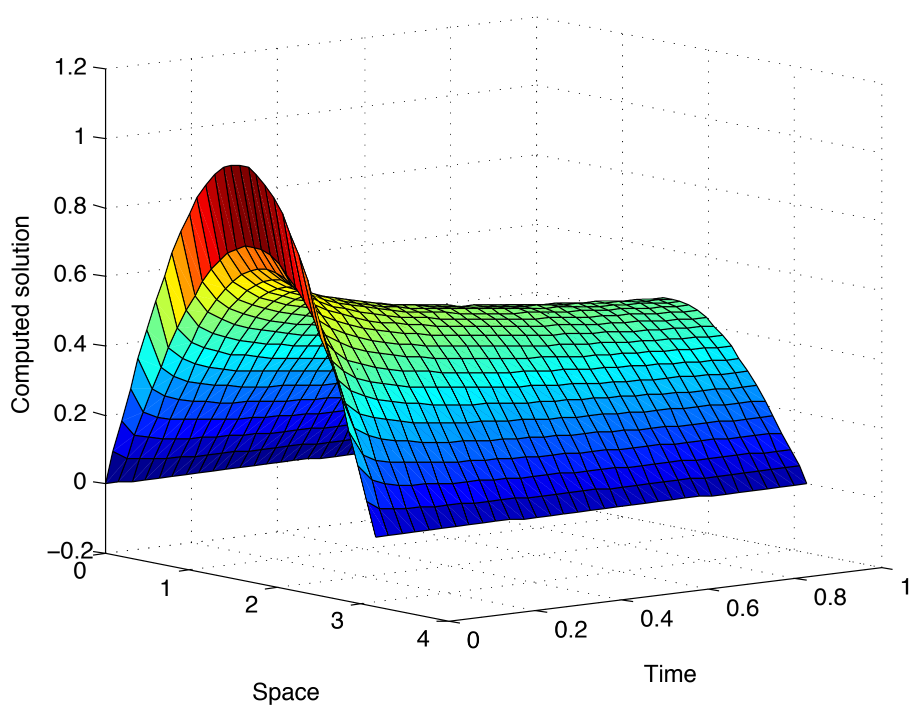
Book of Abstracts

4th Annual Conference

Numerical Methods for
Fractional-derivative Problems

Beijing Computational Science Research Center, China
(online conference)

22–24 October 2020



Fast computation of fractional integrals and its application

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Tasks like the numerical simulation of mechanical processes with viscoelastic materials require to work with finite element methods in combination with material laws (i.e., stress-strain relationships) based on ordinary differential equations of fractional order with respect to the time variable. Such differential equations then need to be solved at very many different points in the space domain. When traditional approaches are employed, the entire finite element simulation requires $O(MN^2)$ operations, where M is the number of elements and N is the number of time steps taken. Moreover, the amount of active memory required is $O(MN)$. Both these requirements are often considered to be too costly for practical applications.

The so-called nested mesh principle has been proposed as a way to reduce the computational cost to $O(MN \log N)$. This is a significant improvement over the standard approach, but it does not address the memory issue. A modification also has a computational cost of $O(MN \log N)$ but at the same time reduces the memory requirement to $O(M \log N)$. However, neither of these methods can be easily linked to a standard finite element package.

Significantly different approaches have been discussed in [1, 2]. They allow to reduce the computational cost to $O(MN)$ and the memory requirements to $O(M)$, and they can be integrated into standard FE packages in a relatively easy manner. We compare these novel fast schemes to the classical ones and explain the advantages and disadvantages of each approach.

KEY WORDS: fractional differential equation, numerical solution, fast algorithm

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¶Supported in part by the German Federal Ministry for Education and Research (Grant No. 01IS17096A).

Regularity, and the spectral method approximation, of the solution to fractional diffusion, advection, reaction equations on a bounded interval in $\mathbb{R}^1$

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The regularity of the solution to a differential equation plays an important role in constructing optimal approximation schemes to the equation. Of interest in this presentation is the regularity, and the spectral method approximation, of the solution to the fractional diffusion, advection, reaction equation:

$$-D\left(rD^{-(2-\alpha)} + (1-r)D^{-(2-\alpha)*}\right)Du(x) + b(x)Du(x) + c(x)u(x) = f(x), \quad x \in (0,1), \tag{1}$$

$$\text{subject to } u(0) = u(1) = 0, \tag{2}$$

where $1 < \alpha < 2$, and $0 \leq r \leq 1$.

Equations (1),(2), with $r = 1/2$, $b(x) = c(x) = 0$ represents the fractional Laplace equation. Recently regularity results and an optimal order finite element approximation scheme for the fractional Laplace equation were given by Acosta and Borthagaray [1]. This work was extended to the fractional Laplace equation with convection and reaction terms by Hao and Zhang in [2], where they also presented an optimal order spectral approximation scheme.

In this presentation we discuss generalizations of [1] and [2] to (1),(2).

