + C_n^- \mathrm{e}^{\mathrm{i}\alpha_n \cdot x' - \mathrm{i}\beta_n x_3}, \quad x \in \Omega_b$$

where $C_n^{\pm} \in \mathbb{C}^3$ are complex vectors satisfying the orthogonal relations

$$C_n^{\pm} \cdot (\alpha_n, \pm\beta_n) = 0 \quad \text{for all} \quad n \in \mathbb{Z}^2. \tag{3.10}$$

Multiplying (3.9) by $\bar{Q}^{(\omega)}$ and integrating by parts yields

$$\int_{\Omega_b} \left[(q-1) E^{(\omega)} - (\tilde{q}-1) \tilde{E}^{(\omega)} \right] \cdot \bar{Q}^{(\omega)} \mathrm{d}x = 0, \tag{3.11}$$

where we have used the boundary conditions in (3.9) and the $-\alpha$-quasiperiodicity of $\bar{Q}^{(\omega)}$.

Denote $m = 1 - q$ and $\tilde{m} = 1 - \tilde{q}$, both of which are compactly supported in Ω_b. By the periodic analogue of the Lippman Schwinger integral equation, we obtain (see [11] for bounded penetrable scatterers)

$$\begin{aligned} E^{(\omega)}(x) &= E^{\text{inc}}(x;\omega) - \omega^2 \mu_0 \varepsilon_0 \int_{\mathbb{R}^3} \Phi_\alpha^{(\omega)}(x,y) m(y) E^{(\omega)}(y) \mathrm{d}y \\ &\quad - \nabla_x \int_{\mathbb{R}^3} \frac{1}{q(y)} [\nabla_y q(y)] \cdot E^{(\omega)}(y)\, \Phi_\alpha^{(\omega)}(x,y) \mathrm{d}y \\ &= E^{\text{inc}}(x;\omega) - \omega^2 \mu_0 \varepsilon_0 \int_{\Omega_b} \Phi_\alpha^{(\omega)}(x,y) m(y) E^{(\omega)}(y) \mathrm{d}y \\ &\quad - \nabla_x \int_{\Omega_b} \frac{1}{q(y_1)} [\nabla_y q(y_1)] \cdot E^{(\omega)}(y)\, \Phi_\alpha^{(\omega)}(x,y) \mathrm{d}y, \end{aligned}$$

where $\Phi_\alpha^{(\omega)}(x,y)$ is the α-quasiperiodic Green's function to the Helmholtz equation, given by

$$\Phi_\alpha^{(\omega)}(x,y) = \sum_{n \in \mathbb{Z}^2} \frac{1}{8\pi^2 \beta_n} \mathrm{e}^{\mathrm{i}(\alpha_n \cdot (x'-y') + \beta_n |x_3 - y_3|)}, \quad x - y \neq (2n\pi, 0) \text{ for all } n \in \mathbb{Z}^2.$$

Since

$$\beta_n = \beta_n(\omega) \to \begin{cases} \mathrm{i}|n| & \text{if} \quad |n| \neq 0, \\ \omega\sqrt{\mu_0\varepsilon_0}\cos\theta & \text{if} \quad |n| = 0, \end{cases} \qquad \text{as } \omega \to 0,$$

by straightforward calculations we obtain

$$\begin{aligned} &\lim_{\omega\to 0} \omega^2\mu_0\varepsilon_0\Phi^{(\omega)}(x,y) \\ &= \lim_{\omega\to 0}\left(\frac{\omega\sqrt{\mu_0\varepsilon_0}}{8\pi^2\cos\theta} + \omega^2\mu_0\varepsilon_0\sum_{|n|\neq 0}\frac{1}{\mathrm{i}8\pi|n|}\mathrm{e}^{\mathrm{i}n\cdot(x'-y')-|n||x_3-y_3|}\right) \\ &= 0. \end{aligned}$$

