Uniqueness and stability for the recovery of a time-dependent source and initial conditions in elastodynamics

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Abstract

This paper is concerned with inverse source problems for the time-dependent Lamé system and the recovery of initial data in an unbounded domain corresponding to the exterior of a bounded cavity or the full space $\mathbb{R}^3$. If the time and spatial variables of the source term can be separated with compact support, we prove that the vector valued spatial source term can be uniquely determined by boundary Dirichlet data in the exterior of a given cavity. If the cavity is absent, uniqueness and stability for recovering source terms depending on the time variable and two spatial variables in the whole space are also obtained using partial Dirichlet boundary data.

Keywords: Linear elasticity, inverse source problems, time domain, uniqueness, stability estimate.

1 Introduction

1.1 Statement of the problem

Consider the radiation of an elastic source F outside a cavity D described by the system

$$\rho(x)\partial_{tt}U(x,t) = \mathcal{L}_{\lambda,\mu}U(x,t) + F(t,x), \quad x=(x_1,x_2,x_3)\in\mathbb{R}^3\backslash\overline{D},\ t>0 \tag{1.1}$$

where ρ denotes the density, $U=(u_1,u_2,u_3)^\top$ is the displacement vector, $D\subset\mathbb{R}^3$ represents the region of the cavity and $\mathcal{L}_{\lambda,\mu}U$ stands for the Lamé operator defined by

$$\mathcal{L}_{\lambda,\mu}U := -\mu(x)\nabla\times\nabla\times U + (\lambda(x)+2\mu(x))\nabla\nabla\cdot U + (\nabla\cdot U)\nabla\lambda(x) + ((\nabla U)+(\nabla U)^T)\nabla\mu(x). \tag{1.2}$$

Here we assume that

$$\rho(x)\geq\rho_0>0, \quad \mu(x)\geq\mu_0>0, \quad \lambda(x)\geq 0, \quad x\in\mathbb{R}^3. \tag{1.3}$$

Together with the governing equation, we impose the initial conditions

$$U(x,0)=V_0(x), \quad \partial_t U(x,0)=V_1(x), \qquad x\in\mathbb{R}^3\backslash\overline{D}, \tag{1.4}$$

and the traction-free boundary condition on ∂D:

$$\mathcal{T}U(x,t)=0, \qquad (x,t)\in\partial D\times\mathbb{R}^+, \tag{1.5}$$

where $\mathcal{T}U$ is the stress boundary condition defined by (2.2) (see Section 2). In this paper we consider the inverse problem of determining the source term F and the initial conditions V_0 and V_1 from knowledge of U on the surface $\partial B_R = \{x \in \mathbb{R}^3 : |x| = R\}$ with $R > 0$ sufficiently large. According to [7, Remark 4.5], even for $V_0 = V_1 = 0$, there is an obstruction for the recovery of general time-dependent source terms F. Facing this obstruction we consider this problem for some specific type of source terms.

1.2 Motivations

We recall that the Lamé system (1.1)-(1.2) is frequently used for the study of linear elasticity and imaging problems. In this context our inverse problems can be seen as the recovery of an external force due to the source term F or internal data at the time point $t = 0$ given by the initial conditions V_0 and V_1. The elastodynamic source terms corresponding to the product of a spatial function g and a temporal function f can be regarded as an approximation of the elastic pulse and are commonly used in modeling vibration phenomena in seismology and teleseismic inversion [2, 42]. This type of sources has been also considered in numerous applications in biomedical imaging (see [3, 4] and the references therein) where our inverse problem can be seen as the recovery of the information provided by the parameter under consideration. We mention also that the recovery of the initial conditions V_0 and V_1 are related to problems of thermoacoustic and photoacoustic tomography. More precisely our problem can be connected to mathematical model of the thermoacoustic tomography (TAT) procedure where one wants to recover the absorption of a biological object subjected to a short radiofrequency pulse (see [1] and the references therein for more detail).

1.3 Known results

Inverse source problems have received much attention over the last thirty years. These problems take different forms and have many applications (environment, imaging, seismology $\cdots$) . For an overview of these problems we refer to [26]. Among the different arguments considered for solving these problems we can mention the approach based on applications of Carleman estimates arising from the work of [14] (see also [33, 34]). This approach has been applied successfully to hyperbolic equations by [47] in order to extend his previous work [46] to a wider class of source terms. More precisely, in [47] the author considered the recovery of source terms of the form $f(x)G(x, t)$, where G is known, while in [46] the analysis of the author is restricted to source terms of the form $\sigma(t)f(x)$, with σ known. More recently, the approach of [47] has been extended by [29] to hyperbolic equations with time-dependent second order coefficients and to less regular coefficients by [48]. We mention also the work of [16, 32] using similar approach for inverse source problems stated for parabolic equations and the result of [43] proved by a combination of geometrical arguments and Carleman estimates. Concerning Lamé system we refer to [22] where a uniqueness result has been stated for the recovery of time-independent source terms by mean of suitable Carleman estimate and we mention also the work of [24, 25, 28] dealing with related problems as well as [10] where an inverse source problem for Biot's equations has been considered. We refer also to the recent work [7] where the recovery of a time-independent source term appearing in the Lamé system in all space has been proved from measurements outside the support of the source under consideration as well as the work of [13] dealing with this inverse source problem for fractional diffusion equations.

In all the above mentioned results the authors considered the recovery of time independent source terms. For the recovery of a source depending only on the time variable we refer to [19] where such problems have been considered for fractional diffusion equations, and for the recovery of some class of sources depending on both space and time variables appearing in a parabolic equation on the half space, we refer to [26, Section 6.3]. For hyperbolic equations, we refer to [12, 41] where the recovery of some specific time-dependent source terms has been considered. For Lamé systems, [7, Theorem 4.2] seems to be the only result available in the mathematical literature where such a problem has been addressed for time-dependent source terms. The result of [7, Theorem 4.2] is stated with source terms depending only on the time variable. To our best knowledge, except the result of [12], dealing with the recovery of discrete in time sources, and the result of the present paper, there is no result in the mathematical literature treating the recovery of a source term depending on both space and time variables appearing in hyperbolic equations.

Finally, for the recovery of initial data, without being exhaustive, we refer to the work of [1, 20] where the recovery of initial conditions has been considered in the context of the TAT procedure.

1.4 Main results

In the present paper we consider three inverse problems related to the recovery of the source term F and the initial conditions V_0 and V_1. First we assume that the cavity $D \neq \emptyset$ is a domain with $\mathcal{C}^3$ boundary ∂D, with connected exterior $\mathbb{R}^3 \backslash \overline{D}$, and we consider source terms of the form

$$F(x,t) = f(t)g(x), \quad x \in \mathbb{R}^3 \backslash \overline{D},\ t \in (0, +\infty), \tag{1.6}$$

with f a real valued function and $g = (g_1, g_2, g_3)^\top : \mathbb{R}^3 \backslash \overline{D} \to \mathbb{R}^3$ a vector valued function. Let $B_R := \{x : |x| < R\}$ be the ball of radius $R > 0$ centered at the origin. Choose $R > 0$ sufficiently large such that $D \subset B_R$. Throughout the paper, it is supposed that the density $\rho \in \mathcal{C}^2(\mathbb{R}^3)$ and the Lamé coefficients $\mu \in \mathcal{C}^3(\mathbb{R}^3)$, $\lambda \in \mathcal{C}^2(\mathbb{R}^3)$ are constant outside some compact set of $\mathbb{R}^3$. We assume that B_R contains the support of g, V_0 and V_1 (that is, $\mathrm{Supp}(g) \cup \mathrm{Supp}(V_0) \cup \mathrm{Supp}(V_1) \subset B_R$), $f \in L^2(0,T)$, $\mathrm{Supp}(f) \subset [0,T]$, $g \in L^2(\mathbb{R}^3 \setminus D)^3$, $V_0 \in H^1(\mathbb{R}^3 \setminus D)^3$ and $V_1 \in L^2(\mathbb{R}^3 \setminus D)^3$. Then, the problem (1.1)-(1.5) admits a unique solution

$$U \in \mathcal{C}^1([0, +\infty); L^2(\mathbb{R}^3 \setminus D))^3 \cap \mathcal{C}([0, +\infty); H^1(\mathbb{R}^3 \setminus D))^3.$$

The proof of this result can be carried out by combining the elliptic regularity properties of $\mathcal{L}_{\lambda,\mu}$ (see e.g., [39, Chapters 4 and 10] and [21, Chapter 5]) with [36, Theorem 8.1, Chapter 3] and [36, Theorem 8.2, Chapter 3] (see also the beginning of Sections 4.1 and 4.2 for more detail). Our first two inverse problems in the exterior of the cavity can be stated as follows.

Inverse Problem 1 (IP1): Assume that $V_0 = V_1 = 0$ and f, D are both known in advance. Determine the spatially dependent function g from the radiated field U measured on the surface $\partial B_R \times \mathbb{R}^+$.
Inverse Problem 2 (IP2): Assume that $g = 0$, D is known in advance. Determine simultaneously the spatially dependent functions V_0 and V_1 from the radiated field U measured on the surface $\partial B_R \times \mathbb{R}^+$.

Obviously, (IP1) is an inverse source problem, while (IP2) aims at recovering the initial value and initial velocity. Below we give a confirmative answer to the uniqueness issue for IP1 and IP2. For IP1 our result can be stated as follows.

Theorem 1. *Let $V_0 = V_1 = 0$, $f \in H_0^1(0,T)$ and F takes the form* (1.6). *Then the boundary data $\{U(x,t) :\ (x,t) \in \partial B_R \times \mathbb{R}^+\}$ uniquely determine g.*

The uniqueness result for (IP2) is stated as follows.

Theorem 2. *Let $F = 0$, $V_1 \in H^1(\mathbb{R}^3 \setminus D)^3$, $V_0 \in H^2(\mathbb{R}^3 \setminus D)^3$, $\mathcal{T}V_0 = 0$ on ∂D. Then the boundary data $\{U(x,t) : (x,t) \in \partial B_R \times \mathbb{R}^+\}$ uniquely determine simultaneously V_0 and V_1.*

For our third inverse problem, we consider the Lamé system without cavity ($D = \emptyset$) and with constant coefficients. We assume here that F takes the form

$$F(\tilde{x}, x_3, t) = g(x_3)\, f(\tilde{x}, t), \quad \tilde{x} \in \mathbb{R}^2,\ x_3 \in \mathbb{R},\ t \in (0, +\infty), \tag{1.7}$$

where the vectorial function $f = (f_1, f_2, 0)^\top$ is compactly supported on $\tilde{B}_R \times [0, T)$ and the scalar function g is supported in $(-R, R)$ for some $R > 0$. Moreover, we assume that g does not vanish identically. Here $\tilde{x} = (x_1, x_2) \in \mathbb{R}^2$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $\tilde{B}_R$ denotes the set $\tilde{B}_R := \{\tilde{x} \in \mathbb{R}^2 : |\tilde{x}| < R\}$. Then our last inverse problem can be stated as follows.

Inverse Problem 3 (IP3): Assume that $V_0 = V_1 = 0$ and g is known in advance. Determine the time and space dependent function f from the radiated field U measured on the surface $\Gamma \times (0, T_1)$, with $T_1 > 0$, $R_1 > 0$ sufficiently large and $\Gamma \subset \partial B_{R_1}$ an open set with positive Lebesgue measurement.

In this paper we give a positive answer to (IP3) both in terms of uniqueness and stability. Our uniqueness result can be stated as follows.

Theorem 3. *Let $D = \emptyset$, $V_0 = V_1 = 0$ and let ρ, λ, μ be constant. Assume that F takes the form* (1.7) *with $f = (f_1, f_2, 0)^\top \in H_0^1(0, T; L^2(\mathbb{R}^2))^3$, $g \in L^2(-R, R)$ and*

$$f(\tilde{x}, 0) = 0, \quad \tilde{x} \in \mathbb{R}^2.$$

We fix $R_1 > \sqrt{2}R$, $\Gamma \subset \partial B_{R_1}$ an arbitrary open set with positive Lebesgue measurement and $T_1 = T + \frac{2R_1\sqrt{\rho}}{\sqrt{\mu}}$. Then the source F can be uniquely determined by $U(x,t)$ measured on $\Gamma \times [0, T_1]$.

By assuming that $\Gamma = \partial B_{R_1}$, we can extend this uniqueness result to a log-type stability estimate.

