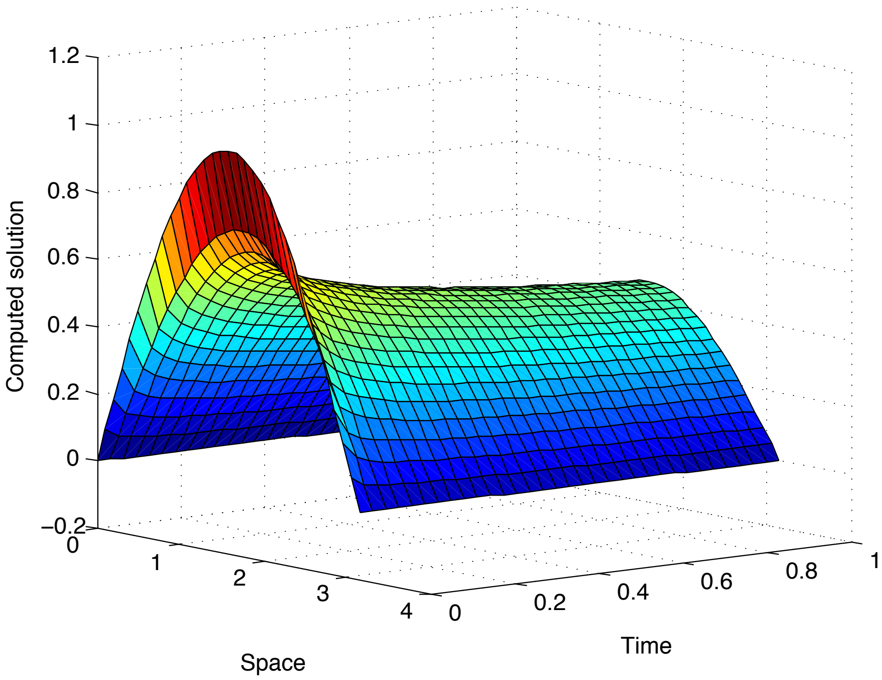
Book of Abstracts

3rd Workshop
Numerical Methods for
Fractional-derivative Problems

Beijing Computational Science Research Center, China

26 & 27 April 2019



3rd CSRC Workshop

Numerical Methods for Fractional-Derivative Problems

26–27 April 2019

All lectures are in Conference Room 1 on 1st Floor (Ground Floor) CSRC building

Each 30-minute time slot allows 25 minutes for the talk *and 5 minutes for questions and change of speakers*

Chairpersons of sessions: see list at the end of this Program

Friday 26 April	
08:15–09:00	*Registration* Enter CSRC at main entrance, then go to the right (past the elevators)
09:00–09:05	*Opening of Conference by Hai-Qing Lin, Director of CSRC*
09:05–10:05	*(Main Speaker)* **Buyang Li**, Convolution quadrature for fractional-derivative problems, artificial boundary conditions, and boundary integral equations
10:05–10:35	**Zhi Zhou**, Numerical analysis of nonlinear subdiffusion equations
10:35–11:00	*Tea/Coffee Break*
11:00–11:30	**Hong Wang**, On mathematical models by (variable-order) time-fractional diffusion equations
11:30–12:00	**Xiangcheng Zheng**, Wellposedness and regularity of variable order time fractional diffusion equations
12:00–12:30	**Hui Liang**, Collocation methods for general Riemann-Liouville two-point boundary value problems
12:35	*Lunch (CSRC Canteen, in basement)*
13:45–13:55	*Group photo in foyer inside CSRC main entrance*
14:00–15:00	*(Main Speaker)* **William McLean**, Discontinuous Galerkin time-stepping
15:00–15:30	**Jiwei Zhang**, Nonuniform time-stepping approaches for reaction-subdiffusion problems
15:30–16:00	*Tea/Coffee Break*
16:00–16:30	**Hong-lin Liao**, Sharp H^1-norm error estimates of two time-stepping schemes for reaction-subdiffusion problems
16:30–17:00	**Xiaoping Xie**, Convergence analysis of a Petrov-Galerkin method for fractional wave problems with nonsmooth data
17:00–17:30	**Yubin Yan**, Detailed error analysis for a fractional Adams method with graded meshes
17:30–18:00	**Beiping Duan**, A rational approximation for the Mittag-Leffler function and its application to design a parallel algorithm for time fractional diffusion equation
18:00	*Dinner (CSRC Canteen, in basement)*