This together with the uniform boundedness of $||E^{(\omega)}||_{H(\mathrm{curl},\Omega_b)}$ for $\omega \in (0,\delta]$ (which follows from the well-posedness of forward scattering) implies that

$$\lim_{\omega\to 0}\left[\omega^2\mu_0\varepsilon_0\int_{\mathbb{R}^3}\Phi_\alpha^{(\omega)}(x,y)m(y)E^{(\omega)}(y)\mathrm{d}y\right] = 0 \quad \text{in} \quad L^2(\Omega_b)^3.$$

On the other hand, we have

$$\begin{aligned} \nabla_x\Phi_\alpha^{(\omega)}(x,y) = &\sum_{|n|\neq 0}\frac{\mathrm{i}(\alpha_n, \mathrm{sign}(x_3-y_3)\beta_n)}{8\pi^2\beta_n}\mathrm{e}^{\mathrm{i}(\alpha_n\cdot(x'-y')+\beta_n|x_3-y_3|)} \\ &+ \frac{\mathrm{i}(d_1,d_2,\mathrm{sign}(x_3-y_3)\cos\theta)}{8\pi^2\cos\theta}\mathrm{e}^{\mathrm{i}(\alpha\cdot(x'-y')+\kappa\cos\theta|x_3-y_3|)}. \end{aligned}$$

Taking the limit $\omega \to 0^+$ yields

$$\begin{aligned} \lim_{\omega\to 0}\nabla_x\Phi_\alpha^{(\omega)}(x,y) = &\sum_{|n|\neq 0}\frac{(\mathrm{i}n, -\mathrm{sign}(x_3-y_3)|n|)}{8\mathrm{i}\pi^2|n|}\mathrm{e}^{\mathrm{i}\alpha\cdot(x'-y')-|n||x_3-y_3|} \\ &+ \frac{\mathrm{i}(d_1,d_2,\mathrm{sign}(x_3-y_3)\cos\theta)}{8\pi^2\cos\theta} \\ :=&\nabla_x\tilde{\Phi}^{(0)}(x,y), \end{aligned}$$

with

$$\begin{aligned} \tilde{\Phi}^{(0)}(x,y) := &\sum_{|n|\neq 0}\frac{1}{8\mathrm{i}\pi^2|n|}\mathrm{e}^{\mathrm{i}n\cdot(x'-y')-|n||x_3-y_3|} \\ &+ \frac{\mathrm{i}}{8\pi^2\cos\theta}\left[(x'-y')\cdot(d_1,d_2)+\cos\theta|x_3-y_3|\right]. \end{aligned}$$

Hence, we obtain

$$E^{(0)}(x) := \lim_{\omega\to 0}E^{(\omega)}(x) = p - \int_{\Omega_b}\frac{1}{q(y_1)}\left[\nabla_y q(y_1)\right]\cdot E^{(0)}(y)\left[\nabla_x\tilde{\Phi}^{(0)}(x,y)\right]\mathrm{d}y.$$

In addition, the limiting function $Q^{(0)}(x) := \lim_{\kappa\to 0}Q^{(\omega)}(x)$ satisfies

$$\nabla\times(\nabla\times Q^{(0)}) = 0, \tag{3.12}$$

and $Q^{(0)}(x)$ is 2π-biperiodic in the variable x'. In general, the function $Q^{(0)}$ can be expanded into the series

$$Q^{(0)}(x) = \sum_{n\in\mathbb{Z}^2} C_n^+ e^{in\cdot x' - i|n|x_3} + C_n^- e^{in\cdot x' + i|n|x_3}, \quad x \in \Omega_b,$$

with the coefficients $C_n^\pm$ satisfying (3.10). Letting $\omega \to 0$ in (3.11) we obtain