Theorem 4. *Let $R_1 > \sqrt{2}R$, $T_1 > T + \frac{2R_1\sqrt{\rho}}{\sqrt{\mu}}$, $V_0 = V_1 = 0$, ρ, λ, μ be constant and assume that $D = \emptyset$, $f \in H^3(\mathbb{R} \times \mathbb{R}^2)^3 \cap H^4(0, T; L^2(\mathbb{R}^2))^3$ satisfies*

$$f(\tilde{x}, 0) = \partial_t f(\tilde{x}, 0) = \partial_t^2 f(\tilde{x}, 0) = \partial_t^3 f(\tilde{x}, 0) = 0, \quad \tilde{x} \in \mathbb{R}^2.$$

Assume also that g has a constant sign (that is, either $g \geq 0$ or $g \leq 0$) and that there exists $M > 0$ such that

$$\|f\|_{H^3(\mathbb{R}\times\mathbb{R}^2)^3} + \|f\|_{H^4(0,T;L^2(\mathbb{R}^2))^3} \leq M. \tag{1.8}$$

Then, there exists $C > 0$ depending on M, R_1, ρ, λ, μ, T, R, T_1 and $\|g\|_{L^1(\mathbb{R})}$ such that

$$||f||_{L^2((0,T)\times\tilde{B}_R)} \leq C \left(\|U\|_{H^3(0,T_1;H^{3/2}(\partial B_{R_1}))^3} + \left| \ln \left(\|U\|_{H^3(0,T_1;H^{3/2}(\partial B_{R_1}))^3} \right) \right|^{-1} \right). \tag{1.9}$$

1.5 Comments about our results

Let us first remark that to our best knowledge Theorem 1 is the first result of recovery of source terms stated for the Lamé system outside a cavity. Indeed, it seems that all other known results have been stated on a bounded domain (e.g. [22]) or in the full space $\mathbb{R}^3$ (e.g. [7]). In addition, comparing to results using Carleman estimates like [22, 43, 47, 48] we make no assumption on the sign of the known part of the source term under consideration at $t = 0$. Indeed, for a source term F of the form (1.6), such assumptions will be equivalent to the requirement that $f(0) > 0$. From the practical point of view, this means that the results of [22, 43, 47, 48] can only be applied to the determination of a source term associated to a phenomenon which has appeared before the beginning of the measurement. This restriction excludes applications where one wants to determine a phenomenon with measurements that start before its appearance. By removing this restriction in Theorem 1 we make our result more suitable for applications in that context.

Let us observe that Theorem 2 seems to be the first result dealing with the recovery of initial conditions stated for the Lamé system outside a cavity. Our result, which is connected to inverse tomography problems related to the TAT procedure studied by [1, 18, 20], is stated, for what seems to be the first time, without any assumptions on the propagation of the singularities of the solutions of our problem. More precisely, we remove the non-trapping condition which seems to be considered in all other papers studying this problem (see [1, 18, 20]). For wave equations, with suitable assumptions, the non-trapping condition leads to a suitable decay in time of the solution restricted to any compact set with respect to the space variable, also called the uniform local energy decay (see [44, Chapter X] and [45]) which seems to be one of the main requirement in the approach developed so far for treating this problem (see [1, 18, 20] as well as more recent papers dealing with the TAT procedure). Here the uniform local energy decay replaces the strong Huygens principle. According to [30] (see also [23]), for the Lamé system in an exterior domain with free stress boundary condition there is no hope to derive the uniform local energy decay used by [1, 18, 20]. Even the result of [9], proving some specific type of logarithmic local energy decay designed for smooth initial conditions, seems insufficient for the approach developed by [1, 18, 20] and the arguments used in the present paper seems to be the first one which can be applied to this problem. Moreover, we mention that our result seems to be the first one dealing with the simultaneous recovery of the initial value and the initial velocity for hyperbolic equations in an unbounded domain with non-constant coefficients (to our best knowledge, all other results consider only the recovery of the initial value with the initial velocity fixed at zero).

To our best knowledge, even for a bounded domain, Theorems 3 and 4 seem to be the first results of unique and stable recovery of a source term depending on both time and space variables appearing in a hyperbolic equation. Indeed, it seems that only results dealing with recovery of source terms depending only on the time variable (see [7, 41]) or space variable (see [7, 22, 43, 47, 48]) are available in the mathematical literature with the exception of [12] where the recovery of discrete in time sources has been considered. Therefore, the results of Theorems 3 and 4 are not only new for the Lamé system but also for more general hyperbolic equations. Moreover, it is worthy mentioning that the source term stated in Theorem 3 covers the type of moving sources whose orbit lies on the ox_1x_2-plane. Hence, Theorem 3 provides insights into how to handle inverse moving source problems; see Remark 1 for details.

We mention also that the stability result of Theorem 4 requires a result of stability in the unique continuation already considered by [11, 15, 31] for the recovery of time-dependent coefficients. Note also that, in contrast of Theorems 1 and 2, thanks to the strong Huygens principle we can state Theorems 3 and 4 at finite time.

In Corollaries 5 and 6 we prove that the results of Theorems 1 and 2 can be reformulated in terms of partial recovery of the source term or the initial data from measurements on a subdomain where the source term or the initial data are known. This situation may for instance occur in several applications where the source under consideration has large support and the data considered in

Theorems 1 and 2 are not accessible. What we prove in Corollaries 5 and 6 is that even in such context one can expect recovery of partial information of the source term under consideration by measurements located on some subdomain where the source is known.

Both Theorems 1, 2 and Corollaries 5, 6 remain valid if the cavity D is absent or if D is a rigid elastic body (i.e., U vanishes on ∂D). The proofs can be carried out by investigating the eigensystem of the Lamé equation with the Dirichlet boundary condition in place of the traction-free boundary condition.

All the results of this paper can be applied to the wave equation (see Remark 1). Actually, the proof for the wave equation will be easier in several aspects and the particular treatment for the Lamé sytem leads to some difficulties inherent to this type of systems (see for instance the proof of Theorems 3 and 4).

1.6 Outline

This paper is organized as follows. In Section 2 we study the inverse problems (IP1) and (IP2). More precisely, we prove Theorems 1 and 2 as well as their Corollaries 5 and 6. In Section 3 we treat the inverse problem (IP3). We start with the uniqueness result stated in Theorem 3. Then, we extend this result by proving the stability estimate stated in Theorem 4. For the readers' convenience, some results related to solutions of the problem (1.1)-(1.5) are given in the appendix.

2 Inverse source problem with traction-free boundary condition

This section is devoted to the uniqueness results of inverse problems (IP1) and (IP2) that are stated in Theorems 1 and 2. More precisely, we consider the radiation of an elastic source in an inhomogeneous medium in the exterior of a cavity D (see Figure 1):

$$\rho(x)\partial_{tt}U(x,t) = \mathcal{L}_{\lambda,\mu}U(x,t) + f(t)\,g(x), \quad x = (x_1,x_2,x_3) \in \mathbb{R}^3\backslash\overline{D},\ t>0, \tag{2.1}$$

where $\rho \in \mathcal{C}^2(\mathbb{R}^3)$ and $\mathcal{L}_{\lambda,\mu}U$ stands for the Lamé operator given by (1.2).

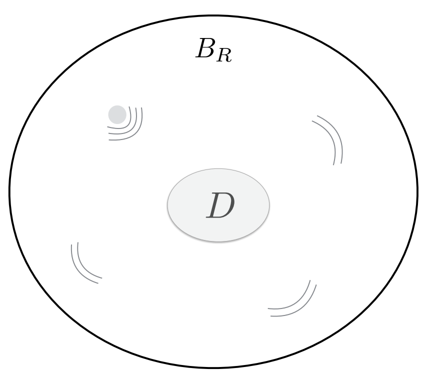

Figure 1: Radiation of a source in an inhomogeneous isotropic elastic medium in the exterior of a cavity. Suppose that the cavity D is known. The inverse problem is to determine the source term from the data measured on $\partial B_R = \{x \in \mathbb{R}^3 : |x| = R\}$.

In this section, we assume that (1.3) is fulfilled. Together with the governing equation (2.1), we fix the initial conditions (1.4) at $t = 0$ and the traction-free boundary condition $\mathcal{T}U$ on ∂D given by

$$\mathcal{T}U := \sigma(U)\nu = 0 \qquad \text{on} \quad \partial D \times \mathbb{R}^+, \tag{2.2}$$

where $\nu = (\nu_1, \nu_2, \nu_2)$ stands for the unit normal direction pointing into the exterior of D and the stress tensor $\sigma(U)$ is given by

$$\sigma(U) := \lambda \operatorname{div} U \, \mathbf{I}_3 + 2\mu E(U), \quad E(U) := \frac{1}{2}((\nabla U) + (\nabla U)^T). \tag{2.3}$$

Note that $\mathbf{I}_3$ means the 3-by-3 unit matrix and that the conormal derivative $\sigma(U)\nu$ corresponds to the *stress vector* or *surface traction* on ∂D. With these notation the Lamé operator (1.2) can be written as $\mathcal{L}_{\lambda,\mu}U = \operatorname{div} \sigma(U)$.

We suppose that $D \subset \mathbb{R}^3$ is a bounded domain with $\mathcal{C}^3$-smooth boundary ∂D and with connected exterior $\mathbb{R}^3 \backslash \overline{D}$. Below we give a confirmative answer to the uniqueness issue for IP1 and IP2. We start with IP1 and more precisely the proof of Theorem 1. If the cavity D is absent and the background medium is homogeneous isotropic, it was shown in [7] via strong Huygens principle and Fourier transform that the boundary data of Theorem 1 can be used to uniquely determine g. According to [30], in the context of Theorem 1, the strong Huygens principle is not valid and we can not even expect a uniform local energy decay (see [30, Corollary 0.3]). It seems that, only some specific type of logarithmic local energy decay, stated in [9] for smooth initial data, are available for our problem, but again such type of decay are not integrable on $\mathbb{R}^+$ with respect to the time variable. For this purpose, we use the Laplace transform in place of the Fourier transform. Below we shall present a proof valid not only in three dimensions but also in two dimensions.

Proof of Theorem 1. Assuming $U(x,t) = 0$ for $|x| = R$ and $t \in \mathbb{R}^+$, we need to prove that $g \equiv 0$. For $R_1 > R$, we fix $\Omega_1 := B_{R_1} \backslash \overline{D}$ and, using the fact that $\mathbb{R}^3 \setminus \overline{D}$ is connected, without loss of generality we can assume that R_1 is chosen in such a way that Ω_1 is connected. Since $\operatorname{Supp}(g) \subset B_R$, the wave field U fulfills the homogeneous initial and boundary conditions of the Lamé system in the exterior of B_R:

$$\begin{cases} \rho(x)\partial_{tt}U(x,t) - \mathcal{L}_{\lambda,\mu}U(x,t) = 0 & \text{in} \quad \mathbb{R}^3 \backslash \overline{B}_R \times \mathbb{R}^+, \\ U(x,0) = \partial_t U(x,0) = 0 & \text{in} \quad \mathbb{R}^3 \backslash B_R, \\ U(x,t) = 0 & \text{on} \quad \partial B_R \times \mathbb{R}^+. \end{cases} \tag{2.4}$$

Applying the elliptic regularity properties of $\mathcal{L}_{\lambda,\mu}$ (see e.g., [39, Chapters 4 and 10] and [21, Chapter 5]) and the results of [36, Theorem 8.1, Chapter 3], [36, Theorem 8.2, Chapter 3] (see also the beginning of Section 4.1 and 4.2 for more detail), one can prove the unique solvability of the initial boundary value problem (2.4). Consequently, we deduce that $U \equiv 0$ in $(\mathbb{R}^3 \backslash \overline{B}_R) \times \mathbb{R}^+$. In particular, since $R_1 > R$, we get the vanishing of the traction of U on $|x| = R_1$, that is,

$$\mathcal{T}U(x,t) = 0 \qquad \text{on} \quad \partial B_{R_1} \times \mathbb{R}^+.$$

Therefore, we deduce that $U \in \mathcal{C}^1([0,+\infty); L^2(\Omega_1))^3 \cap \mathcal{C}([0,+\infty); H^1(\Omega_1))^3$ is the unique solution to the boundary value problem

$$\begin{cases} \rho(x)\partial_t^2 U - L_{\lambda,\mu}U = F(x,t) & \text{in} \quad \Omega_1 \times (0,+\infty), \\ U(\cdot,0) = \partial_t U(\cdot,0) = 0 & \text{in} \quad \Omega_1, \\ \mathcal{T}U = 0 & \text{on} \quad \partial\Omega_1, \end{cases} \tag{2.5}$$

with $F(x,t) = f(t)g(x)$. Further, using standard idea for deriving energy estimates, one can prove that the solution has a long time behavior which is at most of polynomial type (see Proposition 9 in the Appendix). This allows us to define the Laplace transform of U with respect to the time variable as following:

$$\hat{U}(x,s) := \int_{\mathbb{R}} U(x,t)\, e^{-s\,t}\, dt, \quad s>0, \quad x \in \Omega_1,$$

and, we have $\hat{U}(\cdot,s) \in H^1(\Omega_1)^3$ for all $s>0$. In the same way, using the fact that $f \in H_0^1(0,T)$, we deduce that $U_1 = \partial_t U \in \mathcal{C}^1([0,+\infty); L^2(\Omega_1))^3 \cap \mathcal{C}([0,+\infty); H^1(\Omega_1))^3$ solves (2.5) with $F(x,t) = f'(t)g(x)$. In addition, repeating the arguments of Proposition 9, we can show that $\widehat{\partial_t U_1}(x,s) = \widehat{\partial_t^2 U}(x,s) \in L^2(\Omega_1)^3$ and $\widehat{\mathcal{L}_{\lambda,\mu} U}(x,s) = \mathcal{L}_{\lambda,\mu}\hat{U}(x,s) \in L^2(\Omega_1)^3$ are both well defined for all $s>0$. It follows that, for all $s>0$, $\widehat{\partial_t^2 U}(x,s) = s^2\hat{U}(x,s)$ and $\hat{U}(x,s)$ solves the boundary value problem

$$\begin{cases} -\mathcal{L}_{\lambda,\mu}\hat{U}(x,s) + \rho(x)s^2\hat{U}(x,s) = \hat{F}(x,s) & \text{in} \quad \Omega_1, \\ \mathcal{T}\hat{U}(x,s) = 0 & \text{on} \quad \partial\Omega_1. \end{cases} \tag{2.6}$$

Thus, using the elliptic regularity of the Lamé equation (see e.g. [21, Theorem 5.8.1]), we have $\hat{U}(\cdot,s) \in H^2(\Omega_1)^3$. We set $L^2_\rho(\Omega_1)^3$ the space of measurable functions u taking values in $\mathbb{R}^3$ such that $\rho^{\frac{1}{2}}u \in L^2(\Omega_1)^3$. We associate to $L^2_\rho(\Omega_1)^3$ the scalar product

$$\langle u, v\rangle_{L^2_\varrho(\Omega_1)^3} = \int_{\Omega_1} u(x)\cdot\overline{v(x)}\rho(x)\, dx, \quad u, v \in L^2_\varrho(\Omega_1)^3.$$

This implies that $||u||_{L^2_\varrho(\Omega_1)} := \langle u,u\rangle^{\frac{1}{2}}_{L^2_\varrho(\Omega_1)^3} = ||\varrho^{1/2}u||_{L^2(\Omega_1)}$.