Saturday 27 April	
09:00-10:00	*(Main Speaker)* **Masahiro Yamamoto**, Fundamental studies and applications to inverse problems for time-fractional partial differential equations
10:00–10:30	**Jijun Liu**, On the source identifications for time-fractional order diffusion process
10:30–11:00	*Tea/Coffee Break*
11:00–11:30	**Ting Wei**, Recovering a space-dependent source term in a time-fractional diffusion wave equation
11:30–12:00	**Yikan Liu**, Identification of the temporal component in the source term of a (time-fractional) diffusion equation
12:00–12:30	**Zhiyuan Li**, Inversion for orders of fractional derivatives of diffusion equation
12:35	*Lunch (CSRC Canteen, in basement)*
14:00–15:00	*(Main Speaker)* **Chuanju Xu**, Müntz spectral methods for some problems having low regularity solutions
15:00–15:30	**Li-Lian Wang**, Towards effective spectral and *hp* methods for PDEs with integral fractional Laplacian
15:30–16:00	*Tea/Coffee Break*
16:00–16:30	**Zhongdi Cen**, A mesh equidistribution method for a time-fractional Black-Scholes equation
16:30–16:50	**Hu Chen**, An analysis of the Grünwald–Letnikov scheme for initial-value problems with weakly singular solutions
16:50–17:10	**Chaobao Huang**, Optimal spatial H^1-norm analysis of a finite element method for a time-fractional diffusion equation
17:10–17:30	**Xiangyun Meng**, Green's functions, positive solutions, and a Lyapunov inequality for a Caputo fractional-derivative boundary value problem
18:00	*Dinner (CSRC Canteen, in basement)*

Chairpersons of sessions	
Friday 09:00–10:30	Yubin Yan
Friday 11:00–12:30	William McLean
Friday 14:00–15:30	Hong-lin Liao
Friday 16:00–18:00	Hong Wang
Saturday 09:00–10:30	Martin Stynes
Saturday 11:00–12:30	Masahiro Yamamoto
Saturday 14:00–15:30	Buyang Li
Saturday 16:00–17:30	Chuanju Xu

A mesh equidistribution method for a time-fractional Black-Scholes equation

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The time-fractional Black-Scholes equation of Jumarie is considered. The solution of the time-fractional Black-Scholes equation may be singular near certain domain boundaries, which leads to numerical difficulty. The upwind difference method is used to discretize the time-fractional Black-Scholes equation. The stability properties and a posteriori error analysis for the discrete scheme are studied. Then, a posteriori adapted mesh based on a posteriori error analysis is established by equidistributing arc-length monitor function. Numerical experiments are given to confirm the theoretical results.

KEY WORDS: Fractional differential equation, Black-Scholes equation, a posteriori error estimate, mesh equidistribution

An analysis of the Grünwald–Letnikov scheme for initial-value problems with weakly singular solutions

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A convergence analysis is given for the Grünwald-Letnikov discretisation of a Riemann-Liouville fractional initial-value problem on a uniform mesh $t_m = m\tau$ with $m = 0, 1, \ldots, M$. For given smooth data, the unknown solution of the problem will usually have a weak singularity at the initial time $t = 0$. Our analysis is the first to prove a convergence result for this method while assuming such non-smooth behaviour in the unknown solution. In part our study imitates previous analyses of the L1 discretisation of such problems, but the introduction of some additional ideas enables exact formulas for the stability multipliers in the Grünwald-Letnikov analysis to be obtained (the earlier L1 analyses yielded only estimates of their stability multipliers). Armed with this information, it is shown that the solution computed by the Grünwald-Letnikov scheme is $O(\tau t_m^{\alpha-1})$ at each mesh point t_m; hence the scheme is globally only $O(\tau^\alpha)$ accurate, but it is $O(\tau)$ accurate for mesh points t_m that are bounded away from $t = 0$. Numerical results for a test example show that these theoretical results are sharp.