$$\begin{aligned}
0 &= \int_{\Omega_b} (mE^{(0)} - \tilde{m}\tilde{E}^{(0)}) \cdot \bar{Q}^{(0)} dx \\
&= \int_{\Omega_b} (m - \tilde{m})\, p \cdot \bar{Q}^{(0)} dx \\
&\quad + \int_{\Omega_b} m(x) \Big(\nabla_x \int_{\Omega_b} \frac{1}{q(y)} [\nabla_y q(y)] \cdot E^{(0)}(y)\, \tilde{\Phi}^{(0)}(x, y) dy\Big) \cdot \bar{Q}^{(0)}(x)\, dx \\
&\quad - \int_{\Omega_b} \tilde{m}(x) \Big(\nabla_x \int_{\Omega_b} \frac{1}{\tilde{q}(y)} [\nabla_y \tilde{q}(y)] \cdot \tilde{E}^{(0)}(y) \tilde{\Phi}^{(0)}(x, y)\, dy\Big) \cdot \bar{Q}^{(0)}(x) dx. \qquad (3.13)
\end{aligned}$$

Below we shall choose

$$Q^{(0)}(x) = e^{in_1 x_1 + |n_1| x_3} (0, 1, 0)^\top, \quad x \in \Omega_b,$$

for some $n_1 \in \mathbb{Z}$, which is obviously divergence free. Straightforward calculations show that

$$\nabla_x \cdot (m\bar{Q}^{(0)}) = \nabla_x m \cdot \bar{Q}^{(0)} + m(\nabla \cdot \bar{Q}^{(0)}) = 0, \quad x \in \Omega_b,$$

since m depends only on x_1 in Ω_b. Using integration by parts we have

$$\begin{aligned}
&\int_{\Omega_b} m \Big(\nabla_x \int_{\Omega_b} \frac{1}{q} \nabla_y q \cdot E^{(0)} \tilde{\Phi}^{(0)}(x, y) dy\Big) \cdot \bar{Q}^{(0)} dx \\
&= - \int_{\Omega_b} \Big(\int_{\Omega_b} \frac{1}{q} \nabla_y q \cdot E^{(0)} \tilde{\Phi}^{(0)}(x, y) dy\Big) \nabla_x \cdot (m\bar{Q}^{(0)}) dx \\
&= 0.
\end{aligned}$$

Similarly we have

$$\int_{\Omega_b} \tilde{m} \Big(\nabla_x \int_{\Omega_b} \frac{1}{\tilde{q}} \nabla_y \tilde{q} \cdot \tilde{E}^{(0)} \tilde{\Phi}^{(0)}(x, y) dy\Big) \cdot \bar{Q}^{(0)} dx = 0.$$

Hence, from (3.13) we obtain

$$\int_{\Omega_b} (m - \tilde{m})\, p \cdot \bar{Q}^{(0)} dx = 0. \qquad (3.14)$$

Since both m and $\tilde{m}$ are 2π-periodic in x_1, $m - \tilde{m}$ admits the Fourier series expansion $(m - \tilde{m})(x) = \sum_{l\in\mathbb{Z}} a_l e^{ilx_1}$ for $x \in \Omega_b$, where a_l are the Fourier coefficients. It then follows from (3.14) that

$$
\begin{aligned}
0 &= \int_{\Omega_b} (m - \tilde{m})\, p \cdot \bar{Q}^{(0)} \mathrm{d}x \\
&= \int_{-b}^{b} \mathrm{d}x_3 \int_0^{2\pi} \mathrm{d}x_2 \int_0^{2\pi} p_2 \Big(\sum_{l \in \mathbb{Z}} a_l \mathrm{e}^{\mathrm{i}lx_1} \Big) \mathrm{e}^{-\mathrm{i}n_1x_1 + |n_1|x_3} \mathrm{d}x_1 \\
&= 4\pi^2 a_{n_1} p_2 \int_{-b}^{b} \mathrm{e}^{|n_1|x_3} \mathrm{d}x_3,
\end{aligned}
$$

which implies that $a_{n_1} = 0$, since $p_2 \neq 0$ and the integral $\int_{-b}^{b} \mathrm{e}^{|n_1|x_3} \mathrm{d}x_3$ is a positive number. By the arbitrariness of $n_1 \in \mathbb{Z}$, we have $m - \tilde{m} = 0$ which completes the proof. □