We recall that the elliptic operator $-\rho^{-1}\mathcal{L}_{\lambda,\mu}$, with the traction-free boundary condition on $\partial\Omega_1 = \partial D \cup \partial B_{R_1}$, is a selfadjoint operator acting on $L^2_\varrho(\Omega_1)^3$ with a compact resolvent. Therefore, the spectrum of this operator is purely discrete. Denote by $\gamma_\ell \geq 0$, $l \in \mathbb{N}^+$, the increasing sequence of eigenvalues of the elliptic operator $-\rho^{-1}\mathcal{L}_{\lambda,\mu}$ in Ω_1 with the traction-free boundary condition on $\partial\Omega_1$, and denote by $\phi_{\ell,\kappa}$, $\kappa = 1,2,\cdots,m_\ell$, the orthonormal eigenfunctions associated with γ_l. By the ellipticity of the Lamé operator and the fact that the coefficients $\rho, \lambda \in \mathcal{C}^2(\overline{\Omega_1})$, $\mu \in \mathcal{C}^3(\overline{\Omega_1})$, we deduce that $\phi_{\ell,\kappa} \in H^2(\Omega_1)^3$, $\kappa = 1,2,\cdots,m_\ell$, $\ell \in \mathbb{N}^+$. Note that the set of eigenfunctions $\{\phi_{\ell,\kappa} : \ell \in \mathbb{N}^+, m = 1,2,\cdot,m_\ell\}$ forms a basis of $L^2_\rho(\Omega_1)^3$, and that the norm of $L^2_\varrho(\Omega_1)^3$ is equivalent to the one of $L^2(\Omega_1)^3$, since $\rho \in \mathcal{C}^2(\overline{\Omega_1})$ and $\rho > \rho_0 \geq 0$. Therefore, the unique solution to (2.6) can be represented as

$$\hat{U}(x,s) = \hat{f}(s) \sum_{\ell \in \mathbb{N}^+} \frac{\sum_{\kappa=1}^{m_\ell} \langle \rho^{-1}g, \phi_{\ell,\kappa}\rangle\, \phi_{\ell,\kappa}(x)}{s^2+\gamma_\ell}, \quad s>0,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product over $L^2_\varrho(\Omega_1)^3$ and the above series converges in the sense of $L^2_\varrho(\Omega_1)^3$. Using the fact that $f \in L^1(\mathbb{R}_+)$ is supported in $[0,T]$ and it does not vanish identically, we deduce that the function $\hat{f}$ is holomorphic in $s \in \mathbb{C}$ and not identically zero. Thus, there exists an interval $I \subset (0,+\infty)$ such that $|\hat{f}(s)| > 0$ for $s \in I$. Moreover, the sequence

$$\hat{U}_N(x,s) := \hat{f}(s) \sum_{\ell=1}^{N} \frac{\sum_{\kappa=1}^{m_\ell} \langle \rho^{-1}g, \phi_{\ell,\kappa}\rangle\, \phi_{\ell,\kappa}(x)}{s^2+\gamma_\ell}$$

converges to $\hat{U}(x,s)$ as $N \to \infty$ in the sense of $L^2_\varrho(\Omega_1)^3$. Since $L^2_\varrho(\Omega_1)^3$ is embedded continuously into $L^2(D_{R_1})^3$, with $D_{R_1} := \{x \in \mathbb{R}^3 : \ R < |x| < R_1\}$, we deduce that $\hat{U}_N(x,s)|_{x\in D_{R_1}}$ also converges to $\hat{U}(x,s)|_{x\in D_{R_1}}$ in the sense of $L^2(D_{R_1})^3$. These properties together with the fact that $\hat{U}(s)\equiv 0$ in D_{R_1}, $|\hat{f}(s)|>0$ for $s\in I$ imply that, for all $s\in I$, we have

$$\sum_{\ell\in\mathbb{N}^+}\frac{\sum_{\kappa=1}^{m_\ell}\langle\rho^{-1}g,\phi_{\ell,\kappa}\rangle\ \phi_{\ell,\kappa}(x)}{s^2+\gamma_\ell}=0\quad\text{for a.e. } x\in D_{R_1}.$$

On the other hand, the function

$$G(x,z):\quad z\to\sum_{\ell\in\mathbb{N}^+}^{\infty}\frac{\sum_{\kappa=1}^{m_\ell}\ \langle\rho^{-1}g,\phi_{\ell,\kappa}\rangle\phi_{\ell,\kappa}(x)}{z+\gamma_\ell},\quad z\in\mathbb{C}\backslash\{-\gamma_\ell:\ell\in\mathbb{N}^+\}$$

can be regarded as a holomorphic function in the variable z taking values in $L^2(D_{R_1})^3$. Hence, by unique continuation for holomorphic functions we deduce that the condition

$$G(x,s^2)=U(x,s)|_{x\in D_{R_1}}=0\qquad\text{for all}\qquad s\in I$$

implies that

$$G(x,z)=0\qquad\text{for all}\qquad z\in\mathbb{C}\backslash\{-\gamma_\ell:\ell\in\mathbb{N}^+\}.$$

It then follows that

$$(z+\gamma_j)\ G(x,z)\equiv 0,\quad z\in\mathbb{C}\backslash\{-\gamma_\ell:\ell\in\mathbb{N}^+\},\quad j\in\mathbb{N}^+,\ x\in D_{R_1}.\tag{2.7}$$

Therefore, letting $z\to-\gamma_j$ in (2.7) yields

$$\phi_j(x):=\sum_{\kappa=1}^{m_j}\langle\rho^{-1}g,\phi_{j,\kappa}\rangle\ \phi_{j,\kappa}(x)=0\qquad\text{for}\qquad x\in D_{R_1}.$$

On the other hand, we deduce that ϕ_j satisfies the elliptic equation

$$\mathcal{L}_{\lambda,\mu}\phi_j(x)+\rho(x)\gamma_j\phi_j(x)=0,\quad x\in\Omega_1.$$

Combining this with the unique continuation theorem of [5] and the fact that Ω_1 is connected, we get

$$\sum_{\kappa=1}^{m_j}\langle\rho^{-1}g,\phi_{j,\kappa}\rangle\phi_{j,\kappa}(x)=0,\qquad\qquad x\in\Omega_1.$$

Since $\{\phi_{j,\kappa}:\ \kappa=1,2,\cdots,m_j\}$ is an orthonormal family of vectors of $L^2_\varrho(\Omega_1)$, it follows that

$$\langle\rho^{-1}g,\phi_{j,\kappa}\rangle=0,\quad\kappa=1,2,\cdots,m_j.$$

Finally, by the arbitrariness of $j\in\mathbb{N}^+$ and the fact that $\mathrm{Supp}(g)\subset\overline{\Omega_1}$, we obtain

$$g\equiv\rho(\rho^{-1}g)\equiv 0.$$

□

We remark that the boundary surface data on ∂B_R are utilized in the proof of Theorem 1. As a corollary, we prove that interior volume observations can also be used to extract partial information of the spatial source term. Below we consider again the problem (2.1)-(2.2), with f,g being given as in Theorem 1.

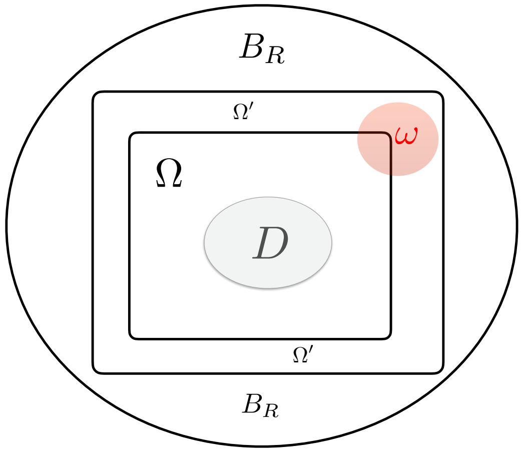

Figure 2: Suppose that g is known in Ω' and the data are collected on ω. The inverse problem is to determine the value of g on Ω.

Corollary 5. *Suppose that f is given, $V_0 = V_1 = 0$ and let Ω be a $\mathcal{C}^3$-smooth domain satisfying $D \subset \Omega \subset B_R$ for some $R > 0$. Let $\Omega' \subset B_R \backslash \overline{\Omega}$ be a neighborhood of $\partial\Omega$ such that (i) $\partial\Omega \subset \partial\Omega'$; (ii) Ω' is connected; (iii) g is known in Ω'; see Figure 2. Let $\omega \subset (B_R \backslash \overline{D})$ be an open set satisfying the condition $\omega \cap \Omega' \neq \emptyset$. Then the wave fields $U(x,t)$ measured on the volume $\omega \times \mathbb{R}^+$ uniquely determine $g|_\Omega$.*

Proof. Assuming that $g_{|\Omega'} \equiv 0$, we need to prove that the condition $U(x,t) = 0$ for $(x,t) \in \omega \times \mathbb{R}^+$ implies that $g|_\Omega \equiv 0$. Taking the Laplace transform $\hat{U}(x,s)$ of $U(x,t)$ with respect to the time variable, we deduce that, for all $s > 0$, $\hat{U}(x,s)$ solves

$$\begin{cases} -\mathcal{L}_{\lambda,\mu}\hat{U}(x,s) + s^2\rho(x)\hat{U}(x,s) = 0 & \text{in} \quad \Omega', \\ \hat{U}(x,s) = 0 & \text{on} \quad \omega. \end{cases}$$

Note that in this step, we make use of the a priori information that g is known in Ω'. Applying the unique continuation for Lamé system (see [5]), we get

$$\hat{U}(x,s) = 0 \quad \text{for all} \qquad x \in \Omega', \; s > 0.$$

Now using the fact that Ω' is a neighborhood of $\partial\Omega$, we deduce that, for all $s > 0$, $\mathcal{T}\hat{U}(x,s) = 0$ on $\partial\Omega$. Therefore, repeating the arguments of Theorem 1 with $\Omega_1 = \Omega$, we deduce that $g|_\Omega = 0$. □

Corollary 5 shows that, the observation data on the volume ω in a neighbourhood of Ω determines uniquely the source term g on Ω. This gives partial information of g only. However, in the special case that $\Omega \subset \mathrm{Supp}(g)$ (for instance, $\Omega = B_R$ and $\omega \subset \mathbb{R}^3 \backslash \overline{B}_R$), one may deduce from Corollary 5 that g can be uniquely determined by the data of U on $\omega \times \mathbb{R}^+$.

Now let us turn to the inverse problem (IP2) and more precisely to the proof of Theorem 2 which follows a path similar to Theorem 1.

Proof of Theorem 2. Assuming $U(x,t) = 0$ for $|x| = R$ and $t \in \mathbb{R}^+$, we need to prove that $V_0 = V_1 \equiv 0$. Since $\mathrm{Supp}(V_0) \cup \mathrm{Supp}(V_1) \subset B_R$, in a similar way to Theorem 1, we can prove that $U \equiv 0$ in $(\mathbb{R}^3 \backslash \overline{B}_{R_1}) \times \mathbb{R}^+$ for $R_1 > R$. Consequently, we get the vanishing of the traction of U on $|x| = R_1$. We fix $\Omega_1 := B_{R_1}\backslash\overline{D}$ and, again, we assume that R_1 is chosen in such way that Ω_1 is connected which is possible since $\mathbb{R}^3\backslash\overline{D}$ is connected. Therefore, we deduce that $U \in \mathcal{C}^2([0,+\infty); L^2(\Omega_1))^3 \cap \mathcal{C}([0,+\infty); H^1(\Omega_1))^3$ is the unique solution to the boundary value problem

$$\begin{cases} \rho(x)\partial_t^2 U - L_{\lambda,\mu}U = 0 & \text{in} \quad \Omega_1 \times (0,+\infty), \\ U(\cdot,0) = V_0, \quad \partial_t U(\cdot,0) = V_1 & \text{in} \quad \Omega_1, \\ \mathcal{T}U = 0 & \text{on} \quad \partial\Omega_1. \end{cases} \tag{2.8}$$