KEY WORDS: Riemann-Liouville derivative, Grünwald-Letnikov scheme, weak singularity, convergence analysis

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1. Hu Chen, Finbarr Holland, and Martin Stynes, An analysis of the Grünwald–Letnikov scheme for initial-value problems with weakly singular solutions, Appl. Numer. Math., 139 (2019) 52–61.

¶This work was supported in part by the National Natural Science Foundation of China under grant 91430216; Chinese Postdoc Foundation Grant 2018M631316 and the National Natural Science Foundation of China young scientists fund Grant 11801026.

A rational approximation for the Mittag-Leffler function and its application to design a parallel algorithm for time fractional diffusion equation

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In this paper we aim at developing high accurate and parallel algorithm for time-fractional diffusion equations in temporal direction. Firstly we expand the source term by a power series with respect to t then the solution can be expressed by some combination of Mittag-Leffler functions $E_{\alpha,\beta}(-\lambda t)$ with α the order of derivative and $\beta \geq 1$. After we apply standard finite element method in space and a diagonal rational interpolation for $E_{\alpha,\beta}(-t)$ on $[0,\infty)$ obtained by a domain transplant skill, a rational-FE scheme is obtained. Different from most of the exiting methods for evolution equations, the scheme is completely parallel, that is one can compute the numerical solution for different t at the same time, which actually removes the memory effect. And the cost of the algorithm is inverting a sparse matrix N times with N the approximation order in time. Furthermore, the scheme converges super-algebraically in temporal direction, which make the computation quite acceptable. Theoretical analysis is provided to demonstrate the convergency of the scheme and extreme numerical experiments are carried out to verify the robustness and efficiency of the scheme.

KEY WORDS: Rational approximation, time-fractional diffusion equation, interpolation error, error analysis

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¶The author sincerely appreciate the discussion with Professor R. Lazarov and his help during the visit at Texas A&M University.

Optimal spatial H^1-norm analysis of a finite element method for a time-fractional diffusion equation

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A time-fractional initial-boundary value problem $D_t^\alpha u - \Delta u = f$, where D_t^α is a Caputo fractional derivative of order $\alpha \in (0,1)$, is considered on the space-time domain $\Omega \times [0,T]$, where $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded Lipschitz domain. Typical solutions $u(x,t)$ of such problems have components that behave like a multiple of t^α as $t \to 0^+$, so the integer-order temporal derivatives of u blow up at $t = 0$. The numerical method of the paper uses a standard finite element method in space on a quasiuniform mesh and considers both the L1 discretisation and Alikhanov's L2-1_σ discretisation of the Caputo derivative on suitably graded temporal meshes. *Optimal error bounds in* $H^1(\Omega)$ are proved; no previous analysis of a discretisation of this problem using finite elements in space has established such a bound. Furthermore, the optimal grading of the temporal mesh can be deduced from our analysis. Numerical experiments show that our error bounds are sharp.

KEY WORDS: Fractional diffusion equation, finite element method, ioptimal error in $H^1(\Omega)$, graded mesh

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1. Chaobao Huang and Martin Stynes, Optimal spatial H^1-norm analysis of a finite element method for a time-fractional diffusion equation. (Submitted for publication).

¶The research of Martin Stynes is supported in part by the National Natural Science Foundation of China under grants 91430216 and NSAF U1530401.

Convolution quadrature for fractional-derivative problems, artificial boundary conditions, and boundary integral equations

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We report some recent progress on the application of convolution quadrature to solving time-fractional evolution equations, PDEs with artificial boundary conditions, and boundary integral equation formulation of wave equation in unbounded domain. Fast algorithms for evaluating convolution quadrature are also discussed.

KEY WORDS: convolution quadrature, fractional derivative, artificial boundary condition, boundary integral equations, fast approximation

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4. B. Jin, B. Li, and Z. Zhou: Correction of high-order BDF convolution quadrature for fractional evolution equations. *SIAM J. Sci. Comput.* 39 (2017), pp. A3129–A3152.