3.4. Remark

We present a uniqueness result for inverse time-harmonic electromagnetic scattering from a bounded penetrable scatterer whose contrast support is given by $\Omega = (-a, a) \times (-b, b) \times (-c, c)$. The refractive index function is supposed to satisfy $q(x) = 1$ for $x \in \mathbb{R}^3 \backslash \overline{\Omega}$, $q(x) = q(x_1)$ in Ω and that $q \in H^1(\Omega)$. Moreover, the values of b and c are supposed to be the prior information that we know in advance. The scattered fields E^{sc} and H^{sc} are required to satisfy the Silver-M*ü* ller radiation condition

$$
\lim_{r \to \infty} (H^{\mathrm{sc}} \times x - rE^{\mathrm{sc}}) = 0 \quad r = |x|
$$

uniformly for all directions $x/|x|$. Hence, the electric field E satisfies the integral equation

$$
\begin{aligned}
E = E^{\mathrm{inc}} &- \kappa^2 \int_{\Omega} \mathrm{G}(x, y) m(y) E(y) \mathrm{d}y \\
&+ \nabla_x \int_{\Omega} \frac{1}{q(y)} \nabla_y q(y) \cdot E(y) \mathrm{G}(x, y) \mathrm{d}y, \quad x \in \mathbb{R}^3,
\end{aligned}
$$

where $m := 1 - q$, $\mathrm{G}(x, y)$ is the free-space fundamental solution to the Helmholtz equation in three dimensions. Denote B_R an open ball in $\mathbb{R}^3$ with radius $R > 0$ such that $\Omega \subset B_R$.

By using similar arguments as those of theorem 3.3, we can prove the following theorem:

Theorem 3.4. *Let $E^{\mathrm{inc}}(x) = p \exp(\mathrm{i}kx \cdot d)$, $p \cdot d = 0$, be an incident plane wave such that the second or the third component of p does not vanish. Then the multiple-frequency data $\{\nu \times E^{(\omega)}(x) : |x| = R, \omega \in \mathrm{I}\}$ uniquely determine $q|_{\Omega}$. Here I is an open interval and $\nu = \frac{x}{|x|}$ is the normal vector at $|x| = R$.*

Proof. By arguing the same as in (3.13) we get

$$
\int_{\Omega} (m - \tilde{m})\, p \cdot \bar{Q}^{(0)} \mathrm{d}x = 0, \tag{3.15}
$$

where $Q^{(0)}$ is any solution to the equation (3.12) in $\mathbb{R}^3$. Choose

$$
Q^{(0)}(x) = \begin{cases} \mathrm{e}^{\mathrm{i}n_1x_1 + |n_1|x_3} (0, 1, 0)^\top & \text{if} \quad p_2 \neq 0, \\ \mathrm{e}^{\mathrm{i}n_1x_1 + |n_1|x_2} (0, 0, 1)^\top & \text{if} \quad p_3 \neq 0. \end{cases}
$$

Then, it is derived from (3.15) that

$$0 = \int_{\mathbb{R}} (m - \tilde{m})(x_1) \mathrm{e}^{-\mathrm{i}n_1 x_1} \, \mathrm{d}x_1 \int_{-b}^{b} \int_{-c}^{c} p \cdot [\overline{Q}^{(0)}(x) \exp(-\mathrm{i}nx_1)] \mathrm{d}x_2 \, \mathrm{d}x_3,$$

implying that

$$0 = \int_{\mathbb{R}} (m - \tilde{m})(x_1) \mathrm{e}^{-\mathrm{i}n_1 x_1} \, \mathrm{d}x_1.$$

This proves $m = \tilde{m}$, due to the arbitrariness of $n_1 \in \mathbb{Z}$. □

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