Then, using arguments similar to Theorem 1, we deduce that the Laplace transform in time $\hat{U}$ of U is well defined on $(0,+\infty)$ and for all $s > 0$ we have $\hat{U}(\cdot,s) \in H^1(\Omega_1)^3$. Combining this with the fact that $V_1 \in H^1(\mathbb{R}^3 \setminus D)^3$, $V_0 \in H^2(\mathbb{R}^3 \setminus D)^3$ with $\mathcal{T}V_0 = 0$ on ∂D, we deduce that $U_1 = \partial_t U \in \mathcal{C}^1([0,+\infty); L^2(\Omega_1))^3 \cap \mathcal{C}([0,+\infty); H^1(\Omega_1))^3$ solves (2.8) with $U_1(\cdot,0) = V_1$ and $\partial_t U_1(\cdot,0) = \rho^{-1}\mathcal{L}_{\lambda,\mu}V_0$. In addition, repeating the arguments of Proposition 9, we can show that $x \mapsto \widehat{\partial_t U_1}(x,s) = \widehat{\partial_t^2 U}(x,s) \in L^2(\Omega_1)^3$ and $x \mapsto \widehat{\mathcal{L}_{\lambda,\mu}U}(x,s) = \mathcal{L}_{\lambda,\mu}\hat{U}(x,s) \in L^2(\Omega_1)^3$ are both well defined for all $s > 0$. It follows that, for all $s > 0$, $x \mapsto \widehat{\partial_t^2 U}(x,s) = s^2\hat{U}(x,s) - V_1 - sV_0$ and $\hat{U}(\cdot,s)$ solves the boundary value problem

$$\begin{cases} -\mathcal{L}_{\lambda,\mu}\hat{U}(x,s) + \rho(x)s^2\hat{U}(x,s) = \rho(x)(sV_0(x) + V_1(x)) & \text{in} \quad \Omega_1, \\ \mathcal{T}\hat{U}(x,s) = 0 & \text{on} \quad \partial\Omega_1. \end{cases} \tag{2.9}$$

Using the notation of Theorem 1 and applying similar arguments, we deduce that, for all $s > 0$, this problem admits a unique solution $\hat{U}(\cdot,s) \in H^2(\Omega_1)^3$ taking the form

$$\hat{U}(x,s) = \sum_{\ell\in\mathbb{N}^+} \frac{\left[s\left(\sum_{\kappa=1}^{m_\ell}\langle V_0,\phi_{\ell,\kappa}\rangle\phi_{\ell,\kappa}(x)\right) + \left(\sum_{\kappa=1}^{m_\ell}\langle V_1,\phi_{\ell,\kappa}\rangle\phi_{\ell,\kappa}(x)\right)\right]}{s^2+\gamma_\ell}, \quad s > 0,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product over $L^2_\varrho(\Omega_1)^3$ and the above series converges in the sense of $L^2_\varrho(\Omega_1)^3$. Then, using the notation of Theorem 1, we get

$$\sum_{\ell\in\mathbb{N}^+} \frac{\left[s\left(\sum_{\kappa=1}^{m_\ell}\langle V_0,\phi_{\ell,\kappa}\rangle\phi_{\ell,\kappa}(x)\right) + \left(\sum_{\kappa=1}^{m_\ell}\langle V_1,\phi_{\ell,\kappa}\rangle\phi_{\ell,\kappa}(x)\right)\right]}{s^2+\gamma_\ell} = 0 \quad \text{for a.e. } x \in D_{R_1},\ s > 0. \tag{2.10}$$

On the other hand, one can easily check that the map

$$z \to \sum_{\ell\in\mathbb{N}^+} \frac{\left[z\left(\sum_{\kappa=1}^{m_\ell}\langle V_0,\phi_{\ell,\kappa}\rangle\phi_{\ell,\kappa}(x)\right) + \left(\sum_{\kappa=1}^{m_\ell}\langle V_1,\phi_{\ell,\kappa}\rangle\phi_{\ell,\kappa}(x)\right)\right]}{z^2+\gamma_\ell}, \quad z \in \mathbb{C}\backslash\{\pm i\sqrt{\gamma_\ell} : \ell \in \mathbb{N}^+\}$$

can be regarded as a holomorphic function in the variable z taking values in $L^2(D_{R_1})^3$. Hence, by unique continuation for holomorphic functions we deduce that the condition (2.10) implies that, for all $z \in \mathbb{C}\backslash\{\pm i\sqrt{\gamma_\ell} : \ell \in \mathbb{N}^+\}$, we have

$$\sum_{\ell\in\mathbb{N}^+} \frac{\left[z\left(\sum_{\kappa=1}^{m_\ell}\langle V_0,\phi_{\ell,\kappa}\rangle\phi_{\ell,\kappa}(x)\right) + \left(\sum_{\kappa=1}^{m_\ell}\langle V_1,\phi_{\ell,\kappa}\rangle\phi_{\ell,\kappa}(x)\right)\right]}{z^2+\gamma_\ell} = 0, \quad \text{for a.e. } x \in D_{R_1}. \tag{2.11}$$

Fixing $j \in \mathbb{N}^+$, multiplying this expression by $z - i\sqrt{\gamma_j}$ and sending $z \to i\sqrt{\gamma_j}$ we get

$$i\sqrt{\gamma_j}\left(\sum_{\kappa=1}^{m_j}\langle V_0,\phi_{j,\kappa}\rangle\phi_{j,\kappa}(x)\right) + \left(\sum_{\kappa=1}^{m_j}\langle V_1,\phi_{j,\kappa}\rangle\phi_{j,\kappa}(x)\right) = 0, \quad x \in D_{R_1}. \tag{2.12}$$

In the same way, multiplying (2.11) by $z+i\sqrt{\gamma_j}$ and sending $z\to -i\sqrt{\gamma_j}$ we get

$$-i\sqrt{\gamma_j}\left(\sum_{\kappa=1}^{m_j}\langle V_0,\phi_{j,\kappa}\rangle\phi_{j,\kappa}(x)\right)+\left(\sum_{\kappa=1}^{m_j}\langle V_1,\phi_{j,\kappa}\rangle\phi_{j,\kappa}(x)\right)=0,\quad z\in x\in D_{R_1}.$$

Combining this with (2.12), we obtain

$$\left(\sum_{\kappa=1}^{m_j}\langle V_1,\phi_{j,\kappa}\rangle\phi_{j,\kappa}(x)\right)=0,\quad x\in D_{R_1},\ j\in\mathbb{N}_+.$$

Then repeating the arguments used at the end of the proof of Theorem 1, we deduce that this condition implies that $V_1\equiv 0$. Then, applying (2.12), we deduce that

$$\left(\sum_{\kappa=1}^{m_j}\langle V_0,\phi_{j,\kappa}\rangle\phi_{j,\kappa}(x)\right)=0,\quad x\in D_{R_1},\ j\geq 2.$$

Note that the term for $j=1$ is not involved in the previous relation, because $\gamma_1=0$ under our traction-free boundary condition. In any case transferring all these information in (2.11) we get

$$\frac{z\left(\sum_{\kappa=1}^{m_1}\langle V_0,\phi_{1,\kappa}\rangle\phi_{1,\kappa}(x)\right)}{z^2+\gamma_1}=\frac{1}{z}\left(\sum_{\kappa=1}^{m_1}\langle V_0,\phi_{1,\kappa}\rangle\phi_{1,\kappa}(x)\right)=0$$

for $z\in\mathbb{C}\backslash\{\pm i\sqrt{\gamma_\ell}:\ell\in\mathbb{N}^+\}$ and $x\in D_{R_1}$, which clearly implies that

$$\sum_{\kappa=1}^{m_1}\langle V_0,\phi_{1,\kappa}\rangle\phi_{1,\kappa}(x)=0,\quad x\in D_{R_1}.$$

Thus, we have

$$\left(\sum_{\kappa=1}^{m_j}\langle V_0,\phi_{j,\kappa}\rangle\phi_{j,\kappa}(x)\right)=0,\quad x\in D_{R_1},\ j\in\mathbb{N}_+$$

and again we deduce that $V_0\equiv 0$. This completes the proof of the theorem. □

In a similar way to Corollary 5, the result of Theorem 2 can be reformulated with internal data as follows.

Corollary 6. *Suppose that $g=0$ and let Ω be a $\mathcal{C}^3$-smooth domain satisfying $D\subset\Omega\subset B_R$ for some $R>0$. Let $\Omega'\subset B_R\backslash\overline{\Omega}$ be a neighborhood of $\partial\Omega$ such that (i) $\partial\Omega\subset\partial\Omega'$; (ii) Ω' is connected; (iii) V_0 and V_1 are known in Ω'. Let $\omega\subset(B_R\backslash D)$ be an open set satisfying the condition $\omega\cap\Omega'\neq\emptyset$. Then the wave fields $U(x,t)$ measured on the volume $\omega\times\mathbb{R}^+$ uniquely determine $V_0|_\Omega$ and $V_1|_\Omega$.*

3 Determination of the source term $g(x_3)f(\tilde{x},t)$

In the previous section, we established uniqueness of recovering a spatial source term in an inhomogeneous background medium with or without embedded obstacles. However, the dependance of the source term on time and spatial variables are completely separated. The counterexamples constructed in [7] show that it is impossible to recover general source terms of the form $F(x,t)$ from the boundary observation on $\partial B_R\times(0,\infty)$. This implies that a priori information on the source term is always necessary for a uniqueness proof. In this section we restrict our discussions to the inverse problem (IP3) for treating an alternative source term of the form $g(x_3)f(\tilde{x},t)$, where the vectorial function $f=(f_1,f_2,0)$ is compactly supported on $\tilde{B}_R\times[0,T)$ and the scalar function g is supported in $(-R,R)$ for some $R>0$.

In this section, we assume that $D=\emptyset$ and suppose that background medium is homogeneous with constant Lamé coefficients λ, μ and a constant density function ρ. Below we shall consider the initial value problem

$$\begin{cases}\rho\partial_{tt}U(x,t)=\mathcal{L}_{\lambda,\mu}U(x,t)+g(x_3)\,f(\tilde{x},t), & x\in\mathbb{R}^3,\ t>0,\\ U(x,0)=\partial_t\,U(x,0)=0, & x\in\mathbb{R}^3.\end{cases}\tag{3.1}$$

The function $g(x_3)\,f(\tilde{x},t)$ can be used to model source terms which mainly radiate over the ox_1x_2-plane and $g(x_3)$ can be regarded as an approximation of the delta function $\delta(x_3)$ in the x_3-direction. Suppose that the function g is known in advance. Our inverse problem in this section is concerned with the recovery of f from $U(x,t)$ measured on $\Gamma\times(0,T_1)$ for some $T_1>0$. Note that Γ is an open subset of ∂B_{R_1} for some $R_1>\sqrt{2}R$. The proof of the uniqueness result, stated in Theorem 3, together with the stability result stated in Theorem 4, will be presented in the subsequent two subsections.

3.1 Proof of Theorem 3

By Lemma 1 in the appendix, the boundary value problem (3.1) admits a unique solution $U\in\mathcal{C}^2([0,+\infty);L^2(\mathbb{R}^3))^3\cap\mathcal{C}([0,+\infty);H^2(\mathbb{R}^3))^3$. Below we state a uniqueness result with partial boundary data measured over a finite time. More precisely, assuming that the condition

$$U(x,t)=0,\quad x\in\Gamma,\ t\in[0,T_1]\tag{3.2}$$

holds true, we will prove that $f=0$. Note first that, since $\mathrm{Supp}(f)\subset\tilde{B}_R\times[0,T)$ and since $U(\cdot,0)=\partial_tU(\cdot,0)=0$, the extension of U by 0 to $\mathbb{R}^3\times\mathbb{R}$ solves the problem

$$\begin{cases}\rho\partial_{tt}U(x,t)=\mathcal{L}_{\lambda,\mu}U(x,t)+g(x_3)\,f(\tilde{x},t), & x\in\mathbb{R}^3,\ t\in\mathbb{R},\\ U(x,0)=\partial_t\,U(x,0)=0, & x\in\mathbb{R}^3.\end{cases}$$

From now on, we denote by U the extension of U by 0 to $\mathbb{R}^3\times\mathbb{R}$. Let us observe that, since f is compactly supported on $\tilde{B}_R\times[0,T)$ and g is supported in $(-R,R)$, for

$$F(\tilde{x},x_3,t):=f(\tilde{x},t)g(x_3),\quad \tilde{x}\in\mathbb{R}^2,\ x_3\in\mathbb{R},\ t\in[0,+\infty),$$

one can easily check that $\mathrm{Supp}(F)\subset B_{\sqrt{2}R}\times[0,T)$. Therefore, by the strong Huygens principle (see e.g. [7, Lemma 2.1]), for any $r\geq0$, it holds that $U(x,t)=0$ for all $|x|<R_1+r$ and $t>T+\frac{2(R_1+r)\sqrt{\rho}}{\sqrt{\mu}}$. It follows that, for all $r\geq0$, we have

$$U(x,t)=0,\quad |x|<R_1+r,\ t\notin\left[0,T+\frac{2(R_1+r)\sqrt{\rho}}{\sqrt{\mu}}\right].\tag{3.3}$$