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Inversion for orders of fractional derivatives of diffusion equation

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In this talk, the diffusion equation with distributed order time-fractional derivatives is discussed. The fractional derivative is inherently nonlocal in time with history dependence, which makes the crucial differences between fractional models and classical models. In this talk, we are mainly concerned the inverse problems for determining the weight function in the distributed order fractional derivatives. By using the eigenfunction expansion and Laplace transform argument, we show that one interior point observation can uniquely determine the weight function under some suitable assumptions. This talk is based on [1] and [2].

KEY WORDS: time-fractional diffusion equation, inverse problem, Laplace transform

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Collocation methods for general Riemann-Liouville two-point boundary value problems

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General Riemann-Liouville linear two-point boundary value problems of order α_p, where $n-1 < \alpha_p < n$ for some positive integer n, are investigated on the interval $[0,b]$. It is shown first that the natural degree of regularity to impose on the solution y of the problem is $y \in C^{n-2}[0,b]$ and $D^{\alpha_p-1}y \in C[0,b]$, with further restrictions on the behaviour of the derivatives of $y^{(n-2)}$ (these regularity conditions differ significantly from the natural regularity conditions in the corresponding Caputo problem). From this regularity it is deduced that the most general choice of boundary conditions possible is $y(0) = y'(0) = \cdots = y^{(n-2)}(0) = 0$ and $\sum_{j=0}^{n_1} \beta_j y^{(j)}(b_1) = \gamma$ for some constants β_j and γ, with $b_1 \in (0,b]$ and $n_1 \in \{0,1,\ldots,n-1\}$. A wide class of transformations of the problem into weakly singular Volterra integral equations (VIEs) is then investigated; the aim is to choose the transformation that will yield the most accurate results when the VIE is solved using a collocation method with piecewise polynomials. Error estimates are derived for this method and for its iterated variant. Numerical results are given to support the theoretical conclusions.

KEY WORDS: Fractional derivative, Riemann-Liouville derivative, two-point boundary value problem, Volterra integral equation, collocation methods

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1. Hui Liang and Martin Stynes, Collocation methods for general Riemann-Liouville two-point boundary value problems, Adv. Comput. Math. 45 (2019), no. 2, 897–928. DOI: 10.1007/s10444-018-9645-1

¶This work was supported in part by the National Natural Science Foundation of China under grants 11771128 and 91430216.

Sharp H^1-norm error estimates of two time-stepping schemes for reaction-subdiffusion problems

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Due to the intrinsically initial singularity of solution and the discrete convolution form in numerical Caputo derivatives, the traditional H^1-norm analysis (corresponding to the case for a classical diffusion equation) to the time approximations of a fractional subdiffusion problem always leads to suboptimal error estimates (a loss of time accuracy). To recover the theoretical accuracy in time, we propose an improved discrete Grönwall inequality and apply it to the numerical analysis of the nonuniform L1 formula and Alikhanov scheme. With the help of a time-space error-splitting technique and the global consistency analysis, sharp H^1-norm error estimates of the two nonuniform approaches are established for a reaction-subdiffusion problems. Numerical experiments are included to confirm the sharpness of our analysis.

KEY WORDS: reaction-subdiffusion problems, initial singularity, discrete Grönwall inequality, time-space error-splitting technique, sharp H^1-norm error estimate

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3. J. Ren, H.-L. Liao, J. Zhang and Z. Zhang, Sharp H^1-norm error estimates of two time-stepping schemes for reaction-subdiffusion problems, 2018, arXiv: 1811.08059v1.

¶This work was supported in part by a grant 1008-56SYAH18037 from NUAA Scientific Research Starting Fund of Introduced Talent, a grant DRA2015518 from 333 High-level Personal Training Project of Jiangsu Province; the NSFC grant 11601119, the program No.18HASTIT027 for HASTIT, Young talents Fund of HUEL; the NSFC grants 11771035, 11471031, 91430216 and NSAF U1530401.