In particular, using the fact that $T_1=T+\frac{2R_1\sqrt{\rho}}{\sqrt{\mu}}$, we get

$$U(x,t)=0,\quad |x|<R_1,\ t\notin[0,T_1]\,.\tag{3.4}$$

In view of (3.3), applying the Fourier transform in time to U gives

$$\mathcal{L}_{\lambda,\mu}\hat{U}(x,\omega)+\omega^2\rho\hat{U}(x,\omega)=-g(x_3)\,\hat{f}(\tilde{x},\omega),\qquad x\in\mathbb{R}^3,\ \omega\in\mathbb{R}, \tag{3.5}$$

where

$$\hat{U}(x,\omega):=\int_{\mathbb{R}}U(x,t)e^{-i\omega t}dt,\qquad \omega\in\mathbb{R}$$

satisfies the Kupradze radiation condition as $|x|\to\infty$ (see [7, 35]) for any fixed $\omega\in\mathbb{R}$. Here $\hat{f}(\tilde{x},\omega)$ denotes the Fourier transform of $f(\tilde{x},t)$ with respect to the time variable. Combining (3.2) with (3.4), we obtain the boundary condition $\hat{U}(x,\omega)=0$, $x\in\Gamma$, $\omega\in\mathbb{R}$. Since, for all $\omega\in\mathbb{R}$, the support of the function $x\mapsto\hat{f}(\tilde{x},\omega)g(x_3)$ is contained into B_{R_1}, by elliptic interior regularity, we deduce that $x\mapsto\hat{U}(x,\omega)$ is analytic with respect to the spatial variable x in a neighborhood of ∂B_R. By analyticity of both the surface ∂B_{R_1} and the function $\hat{U}(\cdot,\omega)$, we get the vanishing of $\hat{U}(x,\omega)$ on the whole boundary ∂B_{R_1} for any $\omega\in\mathbb{R}$. In view of the uniqueness to the Dirichlet boundary value problem in the unbounded domain $|x|>R_1$ (see e.g., [8]), we get

$$\hat{U}(x,\omega)=0,\qquad |x|>R_1,\quad \omega\in\mathbb{R}.$$

Consequently, we have $\mathcal{T}\hat{U}(x,\omega)=0$ on ∂B_{R_1}. Since the source term $f=(f_1,f_2,0)^\top$ is compactly supported on $\tilde{B}_R$, by Hodge decomposition the function $\hat{F}$ can be spatially decomposed into the form

$$\hat{f}(\tilde{x},\omega)=\begin{pmatrix}\nabla_{\tilde{x}}\,\hat{f}_p(\tilde{x},\omega)\\0\end{pmatrix}+\begin{pmatrix}\nabla_{\tilde{x}}^{\perp}\,\hat{f}_s(\tilde{x},\omega)\\0\end{pmatrix}, \tag{3.6}$$

where $\hat{f}_p(\cdot,\omega)$ and $\hat{f}_s(\cdot,\omega)$ are scalar functions compactly supported on $\tilde{B}_R$ as well. Here $\nabla_{\tilde{x}}=(\partial_1,\partial_2)^\top$, $\nabla_{\tilde{x}}^{\perp}=(-\partial_2,\partial_1)^\top$. For $\xi=(\xi_1,\xi_2)\in\mathbb{R}^2$ satisfying

$$|\xi|>\max(|k_p|,|k_s|),\quad k_p^2:=\frac{\omega^2\rho}{\lambda+2\mu},\quad k_s^2:=\frac{\omega^2\rho}{\mu},$$

we introduce the test functions

$$V_p(x,\omega)=\begin{pmatrix}-i\xi_1\\-i\xi_2\\\sqrt{|\xi|^2-k_p^2}\end{pmatrix}e^{-i\xi\cdot\tilde{x}+\sqrt{|\xi|^2-k_p^2}\,x_3},$$

$$V_s(x,\omega)=\begin{pmatrix}i\xi_2\\-i\xi_1\\0\end{pmatrix}e^{-i\xi\cdot\tilde{x}+\sqrt{|\xi|^2-k_s^2}\,x_3}.$$

The numbers k_p and k_s denote respectively the compressional and shear wave numbers in the frequency domain. One can easily check that $\nabla_{\tilde{x}}^{\perp}\cdot V_p\equiv 0$, $\nabla_{\tilde{x}}\cdot V_s\equiv 0$ in $\mathbb{R}^3$ and, using the fact that

$$\nabla_x\times V_p=0,\quad (\lambda+2\mu)\nabla_x\nabla_x\cdot V_p=-\omega^2\rho V_p,$$

$$\nabla_x\cdot V_s=0,\quad -\mu\nabla_x\times(\nabla_x\times V_s)=-\omega^2\rho V_s,$$

we deduce that V_α ($\alpha=p,s$) satisfies the homogeneous Lamé system in the frequency domain

$$\mathcal{L}_{\lambda,\mu}V_\alpha(x,\omega)+\omega^2\rho\,V_\alpha(x,\omega)=0\quad\text{in}\quad\mathbb{R}^3,\qquad x\in\mathbb{R}^3,\quad \alpha=p,s,$$

for any fixed $\omega \in \mathbb{R}$. Now, taking the scalar product with V_p on both sides of the equation (3.5) and applying Betti's formula, we obtain

$$
\begin{aligned}
&\int_{B_{R_1}} \left(\mathcal{L}_{\lambda,\mu}\hat{U}(x,\omega) + \omega^2\rho\hat{U}(x,\omega)\right) \cdot V_p(x,\omega)\,dx \\
= &\int_{\partial B_{R_1}} \mathcal{T}\hat{U}(x,\omega)\cdot V_p(x,\omega) - \mathcal{T}V_p(x,\omega)\cdot \hat{U}(x,\omega)\,ds(x) \\
= &\ 0,
\end{aligned}
$$

where we have used the vanishing of the Cauchy data of $\hat{U}(\cdot,\omega)$ and $\mathcal{T}\hat{U}(\cdot,\omega)$ on ∂B_{R_1}. On the other hand, making use of (3.6) together with the relation $\nabla_{\tilde{x}}^{\perp}\cdot V_p \equiv 0$ yields

$$
\begin{aligned}
0 &= \int_{B_{R_1}} V_p(x,\omega)\cdot \hat{F}(x)(\tilde{x},\omega) g(x_3) dx \\
&= \int_{B_{R_1}} V_p(x,\omega)\cdot \begin{pmatrix} \nabla_{\tilde{x}}\hat{f}_p(\tilde{x},\omega) \\ 0 \end{pmatrix} g(x_3)dx \\
&= \left(\int_{\tilde{B}_R} \begin{pmatrix} -i\xi_1 \\ -i\xi_2 \\ 0 \end{pmatrix} e^{-i\xi\cdot\tilde{x}} \cdot \begin{pmatrix} \nabla_{\tilde{x}}\hat{f}_p(\tilde{x},\omega) \\ 0 \end{pmatrix} d\tilde{x}\right) \left(\int_{-R}^{R} g(x_3) e^{\sqrt{|\xi|^2-k_p^2}\,x_3}\,dx_3\right) \\
&= |\xi|^2 \left(\int_{\tilde{B}_R} e^{-i\xi\cdot\tilde{x}}\hat{f}_p(\tilde{x},\omega)\,d\tilde{x}\right)\left(\int_{-R}^{R} g(x_3)e^{\sqrt{|\xi|^2-k_p^2}\,x_3}\,dx_3\right)
\end{aligned}
$$

for all $\omega \in \mathbb{R}$ and $\xi \in \mathbb{R}^2$ satisfying $|\xi| > \max(k_p, k_s)$. Since g is compactly supported and lying in $L^1((-R,R))$, the function

$$\mathbb{C} \ni z \mapsto \int_{-R}^{R} g(x_3)e^{z\,x_3}\,dx_3$$

is holomorphic in $\mathbb{C}$. Then, using the fact that g is not uniformly vanishing, for every $\omega \in \mathbb{R}$, we can find an open and not-empty interval $I_\omega \subset (\max(k_p,k_s), +\infty)$ such that

$$\int_{-R}^{R} g(x_3)e^{\sqrt{|\xi|^2-k_p^2}\,x_3}\,dx_3 \neq 0, \quad \xi \in \mathbb{R}^2,\ |\xi| \in I_\omega.$$

Hence, for every $\omega \in \mathbb{R}$, we have

$$\int_{\tilde{B}_R} e^{-i\xi\cdot\tilde{x}}\hat{f}_p(\tilde{x},\omega)\,d\tilde{x} = 0 \quad \text{for all} \quad \xi \in \mathbb{R}^2,\ |\xi| \in I_\omega. \tag{3.7}$$

This implies that, for $\omega \in \mathbb{R}$ and for $\hat{f}_p(\cdot,\omega) : \tilde{x} \mapsto \hat{f}_p(\tilde{x},\omega)$, the Fourier transform $\mathcal{F}_{\tilde{x}}[\hat{f}_p](\xi)$ of $\hat{f}_p(\cdot,\omega)$ with respect to $\tilde{x} \in \mathbb{R}^2$ vanishes for $\xi \in \{\eta \in \mathbb{R}^2 : |\eta| \in I_\omega\}$. On the other hand, since, for all $\omega \in \mathbb{R}$, $\hat{f}_p(\cdot,\omega)$ is supported in $\tilde{B}_R$, the function

$$\xi \mapsto \int_{\mathbb{R}^3} e^{-i\xi\cdot\tilde{x}}\hat{f}_p(\tilde{x},\omega)\,d\tilde{x} = \int_{\tilde{B}_R} e^{-i\xi\cdot\tilde{x}}\hat{f}_p(\tilde{x},\omega)\,d\tilde{x}$$

is real analytic with respect to $\xi \in \mathbb{R}^2$. Then, using the fact that the set $\{\xi \in \mathbb{R}^2 :\ |\xi| \in I_\omega\}$ is an open subset of $\mathbb{R}^2$, it follows from (3.7) that

$$\int_{\tilde{B}_R} e^{-i\xi\cdot\tilde{x}}\hat{f}_p(\tilde{x},\omega)\,d\tilde{x} = 0 \quad \text{for all} \quad \xi \in \mathbb{R}^2.$$

Applying the inverse Fourier transform in $\tilde{x}$, we get $\hat{f}_p(\cdot, \omega) = 0$ for all $\omega \in \mathbb{R}$. Further, applying the inverse Fourier transform in t yields $f_p(\tilde{x}, t) \equiv 0$ for all $\tilde{x} \in \tilde{B}_R$ and $t > 0$. The fact that $f_s \equiv 0$ can be verified analogously by considering the scalar product with V_s on both sides of (3.5). This finishes the proof of the relation $f \equiv 0$ in $\tilde{B}_R \times (0, T)$.

3.2 Proof of Theorem 4

To derive stability estimate of f, we need extra regularity assumptions on the source term. As done in the previous section, we still assume that $\mathrm{Supp}(f) \subset \tilde{B}_R \times [0, T)$, $\mathrm{Supp}(g) \subset (-R, R)$ and $g \in L^2(-R, R)$. In the following lemma, the dynamic data are measured over the whole boundary ∂B_R and the proof is carried out in the time domain without using the Fourier transform. As done in (3.6), we can split f via Hodge decomposition into the form

$$f(\tilde{x}, t) = \begin{pmatrix} \nabla_{\tilde{x}} f_p(\tilde{x}, t) \\ 0 \end{pmatrix} + \begin{pmatrix} \nabla_{\tilde{x}}^{\perp} f_s(\tilde{x}, t) \\ 0 \end{pmatrix}, \tag{3.8}$$

where $f_p(\cdot, t)$ and $f_s(\cdot, t)$ are scalar functions compactly supported on $\tilde{B}_R$. Fixing $\omega > 0$ and $\xi \in \mathbb{R}^2$ such that

$$|\xi|^2 > k_p^2 := \frac{\omega^2 \rho}{\lambda + 2\mu}, \tag{3.9}$$

we introduce the time-dependent test function

$$V_p(x, t; \xi, \omega) \quad = \quad \begin{pmatrix} -i\xi_1 \\ -i\xi_2 \\ \sqrt{|\xi|^2 - k_p^2} \end{pmatrix} e^{-i\xi\cdot\tilde{x} + \sqrt{|\xi|^2 - k_p^2}\, x_3}\, e^{-i\omega t}.$$

In the same way, for

$$|\xi|^2 > k_s^2 := \frac{\omega^2 \rho}{\mu}, \tag{3.10}$$

we introduce the function

$$V_s(x, t; \xi, \omega) \quad = \quad \begin{pmatrix} i\xi_2 \\ -i\xi_1 \\ 0 \end{pmatrix} e^{-i\xi\cdot\tilde{x} + \sqrt{|\xi|^2 - k_s^2}\, x_3}\, e^{-i\omega t}.$$

Then, in a similar way to the proof of the uniqueness result, one can check that V_α ($\alpha = p, s$) are solutions to the homogeneous elastodynamic equation

$$\rho \frac{\partial^2}{\partial t^2} V_\alpha(x, t; \xi, \omega) - \mathcal{L}_{\lambda,\mu} V_\alpha(x, t; \xi, \omega) = 0 \qquad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}^+ \tag{3.11}$$

for any fixed $\xi \in \mathbb{R}^2$ and $\omega \in \mathbb{R}$ satisfying (3.9) or (3.10). Moreover, one can easily check that

$$\nabla_{\tilde{x}}^{\perp} \cdot V_p(x, t) = \nabla_{\tilde{x}} \cdot V_s(x, t) = 0, \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{3.12}$$

Therefore, taking the scalar product with $V_p(x,t;\xi,\omega)$ to the right hand side of the equation (3.1), using (3.11) and applying integration by parts yield

$$
\begin{aligned}
& \int_0^{T_1}\int_{B_{R_1}} \left(\rho\frac{\partial^2}{\partial t^2}U(x,t) - \mathcal{L}_{\lambda,\mu}U(t,x)\right)\cdot V_p(x,t;\xi,\omega)\,dxdt \\
= \;& -\int_0^{T_1}\int_{\partial B_{R_1}} \mathcal{T}U(x,t)\cdot V_p(x,t;\xi,\omega) - \mathcal{T}V_p(t,x;\xi,\omega)\cdot U(x,t;\xi,\omega)\,ds(x)\,ds(x)\,dt \\
& + \int_0^{T_1}\int_{B_{R_1}} \left(\rho\frac{\partial^2}{\partial t^2}U(x,t)\cdot V_p(x,t;\xi,\omega) - \rho\frac{\partial^2}{\partial t^2}V_p(x,t)\cdot U(x,t;\xi,\omega)\right) dxdt \\
= \;& -\int_0^{T_1}\int_{\partial B_{R_1}} \mathcal{T}U(x,t)\cdot V_p(x,t;\xi,\omega) - \mathcal{T}V_p(t,x;\xi,\omega)\cdot U(x,t;\xi,\omega)\,ds(x)\,dt \\
& + \rho\int_{B_{R_1}} \frac{\partial U(x,T_1)}{\partial t}\cdot V_p(x,T_1;\xi,\omega) - \frac{\partial V_p(x,T_1;\xi,\omega)}{\partial t}\cdot U(x,T_1)\,dx.
\end{aligned}
$$

Again recalling the strong Huygens principle, we know that (3.3) holds true and, using the fact that $T_1 > T + \frac{2R_1\sqrt{\rho}}{\sqrt{\mu}}$, we deduce that, $U(x,T_1) = \partial_t U(x,T_1) = 0$, $x \in B_{R_1}$. Hence, the integral over B_{R_1} on the right hand side of the previous identity vanishes. Following estimate (4.6) of Proposition 8 in the appendix, the traction of U on the boundary ∂B_{R_1} can be bounded by the trace of U itself. Hence, the left hand side can be bounded by

$$
\begin{aligned}
& \left|\int_0^{T_1}\int_{B_{R_1}} \left(\rho\frac{\partial^2}{\partial t^2}U(x,t) - \mathcal{L}_{\lambda,\mu}U(t,x)\right)\cdot V_p(x,t;\xi,\omega)\,dxdt\right| \\
= \;& \left|\int_0^{T_1}\int_{\partial B_{R_1}} \mathcal{T}U(x,t)\cdot V_p(x,t;\xi,\omega) - \mathcal{T}V_p(t,x;\xi,\omega)\cdot U(x,t;\xi,\omega)\,ds(x)\,dt\right| \\
\le \;& ||\mathcal{T}U||_{L^2((0,T_1)\times\partial B_{R_1})^3}||V_p||_{L^2((0,T_1)\times\partial B_{R_1})^3} + ||U||_{L^2((0,T_1)\times\partial B_{R_1})^3}||\mathcal{T}V_p||_{L^2((0,T_1)\times\partial B_{R_1})^3} \\
\le \;& C\,\left(\|U\|_{H^3(0,T_1;H^{3/2}(\partial B_{R_1})^3}\,||V_p||_{L^2((0,T_1)\times\partial B_{R_1})^3} + ||U||_{L^2((0,T_1)\times\partial B_{R_1})^3}\,||V_p||_{L^2(0,T_1;H^2(B_{R_1}))}\right) \\
\le \;& C\,\|U\|_{H^3(0,T_1;H^{3/2}(\partial B_{R_1})^3}\,||V_p||_{L^2(0,T_1;H^2(B_{R_1}))} \\
\le \;& C\,\|U\|_{H^3(0,T_1;H^{3/2}(\partial B_{R_1})^3}\,(1+|(\xi,\omega)|^3)e^{R_1\sqrt{|\xi|^2-k_p^2}}
\end{aligned}
$$

for all $|\xi| > k_p$, where $C > 0$ depends on M, R_1, T_1, ρ, λ and μ. On the other hand, using the governing equation (3.1) together with the relations (3.8), (3.12) and using the fact that the sign of

g is constant, we obtain

$$
\begin{aligned}
&\left|\int_0^{T_1}\int_{B_{R_1}}\left(\rho\frac{\partial^2}{\partial t^2}U(x,t)-\mathcal{L}_{\lambda,\mu}U(x,t)\right)\cdot V_p(x,t;\xi,\omega)\,dxdt\right| \\
=\;&\left|\int_0^{T_1}\int_{B_{R_1}}F(\tilde{x},t)g(x_3)\cdot V_p(x,t;\xi,\omega)\,dx\,dt\right| \\
=\;&\left|\int_0^{T_1}\int_{B_{R_1}}\begin{pmatrix}\nabla_{\tilde{x}}f_p(\tilde{x},t)\\ 0\end{pmatrix}g(x_3)\;\cdot V_p(x,t;\xi,\omega)\,dx\,dt\right| \\
=\;&|\xi|^2\left|\int_0^{T_1}\int_{B_{R_1}}f_p(\tilde{x},t)g(x_3)\,e^{-i\xi\cdot\tilde{x}+\sqrt{|\xi|^2-k_p^2}\,x_3}\,e^{-i\omega t}\,dx\,dt\right| \\
=\;&|\xi|^2\left|\left(\int_0^{T_1}\int_{\tilde{B}_R}f_p(\tilde{x},t)e^{-i\xi\cdot\tilde{x}-i\omega t}d\tilde{x}dt\right)\left(\int_{-R}^{R}g(x_3)\,e^{\sqrt{|\xi|^2-k_p^2}\,x_3}\,dx_3\right)\right| \\
\geq\;&|\xi|^2\left|\left(\int_0^{T_1}\int_{\tilde{B}_R}f_p(\tilde{x},t)e^{-i\xi\cdot\tilde{x}-i\omega t}d\tilde{x}dt\right)\right|\,\|g\|_{L^1(\mathbb{R})}\;e^{-(\sqrt{|\xi|^2-k_p^2})R},
\end{aligned}
$$

for all $|\xi|>k_p$. Since f_p is supported on $\tilde{B}_R\times(0,T_1)$, the first integral on the right hand of the last identity is the Fourier transform of f_p with respect to $(\tilde{x},t)$ at the value (ξ,ω), which we denote by $\hat{f}_p(\xi,\omega)$. Combining the previous two relations we obtain

$$
|\hat{f}_p(\xi,\omega)|\leq C\,\frac{(1+|(\xi,\omega)|^3)\,\|U\|_{H^3(0,T_1;H^{3/2}(\partial B_{R_1}))^3}\,e^{2R_1\sqrt{|\xi|^2-k_p^2}}}{|\xi|^2\;\|g\|_{L^1(\mathbb{R})}} \tag{3.13}
$$

for all $|\xi|>k_p(\omega)$. We note that (3.13) gives the estimate of $\hat{f}_p$ over the cone $\{(\xi,\omega)\in\mathbb{R}^3:|\xi|^2>\omega^2\rho/(\lambda+2\mu)\}$. In order to derive from (3.13) a stability estimate of $\hat{f}_p$ on B_r for a large $r>0$, we will use a result of stability in the analytic continuation, following the arguments presented in [15, 31]. Note that in contrast to [15, 31] we also need to overcome the difficulty arising from the fact that the right hand side of (3.13) is singular at $\xi=0$. Below we state a stability estimate for analytic continuation problems; see [6, Theorem 4] (see also [38, 40], where similar results were established).

Proposition 7. *Let $s>0$ and assume that $g:\ B_{2s}\subset\mathbb{R}^3\to\mathbb{C}$ is a real analytic function satisfying*

$$
\left\|\nabla^\beta g\right\|_{L^\infty(B_{2s})}\leq\frac{N\,\beta\,!}{(s\tau)^{|\beta|}},\qquad \beta=(\beta_1,\beta_2,\beta_2)\in\mathbb{N}^3,
$$

for some $N>0$ and $0<\tau\leq 1$. Further let $\Lambda\subset B_{s/2}$ be a measurable set with strictly positive Lebesgue measure. Then,

$$
\|g\|_{L^\infty(B_s)}\leq CN^{(1-b)}\,\|g\|^b_{L^\infty(\Lambda)}\,, \tag{3.14}
$$

where $b\in(0,1)$, $C>0$ depend on τ, $|\Lambda|$ and s.

To proceed with the proof of Theorem 4, we follow the lines in [31] by introducing the function

$$
H_r(\xi,\omega):=\hat{f}_p(r(\xi,\omega))=(2\pi)^{-3/2}\int_{\mathbb{R}^3}f_p(\tilde{x},t)e^{-ir(\omega t+\xi\cdot\tilde{x})}d\tilde{x}dt
$$

for some $r > 1$ and $|(\xi,\omega)| \leq 2s$. In a similar way to [31], we fix $s = [\max(T_1, 2R)]^{-1} + 1$, choose $N = Ce^{3r}$, with C some constant independent of r, and take $\tau = \frac{[\max(T_1,2R)]^{-1}}{s} = (s-1)/s$. Then we obtain

$$\left\|\partial_\omega^n \partial_\xi^\beta H_r\right\|_{L^\infty(B_{2s})} \leq C\frac{e^{3r}\beta! n!}{([\max(T,2R)]^{-1})^{|\beta|+n}} = \frac{N\beta! n!}{(s\tau)^{|\beta|+n}}, \quad n \in \mathbb{N}_+,\ \beta \in \mathbb{N}_+^2. \tag{3.15}$$

In contrast to many other results (actually all the results that we know) established by an application of a stability estimate for analytic continuation problems similar to the one of Problem 7 (see for instance [11, 15, 31]), the right hand side of our initial estimate (3.13) is singular at $\xi = 0$. For this reason, we can not apply Proposition 7 with a set Λ independent of r. However, we need to choose Λ in such a way that the parameters N and b appearing in (3.14) will be independent of r. For this purpose, fixing $c := \frac{\rho}{\lambda+2\mu}$, $d := \frac{s}{2\sqrt{1+c^{-1}}}$ and $a_r \in \left(0, \frac{d}{\sqrt{c}}\right)$, we define

$$\Lambda_r := \left\{(\xi,\omega) \in \tilde{B}_d \times \left[-a_r, \frac{d}{\sqrt{c}}\right] :\ \max(r^{-2}, \sqrt{c}|\omega|) < |\xi|\right\}.$$

It is easy to check that Λ_r is a subset of $B_{s/2}$ in $\mathbb{R}^3$, and it is also a subset of the cone $\{(\xi,\omega) \in \mathbb{R}^3 : |\xi|^2 > \omega^2\rho/(\lambda+2\mu)\}$. We remark that $|\Lambda_r| = \kappa_r(-a_r)$, where

$$\kappa_r : y \mapsto \int_y^{\frac{d}{\sqrt{c}}} \int_{\max(r^{-2},\sqrt{c}|\omega|)<|\xi|<d} d\xi d\omega.$$

Note that $\kappa_r\left(-\frac{d}{\sqrt{c}}\right) = 2\kappa_r(0)$ and one can check that

$$\kappa_r(0) = \frac{2\pi d^3}{3\sqrt{c}} - \frac{2\pi r^{-6}}{3\sqrt{c}}.$$

Thus, there exists $r_0 > 1$ depending only on R, ρ, λ, μ, T, such that

$$\frac{\pi d^3}{2\sqrt{c}} < \kappa_r(0) < \frac{2\pi d^3}{3\sqrt{c}}, \quad r > r_0.$$

Therefore, we have

$$\kappa_r\left(-\frac{d}{\sqrt{c}}\right) = 2\kappa_r(0) > \frac{\pi d^3}{\sqrt{c}} > \frac{2\pi d^3}{3\sqrt{c}} > \kappa_r(0), \quad r > r_0$$

and, from the continuity of the map κ_r, we deduce that we can choose a_r in such way that

$$|\Lambda_r| = \kappa_r(a_r) = \frac{\pi d^3}{\sqrt{c}}, \quad r > r_0.$$

This implies that, with such choice of a_r, the volume $|\Lambda_r|$ depends only on R, ρ, λ, μ and T_1. Consequently, combining (3.15) with Proposition 7, we deduce that

$$|\hat{f}_p(r(\xi,\omega))| = |H_r(\xi,\omega)| \leq Ce^{3(1-b)r}\left(\|H_r\|_{L^\infty(\Lambda_r)}\right)^b, \quad |(\xi,\omega)| < s,\ r > r_0,$$

where $C > 0$, $b \in (0,1)$ depend only on R, ρ, λ, μ and T_1. In addition, applying (3.13), we get

$$\|H_r\|_{L^\infty(\Lambda_r)} \leq Cr^2 e^{c_1 r} \|U\|_{H^3(0,T;H^{3/2}(\partial B_R))^3},$$

where C and c_1 depend only on R, ρ, λ, μ and T_1. Therefore, we can find C, c depending only on R, ρ, λ, μ and T_1 such that

$$|\hat{f}_p(\xi,\omega)| \leq Ce^{cr} \|U\|_{H^3(0,T_1;H^{3/2}(\partial B_R))^3}, \quad |(\xi,\omega)| < r,\ r > sr_0.$$

It follows that

$$\int_{B_r} |\hat{f}_p(\xi,\omega)|^2 d\xi d\omega \leq Ce^{cr} \|U\|_{H^3(0,T_1;H^{3/2}(\partial B_R))^3}, \quad |(\xi,\omega)| < r,\ r > sr_0, \tag{3.16}$$

by eventually replacing the constants C and c. On the other hand, using (1.8) and the fact that $\Delta_{\tilde{x}} f_p = \nabla_{\tilde{x}} \cdot f$, we deduce that $f_p \in H^2(\mathbb{R}^3)$ and $\|f_p\|_{H^2(\mathbb{R}^3)} \leq CM$, with C depending only on T and R. Thus, we find

$$\begin{aligned}\int_{|(\xi,\omega)|>r} |\hat{f}_p(\xi,\omega)|^2 d\xi d\omega &\leq r^{-4} \int_{|(\xi,\omega)|>r} (1+|(\xi,\omega)|^2)^4 |\hat{f}_p(\xi,\omega)|^2 d\xi d\omega \\ &\leq r^{-4} \|f_p\|_{H^2(\mathbb{R}^3)} \leq C^2 r^{-4} M^2.\end{aligned}$$

Combining this with (3.16), for all $r > sr_0$, we find

$$\begin{aligned}\int_{\mathbb{R}^3} |\hat{f}_p(\xi,\omega)|^2 d\xi d\omega &= \int_{B_r} |\hat{f}_p(\xi,\omega)|^2 d\xi d\omega + \int_{|(\xi,\omega)|>r} |\hat{f}_p(\xi,\omega)|^2 d\xi d\omega \\ &\leq C\left(e^{cr} \|U\|_{H^3(0,T_1;H^{3/2}(\partial B_{R_1}))^3} + r^{-4}\right).\end{aligned}$$

Recalling the Plancherel formula, it holds that

$$\|f_p\|_{L^2((0,T_1)\times\tilde{B}_{R_1})} \leq C\left(e^{cr} \|U\|_{H^3(0,T_1;H^{3/2}(\partial B_{R_1}))^3} + r^{-4}\right), \quad r > sr_0.$$

Now, choosing $r = c^{-1}\ln(\|U\|_{H^3(0,T_1;H^{3/2}(\partial B_{R_1}))^3})$, we get for $\|U\|_{H^3(0,T_1;H^{3/2}(\partial B_{R_1}))^3}$ sufficiently small that

$$\|f_p\|_{L^2((0,T_1)\times\tilde{B}_{R_1})} \leq C\left(\|U\|^2_{H^3(0,T_1;H^{3/2}(\partial B_{R_1})^3} + \left|\ln\left(\|U\|_{H^3(0,T_1;H^{3/2}(\partial B_{R_1})^3}\right)\right|^{-2}\right), \tag{3.17}$$

which can be obtained by applying the classical arguments of optimization (see for instance the end of the proof of [31, Theorem 1]). This gives the estimate of F_p by our measurement data taken on ∂B_{R_1}.