On the source identifications for time-fractional order diffusion process

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The inverse problems for PDEs with time fractional derivatives are of great importance, since such derivatives are non-local and consequently can describe some physical phenomena with memory effects. For inverse problems by such PDEs models, we are interested in the reconstructions of the unknown sources from some measurable data of the physical fields. Due to the different form of inversion input data, such inverse problems may be either nonlinear or ill-posed. In this talk, for the governed PDEs with time fractional derivatives, we will introduce our works related to backward problems for diffusion process, the simultaneous reconstructions for the unknown initial status and boundary sources, the unknown source located on part of the medium boundary. The theoretical results by regularizing schemes such as the uniqueness of the inverse problems, the choice strategies of the regularizing parameters, the convergence rate of the regularizing solution, together with the numerical realizations of the proposed reconstruction schemes will be discussed. This is a joint work with Dr. Liang Yan and Prof. M.Yamamoto.

KEY WORDS: fractional equation, source identification, inverse problem, uniqueness, regularization.

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1. J. J. Liu and M. Yamamoto, A backward problem for the time-fractional diffusion equation, Applicable Analysis, Vol.89, No.11, 1769-1788, 2010.
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Identification of the temporal component in the source term of a (time-fractional) diffusion equation

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In this talk, we consider the reconstruction of $\rho(t)$ in the (time-fractional) diffusion equation

$$(\partial_t^\alpha - \triangle)u(x,t) = \rho(t)g(x)$$

by the observation at a single point x_0. We are mainly concerned with the situation of $x_0 \in \operatorname{supp} g$, which is of practical significance in environmental problems. First, the uniqueness is proved by utilizing the strong maximum principle. Next, assuming the finite sign changes of ρ and an extra observation interval, we establish the multiple logarithmic stability for the problem based on a reverse convolution inequality and a lower estimate for positive solutions. Meanwhile, we develop a fixed point iteration for the numerical reconstruction and prove its convergence. The performance of the proposed method is illustrated by several numerical examples. This talk is mainly based on the papers [1, 2, 3].

KEY WORDS: Fractional diffusion equation, inverse source problem, multiple logarithmic stability, reverse convolution inequality

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3. Y. Liu and Z. Zhang, Reconstruction of the temporal component in the source term of a (time-fractional) diffusion equation, *J. Phys. A*, **50**, 2017, 305203 (27pp).

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Discontinuous Galerkin Time-Stepping

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This talk is an overview of the discontinous Galerkin (DG) method as a time-stepping procedure for classical and fractional diffusion problems. After motivating the definition of the procedure, its implementation is outlined and then key properties are illustrated with simple, scalar ODEs and fractional ODEs. Results for classical and fractional diffusion problems in 1D are presented, and an explanation for the observed superconvergence behaviour is sketched. Finally, a general stability analysis is discussed.

Green's functions, positive solutions, and a Lyapunov inequality for a Caputo fractional-derivative boundary value problem

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We consider a nonlinear boundary problem whose highest-order derivative is a Caputo derivative of order α with $1 < \alpha < 2$. Properties of its associated Green's function are derived. These properties enable us to deduce sufficient conditions for the existence of a positive solution to the boundary value problem and to prove a Lyapunov inequality for the problem. Our results sharpen and extend earlier results of other authors.

KEY WORDS: Caputo fractional derivative, nonlinear boundary value problem, Green's function, positive solution, Lyapunov inequality.

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On mathematical models by (variable-order) time-fractional diffusion equations

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Recently, Stynes et al proved that time-fractional diffusion equations (tFDEs) generate solutions with singularity near the initial time t=0, which makes the error estimates in the literature that were proved under full regularity assumptions of the true solutions inappropriate.

From a modeling point of view, the singularities of the solutions to tFDEs at t=0 do not seem physically relevant to the diffusive transport the tFDEs model. The fundamental reason lies between the incompatibility between the nonlocality of tFDEs and the locality of the initial condition.

To eliminate the incompatibility, we propose a modified tFDE model in which the fractional order will vary near the time t=0, which naturally leads to variable-order tFDEs. We will also show that variable-order tFDEs occur naturally in applications. Finally, we briefly discuss the mathematical difficulties in the analysis of variable-order tFDEs, since many widely used Laplace transform based techniques do not apply here.