Using similar arguments, we can prove

$$\|f_s\|_{L^2((0,T)\times\tilde{B}_R)} \leq C\left(\|U\|^2_{H^3(0,T_1;H^{3/2}(\partial B_R)^3} + \left|\ln\left(\|U\|_{H^3(0,T_1;H^{3/2}(\partial B_R)^3}\right)\right|^{-2}\right). \tag{3.18}$$

On the other hand, by interpolation and the upper bound (1.8), we have

$$\begin{aligned}\|f\|_{L^2((0,T_1)\times\tilde{B}_R)} &\leq \|\nabla_{\tilde{x}} f_p\|_{L^2((0,T_1)\times\tilde{B}_R)} + \left\|\nabla_{\tilde{x}}^{\perp} f_s\right\|_{L^2((0,T_1)\times\tilde{B}_R)} \\ &\leq C(\|f_p\|_{H^1((0,T_1)\times\tilde{B}_R)} + \|f_s\|_{H^1((0,T_1)\times\tilde{B}_R)}) \\ &\leq C\left(\|f_p\|^{\frac{1}{2}}_{H^2((0,T_1)\times\tilde{B}_R)} \|f_p\|^{\frac{1}{2}}_{L^2((0,T_1)\times\tilde{B}_R)} + \|f_s\|^{\frac{1}{2}}_{H^2((0,T_1)\times\tilde{B}_R)} \|f_s\|^{\frac{1}{2}}_{L^2((0,T_1)\times\tilde{B}_R)}\right) \\ &\leq C\left(\|f_p\|^{\frac{1}{2}}_{L^2((0,T_1)\times\tilde{B}_R)} + \|f_s\|^{\frac{1}{2}}_{L^2((0,T_1)\times\tilde{B}_R)}\right),\end{aligned}$$

with C depending on M, T_1 and R. Then, combining this with (3.17)-(3.18), we obtain (1.9).

Remark 1. *The uniqueness and stability results presented in Theorems 3 and 4 carry over to the scalar inhomogeneous wave equation of the form*

$$c^{-2}\partial_{tt}U(x,t) = \Delta U(x,t) + f(\tilde{x},t)g(x_3), \qquad (x,t) \in \mathbb{R}^3 \times (0,\infty),$$
$$U(x,0) = U_t(x,0) = 0, \qquad x \in \mathbb{R}^3,$$

where $c > 0$ is a constant, both f and g are compactly supported scalar functions. If the function $g(x_3)$ is a known non-vanishing function, one can determine the source term $f(\tilde{x},t)$ from partial boundary data. In particular, f is allowed to be a moving source with the orbit lying on the ox_1x_2-plane. In the frequency domain, the above wave equation gives rise to an inverse problem of recovering the wave-number-dependent source term $f(\tilde{x},k)$ from the multi-frequency boundary observation data of the inhomogeneous Helmholtz equation

$$\Delta u(x,k) + \frac{k^2}{c^2} u(x,k) = \hat{f}(\tilde{x},k)g(x_3).$$

Progress along these directions will be reported in our forthcoming publications.

4 Appendix

4.1 Well-posedness result and estimation of surface traction

In this subsection, we consider the inhomogeneous Lamé system

$$\begin{cases} \rho(x)\partial_t^2 U - \mathcal{L}_{\lambda,\mu} U = F(x,t), & (x,t) \in \mathbb{R}^3 \times (0,+\infty), \\ U(\cdot,0) = \partial_t U(\cdot,0) = 0, & x \in \mathbb{R}^3, \end{cases} \tag{4.1}$$

where the operator $\mathcal{L}_{\lambda,\mu}$ is given by (2.1) with the density function ρ and the Lamé coefficients λ and μ fulfilling the condition (1.3). We assume that $\mathrm{Supp}(F) \subset B_R \times [0,T)$, with $B_R := \{x \in \mathbb{R}^3 : \ |x| < R\}$. It is well-known that the operator $\mathcal{L}_{\lambda,\mu}$ is an elliptic operator and the standard elliptic regularity holds; see e.g., [39, Chapters 4 and 10] and [21, Chapter 5]. The quadratic form corresponding to $\mathcal{L}_{\lambda,\mu}$ is given by

$$\mathcal{E}(U,V) := \lambda\,(\mathrm{div}\ U)(\mathrm{div}\ V) + 2\mu E(U) : E(V)$$

where the stress tensor E is defined via (2.3), with the notation $A : B := \sum_{i,j=1}^3 a_{ij}b_{ij}$ for $A = (a_{ij})_{i,j=1}^3$, $B = (b_{ij})_{i,j=1}^3$. Hence, for a bounded Lipschitz domain $D \subset \mathbb{R}^3$ there holds the relation (see e.g., [5, Lemma 3])

$$-\int_D \mathcal{L}_{\lambda,\mu} U \cdot \overline{V}\, dx = \int_D \mathcal{E}(U,\overline{V})\, dx - \int_{\partial D} \overline{V} \cdot \mathcal{T}U\, ds \tag{4.2}$$

for all $U, V \in H^2(D)^3$. By the well-known Korn's inequality (see e.g. [39, Theorem 10.2], [17, Chapter 3]), it holds that

$$\int_{\mathbb{R}}^3 \mathcal{E}(U,\overline{U}) + c_1 \left\|U\right\|_{L^2(\mathbb{R}^3)^3} \geq c_2 \left\|U\right\|_{H^1(\mathbb{R}^3)^3}, \quad U \in H^1(\mathbb{R}^3)^3 \tag{4.3}$$

for some constants $c_1, c_2 > 0$. In the particular case of constant Lamé coefficients, we have

$$\mathcal{L}_{\lambda,\mu} U = \mu \Delta U + (\lambda + 2\mu)\nabla(\nabla \cdot U),$$

and

$$\mathcal{E}(U,V) = 2\mu \sum_{j,k=1}^{3} \partial_k U_j \; \partial_k V_j + \lambda\,(\mathrm{div}\; U)(\mathrm{div}\; V) - \mu\,\mathrm{curl}\; U \cdot \mathrm{curl}\; V.$$

In this case, the surface traction can be simplified to be

$$\mathcal{T}U = 2\mu\,\partial_\nu U + \lambda(\mathrm{div}\; U)\,\nu + \mu\,\nu \times \mathrm{curl}\; U \qquad \text{on} \quad \partial D.$$

We refer to the monograph [35] for a comprehensive studies on the Lamé system. Below we state a well-posedness result to the elastodynamic system in unbounded domains by applying the standard arguments of [36, Chapter 8].

Lemma 1. *Let $F \in H^1(\mathbb{R}_+; L^2(\mathbb{R}^3))^3$ be such that $F(\cdot,0) = x \mapsto F(0,x) = 0$. Then problem* (4.1) *admits a unique solution $U \in \mathcal{C}^2([0,+\infty); L^2(\mathbb{R}^3))^3 \cap \mathcal{C}([0,+\infty); H^2(\mathbb{R}^3))^3$.*

Proof. Without lost of generality, we assume that $\rho = 1$. We define on $L^2(\mathbb{R}^3)^3$ the sesquilinear form a with domain $D(a) := H^1(\mathbb{R}^3)^3$ given by

$$a(U_1,U_2) := \int_{\mathbb{R}^3} \mathcal{E}(U_1,U_2)dx.$$

In view of (4.2), by density, we find

$$\langle -\mathcal{L}_{\lambda,\mu} U_1, U_2 \rangle_{H^{-1}(\mathbb{R}^3)^3, H^1(\mathbb{R}^3)^3} = a(U_1,U_2), \quad U_1, U_2 \in H^1(\mathbb{R}^3)^3.$$

Therefore, in view of (4.3), fixing $H = L^2(\mathbb{R}^3)^3$, $V = H^1(\mathbb{R}^3)^3$ and applying [36, Theorem 8.1, Chapter 3] and [36, Theorem 8.2, Chapter 3], we deduce that (4.1) admits a unique solution $U \in \mathcal{C}^1([0,+\infty); L^2(\mathbb{R}^3))^3 \cap \mathcal{C}([0,+\infty); H^1(\mathbb{R}^3))^3$. It remains to prove that $U \in \mathcal{C}^2([0,+\infty); L^2(\mathbb{R}^3))^3 \cap \mathcal{C}([0,+\infty); H^2(\mathbb{R}^3))^3$. For this purpose, we consider $V = \partial_t U$ and, using the fact that $F(\cdot,0) = 0$, we deduce that V solves

$$\begin{cases} \partial_t^2 V - \mathcal{L}_{\lambda,\mu} V = \partial_t \tilde{F}(x,t), & (x,t) \in \mathbb{R}^3 \times (0,+\infty), \\ V(\cdot,0) = \partial_t V(\cdot,0) = 0, & x \in \mathbb{R}^3. \end{cases} \tag{4.4}$$

Using the fact that $\partial_t F \in L^2((0,+\infty) \times \mathbb{R}^3)$ and applying the above arguments we deduce that $V \in \mathcal{C}^1([0,+\infty); L^2(\mathbb{R}^3))^3$ and that $U \in \mathcal{C}^2([0,+\infty); L^2(\mathbb{R}^3))^3$. Therefore, for any $t \in [0,+\infty)$, U is a solution of the boundary value problem

$$-\mathcal{L}_{\lambda,\mu} U(x,t) = -\partial_t^2 U(x,t) + F(x,t), \quad x \in \mathbb{R}^3. \tag{4.5}$$

Since $\partial_t^2 U(t,\cdot) \in L^2(\mathbb{R}^3)^3$, from the elliptic regularity of the operator $-\mathcal{L}_{\lambda,\mu}$ (see e.g. [21, Theorem 5.8.1]), we deduce that $U(t,\cdot) \in H^2(\mathbb{R}^3)^3$. Moreover, for any $t_1, t_2 \in [0,+\infty)$, we have

$$\begin{aligned} &\|U(\cdot,t_1) - U(\cdot,t_2)\|_{H^2(\mathbb{R}^3)^3} \\ &\le C(\|\mathcal{L}_{\lambda,\mu}(U(\cdot,t_1) - U(\cdot,t_2))\|_{L^2(\mathbb{R}^3)^3} + \|U(\cdot,t_1) - U(\cdot,t_2)\|_{L^2(\mathbb{R}^3)^3}) \\ &\le C\left(\left\|\partial_t^2 U(\cdot,t_1) - \partial_t^2 U(\cdot,t_2))\right\|_{L^2(\mathbb{R}^3)^3} + \|U(\cdot,t_1) - U(\cdot,t_2)\|_{L^2(\mathbb{R}^3)^3} + \|F(\cdot,t_1) - F(\cdot,t_2))\|_{L^2(\mathbb{R}^3)^3}\right). \end{aligned}$$

Therefore, using the fact that $U \in \mathcal{C}^2([0,+\infty); L^2(\mathbb{R}^3))^3$ and the fact that F extended by 0 to $\mathbb{R}^3 \times \mathbb{R}$ is lying in $H^1(\mathbb{R}; L^2(\mathbb{R}^3))^3 \subset \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^3))^3$, we deduce that $U \in \mathcal{C}([0,+\infty); H^2(\mathbb{R}^3))^3$. □

Using this result for $F \in H^1(0,T;L^2(\mathbb{R}^3))^3$, we know

$$U(x,t),\ \mathcal{T}U(x,t) \in \mathcal{C}([0,+\infty);L^2(\partial B_R))^3, \quad (x,t) \in \partial B_R \times [0,\infty).$$

With additional smoothness assumptions we can also estimate $\mathcal{T}U_{|\partial B_R\times[0,T_2]}$ by $U_{|\partial B_R\times[0,T_2]}$, with $T_2 > 0$. The main result of this subsection can be stated as follows.

Proposition 8. *Let $T_2 > 0$ and let $F \in H^4(\mathbb{R}_+;L^2(\mathbb{R}^3))^3$ be such that $F(\cdot,0) = \partial_t F(\cdot,0) = \partial_t^2 F(\cdot,0) = \partial_t^3 F(\cdot,0) = 0$. Then problem (4.1) admits a unique solution $U \in \mathcal{C}^4([0,+\infty);L^2(\mathbb{R}^3))^3 \cap \mathcal{C}^3([0,+\infty);H^2(\mathbb{R}^3))^3$ satisfying the estimate*

$$\|\mathcal{T}U\|_{L^2(\partial B_R\times(0,T_2))^3} \leq C\,\|U\|_{H^3(0,T_2;H^{\frac{3}{2}}(\partial B_R))^3}\,, \tag{4.6}$$

with C depending on λ, ρ, μ, T_2 and R.

Proof. Note first that $W = \partial_t^3 U$ solves

$$\begin{cases} \rho\partial_t^2 W - \mathcal{L}_{\lambda,\mu}W = \partial_t^3 F(x,t), & (x,t) \in \mathbb{R}^3 \times (0,+\infty), \\ W(\cdot,0) = \partial_t W(\cdot,0) = 0, & x \in \mathbb{R}^3. \end{cases} \tag{4.7}$$

Using the fact that $\partial_t^3 F \in H^1(\mathbb{R}_+;L^2(\mathbb{R}^3))^3$ with $\partial_t^3 F(\cdot,0) = 0$, we can apply Lemma 1 to deduce that $W \in \mathcal{C}^2([0,+\infty);L^2(\mathbb{R}^3))^3 \cap \mathcal{C}([0,+\infty);H^2(\mathbb{R}^3))^3$. Thus, we have $U \in \mathcal{C}^4([0,+\infty);L^2(\mathbb{R}^3))^3 \cap \mathcal{C}^3([0,+\infty);H^2(\mathbb{R}^3))^3$. This implies that $g := U_{|\partial B_R\times[0,+\infty)} \in \mathcal{C}^3([0,+\infty);H^{\frac{3}{2}}(\partial B_R))^3$. Hence, the restriction of U to $(\mathbb{R}^3\backslash B_R)\times(0,T_2)$ solves the initial boundary value problem

$$\begin{cases} \rho\partial_t^2 U - \mathcal{L}_{\lambda,\mu}U = 0, & (x,t) \in (\mathbb{R}^3\backslash B_R) \times (0,T_2), \\ U(\cdot,0) = \partial_t U(\cdot,0) = 0, & x \in \mathbb{R}^3\backslash B_R, \\ U = g, & (x,t) \in \partial B_R \times (0,T_2). \end{cases} \tag{4.8}$$

Using the fact that $g(\cdot,0) = \partial_t g(\cdot,0) = \partial_t^2 g(\cdot,0) = 0$, from a classical lifting result, one can find $G \in \mathcal{C}^3([0,+\infty);H^2(\mathbb{R}^3\backslash B_R))^3$ such that $G_{|\partial B_R\times(0,T_2)} = g$, $G(\cdot,0) = \partial_t G(\cdot,0) = \partial_t^2 G(\cdot,0) = 0$ and

$$\|G\|_{H^3(0,T_2;H^2(\mathbb{R}^3))^3} \leq C\,\|g\|_{H^3(0,T_2;H^{\frac{3}{2}}(\partial B_R))^3}\,, \tag{4.9}$$

with C depending only on T_2 and R. Therefore, we can split U to $U = V + G$ on $(0,T_2)\times(\mathbb{R}^3\backslash B_R)$, with V the solution of

$$\begin{cases} \rho\partial_t^2 V - \mathcal{L}_{\lambda,\mu}V = -(\rho\partial_t^2 G - \mathcal{L}_{\lambda,\mu}G) := H, & (x,t) \in (\mathbb{R}^3\backslash B_R) \times (0,T_2), \\ V(\cdot,0) = \partial_t V(\cdot,0) = 0, & x \in \mathbb{R}^3\backslash B_R, \\ V = 0, & (x,t) \in \partial B_R \times (0,T_2). \end{cases}$$

Using the fact that $H \in H^1(0,T_2;L^2(\mathbb{R}^3\backslash B_R))$ and $H(\cdot,0) = 0$, in a similar way to Lemma 1 we can prove that $V \in \mathcal{C}^2([0,T_2];L^2(\mathbb{R}^3\backslash B_R))^3 \cap \mathcal{C}([0,T_2];H^2(\mathbb{R}^3\backslash B_R))^3$ satisfying the estimate

$$\|V\|_{L^2(0,T_2;H^2(\mathbb{R}^3\backslash B_R))^3} \leq C\,\|H\|_{H^1(0,T_2;L^2(\mathbb{R}^3\backslash B_R))^3} \leq C\,\|G\|_{H^3(0,T_2;H^2(\mathbb{R}^3))^3}\,,$$

with C depending on λ, ρ, μ, T_2 and R. Combining this with (4.9), we deduce that

$$\|U\|_{L^2(0,T_2;H^2(\mathbb{R}^3\backslash B_R))^3} \leq C\,\|g\|_{H^3(0,T_2;H^{\frac{3}{2}}(\partial B_R))^3}$$

and using the continuity of the trace map, we obtain

$$\|\mathcal{T}U\|_{L^2((0,T_2)\times\partial B_R)^3} \leq C\,\|U\|_{L^2(0,T_2;H^2(\mathbb{R}^3\backslash B_R))^3}\,.$$

Combining the last two estimates we finally obtain (4.6). □

4.2 Long time asymptotic behavior of the solution on a bounded domain

In this subsection we fix Ω_1 to be a bounded $\mathcal{C}^2$ domain of $\mathbb{R}^3$. We consider the bilinear form a with domain $D(a) = H^1(\Omega_1)^3$ given by

$$\begin{aligned} a(U_1, U_2) &= \int_{\Omega_1} \mathcal{E}(U, \overline{V})\, dx \\ &= \int_{\Omega_1} \Big[(\lambda(x) + 2\mu(x))(\nabla \cdot U_1(x))\overline{(\nabla \cdot U_2(x))} - \mu(x)\nabla \times U_1(x) \cdot \overline{\nabla \times U_2(x)}\Big]\, dx. \end{aligned}$$

Then, for $U_1 \in H^1(\Omega_1)^3$ such that $\mathcal{L}_{\lambda,\mu}U_1 \in L^2(\Omega_1)^3$ and $\mathcal{T}U_1 = 0$ on $\partial\Omega_1$, we have

$$a(U_1, U_2) = -\int_{\Omega_1} \mathcal{L}_{\lambda,\mu}U_1(x) \cdot \overline{U_2(x)} dx. \tag{4.10}$$

Fixing $H = L^2(\Omega_1)^3$, $V = H^1(\Omega_1)^3$ and applying [21, Chapter 5], [36, Theorem 8.1, Chapter 3] and [36, Theorem 8.2, Chapter 3], we deduce that, for $F \in L^2((0, +\infty) \times \Omega_1)^3$, $V_0 \in V$ and $V_1 \in H$, the problem

$$\begin{cases} \rho(x)\partial_t^2 U - \mathcal{L}_{\lambda,\mu}U = F(x,t), & (x,t) \in \Omega_1 \times (0, +\infty), \\ U(\cdot, 0) = V_0,\ \partial_t U(\cdot, 0) = V_1, & x \in \Omega_1 \\ \mathcal{T}U(x,t) = 0, & (x,t) \in \partial\Omega_1 \times (0, +\infty), \end{cases} \tag{4.11}$$

admits a unique solution in $\mathcal{C}^1([0, +\infty); L^2(\Omega_1))^3 \cap \mathcal{C}([0, +\infty); H^1(\Omega_1))^3$. Below we show the long time behavior of the solution of (4.11).

Proposition 9. *Let $F \in L^2((0, +\infty) \times \Omega_1)^3$ be such that* $\mathrm{Supp}(F) \subset \overline{\Omega}_1 \times [0, T)$ *and let $V_0 \in H^1(\Omega_1)^3$, $V_1 \in L^2(\Omega)^3$. Then problem* (4.11) *admits a unique solution $U \in \mathcal{C}^1([0, +\infty); L^2(\Omega_1))^3 \cap \mathcal{C}([0, +\infty); H^1(\Omega_1))^3$ satisfying*

$$\|U(t, \cdot)\|_{H^1(\Omega_1)^3} \le \tilde{C}(\|F\|_{L^2((0,T)\times\Omega_1)} + \|V_0\|_{H^1(\Omega_1)^3} + \|V_1\|_{L^2(\Omega_1)^3})(t+1), \quad t > 0, \tag{4.12}$$

with $\tilde{C}$ independent of t.

Proof. Without lost of generality we may assume that $V_0 = V_1 = 0$. Indeed the result with non-vanishing initial conditions can be carry out in a similar way. Let us first assume that $F \in H_0^1(0, T; L^2(\Omega_1))^3$. Repeating the arguments in the proof of Lemma 1, we can prove that the regularity of U can be improved to be

$$U \in \mathcal{C}^2([0, +\infty); L^2(\Omega_1))^3 \cap \mathcal{C}([0, +\infty); H^2(\Omega_1))^3.$$

Now let us consider the energy

$$J(t) := \int_{\Omega_1} \rho(x)|\partial_t U(x,t)|^2 + \mathcal{E}(U(x,t), U(x,t))\, dx.$$

For simplicity, we assume that F takes values in $\mathbb{R}^3$ such that U takes also values in $\mathbb{R}^3$, otherwise our arguments may be extended without any diffuculty to function F taking values in $\mathbb{C}^3$. It is clear that $J \in \mathcal{C}^1([0, +\infty))$ and

$$J'(t) = 2\int_{\Omega_1} \rho(x)\partial_t^2 U(x,t) \cdot \partial_t U(x,t) + \mathcal{E}(U(x,t), \partial_t U(x,t))\, dx.$$

Using the fact that $\mathcal{T}U = 0$ on $[0,+\infty) \times \partial\Omega_1$, we can integrate by parts to obtain (see (4.10))

$$J'(t) = 2\int_{\Omega_1} [\rho(x)\partial_t^2 U(x,t) - \mathcal{L}_{\lambda,\mu}U(x,t)] \cdot \partial_t U(x,t)dx = 2\int_{\Omega_1} F(x,t)\cdot \partial_t U(x,t).$$

Thus, using the fact that $\rho \geq \rho_0 > 0$ we get

$$\begin{aligned}
\int_{\Omega_1} |\partial_t U(x,t)|^2 dx \leq \rho_0^{-1} J(t) &\leq 2\rho_0^{-1}\int_0^t \left| \int_{\Omega_1} F(x,s)\cdot\partial_t U(x,s)dx\right| ds \\
&\leq 2\rho_0^{-1}\int_0^T \int_{\Omega_1} |F(x,s)|\, |\partial_t U(x,s)| dxds \\
&\leq 2\rho_0^{-1} T^{\frac{1}{2}} \|F\|_{L^2((0,T)\times\Omega_1)^3} \|\partial_t U\|_{L^\infty(0,T;L^2(\Omega_1))^3}. \qquad (4.13)
\end{aligned}$$

Combining this with a classical estimate of $\|\partial_t U\|_{L^\infty(0,T;L^2(\Omega_1))^3}$ (e.g. [36, Formula (8.15), Chapter 3]), we deduce that

$$\|\partial_t U\|_{L^\infty(0,T;L^2(\Omega_1))^3} \leq \tilde{C}\, \|F\|_{L^2((0,T)\times\Omega_1)^3}$$

with $\tilde{C}$ depending only on T, Ω_1, ρ, $\mathcal{L}_{\lambda,\mu}$. It then follows from (4.13) that

$$\int_{\Omega_1} |\partial_t U(x,t)|^2 dx \leq \tilde{C}\, \|F\|^2_{L^2((0,T)\times\Omega_1)^3}.$$

Combining this with the fact that

$$U(t,\cdot) = \int_0^t \partial_t U(\cdot,s)ds,$$

we obtain that

$$\begin{aligned}
\|U(\cdot,t)\|_{L^2(\Omega_1)^3} &= \left\| \int_0^t \partial_t U(\cdot,s)ds \right\|_{L^2(\Omega_1)^3} \leq \int_0^t \|\partial_t U(\cdot,s)\|_{L^2(\Omega_1)^3} ds \\
&\leq \tilde{C}\, \|F\|_{L^2((0,T)\times\Omega_1)^3}\, t. \qquad (4.14)
\end{aligned}$$

By density, we can extend this estimate to $F \in L^2((0,+\infty)\times\Omega_1)^3$.

Applying Korn's inequality gives the estimate

$$\begin{aligned}
\|U(\cdot,t)\|^2_{H^1(\Omega_1)^3} &\leq \tilde{C}\left(\|U(\cdot,t)\|^2_{L^2(\Omega_1)^3} + \int_{\Omega_1} \mathcal{E}(U(x,t),U(x,t))dx \right) \\
&\leq \tilde{C}(\|U(\cdot,t)\|^2_{L^2(\Omega_1)^3} + J(t)). \qquad (4.15)
\end{aligned}$$

Taking the scalar product with U on both sides of (4.11) and integrating by part with respect to x over Ω_1, we can estimate $J(t)$ by

$$J(t) = \int_{\Omega_1} F(x,t)\, \cdot U(x,t)\, dx \leq ||F(\cdot,t)||_{L^2(\Omega_1)^3} ||U(\cdot,t)||_{L^2(\Omega_1)^3}. \qquad (4.16)$$

Now, inserting (4.16) into (4.15) and making use of (4.14), we finally obtain

$$\|U(\cdot,t)\|^2_{H^1(\Omega_1)^3} \leq \tilde{C}\, \|F\|^2_{L^2((0,T)\times\Omega_1)^3} (1+t^2),$$

which proves (4.12). □

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