KEY WORDS: Variable-order time-fractional diffusion equation, a modified time-fractional diffusion equation model, mathematical modeling

Towards effective spectral and *hp* methods for PDEs with integral fractional Laplacian

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The anomalous diffusion governed by PDEs involving fractional Laplacian in multi-dimensional bounded domains poses significant challenges in numerical solutions. In particular, the integral fractional Laplacian presents even more notorious numerical difficulties among several different definitions of fractional Laplacian. The numerics actually lags behind the PDE theory and even the numerical analysis (of FEM) [1, 2]. In this talk, we report our recent attempts and results (some of them are preliminary) on spectral and *hp* methods on rectangular domains. The key is to computing the stiffness matrix is in the Fourier domains, where the explicit form of the Fourier transforms of spectral and FEM basis can be derived explicitly. This allows for easy imposition of continuity across elements. The idea is extendable to nonlocal Laplacian. In this talk, we shall also introduce efficient spectral methods using Hermite functions for PDEs with nonlocal Laplacian, and show that such a basis enjoys the de-convolution property and ease of removal of kernel singularities.

This talk is based on ongoing works with Changtao Sheng (NTU), Huiyuan Li (CAS) and/or Jie Shen (Purdue University).

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Recovering a space-dependent source term in a time-fractional diffusion wave equation

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This work is concerned with identifying a space-dependent source function from noisy final time measured data by a variational regularization approach in a time-fractional diffusion wave equation. We provide a regularity of direct problem as well as the existence and uniqueness of adjoint problem. The uniqueness of the inverse source problem is discussed. Using the Tikhonov regularization method, the inverse source problem is formulated into a variational problem and a conjugate gradient algorithm is proposed to solve it. The efficiency and robust of the proposed method are supported by some numerical experiments.

KEY WORDS: Inverse source problem, Tikhonov regularization, conjugate gradient algorithm

Convergence analysis of a Petrov-Galerkin method for fractional wave problems with nonsmooth data

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A Petrov-Galerkin method is analyzed for time fractional wave problems with nonsmooth data. Well-posedness and regularity of the weak solution to the time fractional wave problem are established, and optimal convergence with nonsmooth data is derived. Several numerical experiments are provided to validate the theoretical results.

KEY WORDS: fractional wave problem, nonsmooth data, Petrov-Galerkin method, convergence

REFERENCES

1. Hao Luo, Binjie Li, Xiaoping Xie, Convergence analysis of a Petrov-Galerkin method for fractional wave problems with nonsmooth data, arXiv1901.02799

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Muntz Spectral Methods for Some Problems Having Low Regularity Solutions

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In this talk we will present a new spectral method for a class of equations with non-smooth solutions. The proposed method makes use of the fractional polynomials, also known as Muntz polynomials. We first present some basic approximation properties of the Muntz polynomials, including error estimates for the weighted projection and interpolation operators. Then we will show how to construct efficient spectral methods by using the Muntz polynomials. A detailed convergence analysis will be provided. The potential application of this method covers a large number of problems, including classical elliptic equations, integro-differential equations with weakly singular kernels, fractional differential equations, and so on.

KEY WORDS: Muntz polynomials, spectral methods, low regularity solutions, exponential convergence

AMS SUBJECT CLASSIFICATIONS: 65N35, 65M70, 45D05, 45Exx, 41A10, 41A25

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Fundamental studies and applications to inverse problems for time-fractional partial differential equations

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For $0 < \alpha < 1$, we consider an initial/boundary value problem:

$$\begin{cases} \partial_t^\alpha (u(x,t) - a(x)) = -A(t)u(x,t) + F(x,t), \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $x = (x_1, ..., x_d) \in \mathbb{R}^d$, and ∂_t^α, $0 < \alpha < 1$ denotes a fractional derivative of order α, $\Omega \subset \mathbb{R}^d$ is a smooth bounded domain, and $-A(t)$ is a uniform elliptic operator of the second order whose coefficients are time-dependent. Here a can be considered as initial value after a suitable definition of ∂_t^α.

First we define ∂_t^α in Sobolev spaces and show that ∂_t^α allows us consistent treatments. For example, by a natural way, we can prove the unique existence of the solution to the initial/boundary value problem with regularity properties.

Second we verify that our defined ∂_t^α is well related to the classical Caputo and Riemann-Liouville derivatives via the closures of the operators, so that our approach is consistent with the existing theories as long as data are regular in a suitable sense.

Finally we discuss several problems including an inverse source problem within the framework of our defined ∂_t^α.

Detailed error analysis for a fractional Adams method with graded meshes

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We consider a fractional Adams method for solving the nonlinear fractional differential equation ${}_0^C D_t^\alpha y(t) = f(t, y(t))$, $\alpha > 0$, equipped with the initial conditions $y^{(k)}(0) = y_0^{(k)}, k = 0, 1, \dots, \lceil \alpha \rceil - 1$. Here α may be an arbitrary positive number and $\lceil \alpha \rceil$ denotes the smallest integer no less than α and the differential operator is the Caputo derivative. Under the assumption ${}_0^C D_t^\alpha y \in C^2[0, T]$, Diethelm et al. [1, Theorem 3.2] introduced a fractional Adams method with the uniform meshes $t_n = T(n/N), n = 0, 1, 2, \dots, N$ and proved that this method has the optimal convergence order uniformly in t_n, that is $O(N^{-2})$ if $\alpha > 1$ and $O(N^{-1-\alpha})$ if $\alpha \leq 1$. They also showed that if ${}_0^C D_t^\alpha y(t) \notin C^2[0, T]$, the optimal convergence order of this method cannot be obtained with the uniform meshes. However, it is well known that for $y \in C^m[0, T]$ for some $m \in \mathbb{N}$ and $0 < \alpha < m$, the Caputo fractional derivative ${}_0^C D_t^\alpha y(t)$ takes the form "${}_0^C D_t^\alpha y(t) = ct^{\lceil \alpha \rceil - \alpha} + \textit{smoother terms}$"[1, Theorem 2.2], which implies that ${}_0^C D_t^\alpha y$ behaves as $t^{\lceil \alpha \rceil - \alpha}$ which is not in $C^2[0, T]$. By using the graded meshes $t_n = T(n/N)^r, n = 0, 1, 2, \dots, N$ with some suitable $r > 1$, we show that the optimal convergence order of this method can be recovered uniformly in t_n even if ${}_0^C D_t^\alpha y$ behaves as $t^\sigma, 0 < \sigma < 1$. Numerical examples are given to show that the numerical results are consistent with the theoretical results.

KEY WORDS: Fractional differential equations, Adams method, graded meshes, error estimates

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Nonuniform time-stepping approaches for reaction-subdiffusion problems

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Nonuniform time-stepping methods are promising for Caputo reaction-subdiffusion problems because they would be simple and effectiveness in resolving the initial singularity and other nonlinear behaviors occurred away from the initial time. Compared with traditional local methods for the first-order derivative, the numerical analysis for nonlocal time-stepping schemes on nonuniform time meshes are challenging due to the convolution integral (nonlocal) form of fractional derivative. We develop a general framework for the stability and convergence analysis with three tools: a family of complementary discrete convolution kernels, a discrete fractional Grönwall inequality and a global (convolutional) consistency analysis, which is not limited to a specific time mesh by building a convolution structure of local truncation error. It seems that the present techniques are extendable to the variable-order, distributed-order diffusion equations and other nonlocal-in-time diffusion problems. This framework works for a family of widely-used scheme such as L1 scheme, Alikhanov's scheme (second-order scheme), and fast-algorithm-based L1 scheme and Alikhanov's scheme, et al.

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Wellposedness and regularity of variable order time fractional diffusion equations

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We prove the wellposedness of a nonlinear variable-order fractional ordinary differential equation and the regularity of its solutions, which is determined by the values of the variable order and its high-order derivatives at time $t = 0$. More precisely, we prove that its solutions have full regularity like its integer-order analogue if the variable order has an integer limit at $t = 0$ or exhibits singular behaviors at $t = 0$ like in the case of the constant-order fractional differential equations if the variable order has a non-integer value at time $t = 0$.

We then extend the developed techniques to prove the wellposedness of a variable-order linear time-fractional diffusion equation in multiple space dimensions and the regularity of its solutions, which depends on the behavior of the variable order at $t = 0$ in the similar manner to that of the fractional ordinary differential equations.

KEY WORDS: Variable-order fractional diffusion equation, wellposedness, regularity

Numerical Analysis of Nonlinear Subdiffusion Equations

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We present a general framework for the rigorous numerical analysis of time-fractional nonlinear parabolic partial differential equations, with a fractional derivative of order $\alpha \in$ in time. It relies on three technical tools: a fractional version of the discrete Grönwall-type inequality, discrete maximal regularity, and regularity theory of nonlinear equations. We establish a general criterion for showing the fractional discrete Grönwall inequality, and verify it for the L1 scheme and convolution quadrature generated by BDFs. Further, we provide a complete solution theory, e.g., existence, uniqueness and regularity, for a time-fractional diffusion equation with a Lipschitz nonlinear source term. Together with the known results of discrete maximal regularity, we derive pointwise $L^2(\Omega)$ norm error estimates for semidiscrete Galerkin finite element solutions and fully discrete solutions, without any extra regularity assumption on the solution or compatibility condition on the problem data. The sharpness of the convergence rates is supported by the numerical experiments.

KEY WORDS: nonlinear fractional diffusion equation, discrete fractional Grönwall inequality, L1 scheme, convolution quadrature, error estimate

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Posters

- Shimin Guo, *An efficient spectral-Galerkin method for three-dimensional Riesz-like space fractional nonlinear coupled reaction-diffusion equations*
- Lijing Zhao, *Several spectral methods for the Dirichlet problem for the fractional Laplacian*

An efficient spectral-Galerkin method for three-dimensional Riesz-like space fractional nonlinear coupled reaction-diffusion equations*

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Abstract: Based on the fractional Fick's law, we establish a novel three-dimensional fractional model in the chemical reaction, namely the Riesz-like space fractional nonlinear coupled reaction-diffusion equations. We develop an efficient spectral method for the three-dimensional space fractional model: Using the backward difference method for time stepping and the Legendre-Galerkin spectral method for space discretization, we construct a fully discrete linearized numerical scheme which leads to a algebraic system. Then a direct method based on the matrix diagonalization approach is proposed to solve the linear algebraic system, where the cost of the algorithm is of a small multiple of N^4 (N is the polynomial degree in each spatial coordinate) flops for each time level. This indicates that the three-dimensional space fractional differential equations can be solved with a spectral-Galerkin method in essentially the same cost as the related integer-order differential equations.

In addition, the stability and convergence analysis are rigorously established. We obtain the optimal error estimate in space, and the numerical analysis also shows that the fully discrete scheme is unconditionally stable and convergent of order one in time. Furthermore, numerical experiments are presented to confirm the theoretical claims. As the applications of the proposed method, the fractional-in-space Gray-Scott model is solved to capture the pattern formation with an analysis of the properties of the fractional powers. Simulation results show that the model has very different dynamics with different values of fractional powers. Our results can provide a better understanding of many aspects of heterogeneity in chemical reactions.

Finally, the proposed method and supporting theoretical results can be extended to solve the *multi-component* three-dimensional space fractional nonlinear coupled reaction-diffusion equations.

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Several Spectral Methods for the Dirichlet Problem for the Fractional Laplacian

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We discuss a class of fractional integral spaces in detail, which consist of fractional integration of the functions in L^p space with $p \geq 1$. Comparing with the most existing function spaces working for fractional operators, the striking benefits/resonableness of this kind of fractional integral spaces include: 1. high order derivative of a function belonging to a function space implies that the low order one does; 2. boundary singularities of functions in the function space still remain at the boundary after derivative operation, not propagating the whole domain, even though the fractional derivatives are non-local; 3. tempered fractional operators and the corresponding Riemann-Liouville ones are equivalent to each other in the sense of norms, although their physical meanings are different. We apply these integral spaces into designing weak formulations for one-dim Dirichlet problems with fractional Laplacian and then solve them using spectral methods. After a carefully treatment, the condition numbers of the fractional difference matrices in the scheme can be substantially diminished.

KEY WORDS: Fractional integral spaces, fractional Laplacian, spectral methods, condition number.

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