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Local convergence of the FEM for the integral fractional Laplacian

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We analyze the local convergence behavior of the finite element method (FEM) with quasi-uniform meshes applied to the fractional differential equation $(-\Delta)^s u = f$ on a bounded domain, where $(-\Delta)^s$ denotes the integral fractional Laplacian for $s \in (0, 1)$.

Globally, the rate of convergence of the FEM-error is limited by the regularity of the solution, which typically is reduced due to singularities of solutions of fractional PDEs at the boundary. However, if the quantity of interest is a subpart of the computational domain away from the singularities, we can expect better convergence behavior of the error, as observed for the Laplacian $(-\Delta)$ in [1].

In this talk, we provide sharp local *a-priori* estimates for the non-local operator $(-\Delta)^s$ both in the localized energy norm and the local H^1-norm. Thereby, the local error can be bounded by the local best-approximation error and a global error in a weak norm. Using a duality argument to control the global weak norm, we obtain (sharp) local convergence rates, which imply faster convergence locally for solutions that are locally smooth.

We also present numerical examples that confirm the sharpness of our estimates.

KEY WORDS: Integral fractional Laplacian, finite element method, local error estimates

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The matrix Mittag-Leffler function: applications in fractional calculus and computational aspects

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The Mittag-Leffler (ML) function is known to play a fundamental role in theory and computation of fractional-order differential equations.

In this talk we consider the special case of the ML function with matrix arguments.

We first review the main applications of the matrix ML function in a series of problems in fractional calculus, ranging from the study of stability to control theory, solution of multiterm fractional differential equations, development of methods for fractional-order PDEs and so on.

We hence discuss the problem of the efficient and accurate computation of matrix ML functions, for matrices of large and moderate size, by exploiting the *optimal parabolic contour algorithm* originally devised for the scalar case.

We hence present some recent results for improving efficiency and accuracy of the existing algorithms.

KEY WORDS: Mittag-Leffler function, matrix function, numerical computation, optimal parabolic contour algorithm

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¶This work was supported in part by the italian Istituto Nazionale di Alta Matematica (INdAM) under a GNCS project.

Applications of Riemann-Liouville fractional integrals in the computation of equilibrium measures with power law kernels

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Equilibrium measure problems naturally appear in the mathematical description of particle swarms in which particle behavior may be modeled via attractive and repulsive forces, e.g. bird flocking, ensemble movement of cellular scale organisms and classical particle interactions. Analytic solutions for certain parameter regimes were derived in [3, 4] and some further candidate solutions were introduced in [1] for higher even integer parameters but little is known about the behavior of solutions in high non-integer parameter cases, where discrete particle simulations predict interesting gap formation phenomena. To address this we introduce a sparse spectral method using certain weighted ultraspherical polynomial bases. The recurrence relationship underlying our method relies on specific results from the theory of fractional Riemann-Liouville integrals and Riesz potentials in the sense of the inverse fractional Laplacian [2]. Numerical experiments demonstrate that our method reproduces known analytic results, agrees with the results of independent particle swarm simulations and can be used to study solution behavior and uniqueness in parameter regimes without known analytic results.

KEY WORDS: Riemann-Liouville fractional integral, equilibrium measure, power law kernel, sparse spectral method, ultraspherical polynomials, Riesz potential

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Sine transform based preconditioner for multi-dimensional Riesz fractional diffusion equations

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In this talk, we apply a preconditioner based on the sine transform to the linear system arising from discretized multi-dimensional Riesz spatial fractional diffusion equations and analyse the spectra of the preconditioned matrix in [1]. At first, the finite difference method is employed to approximate the multi-dimensional Riesz fractional derivatives, which will generate symmetric positive definite ill-conditioned multi-level Toeplitz matrices. The preconditioned conjugate gradient method with the sine transform preconditioner is employed to solve the resulting linear system. Theoretically, we prove that the spectra of the preconditioned matrices are uniformly bounded in the open interval $(1/2, 3/2)$ and thus the preconditioned conjugate gradient method converges linearly. Our theoretical results fill in a vacancy in the literature. Numerical examples are presented to demonstrate our new theoretical results in the literature and show the convergence performance of the proposed preconditioner that is better than other existing preconditioners.

KEY WORDS: Riesz fractional derivative, multi-level Toeplitz matrix, sine transform based preconditioner, fractional order zero, preconditioned conjugate gradient method

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¶This work is supported by research grants of the Science and Technology Development Fund, Macau SAR (file no. 0118/2018/A3), and MYRG2018-00015-FST from University of Macau, and the HKRGC GRF 12306616, 12300218, 12300519 and 17201020.

Numerical analysis of inverse problems for time-fractional diffusion

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Over the last two decades, time-fractional diffusion involving a Caputo fractional derivative in time has received much attention in physics and engineering, and the numerical analysis of relevant mathematical models have also witnessed impressive progress, and so is the analysis of relevant inverse problems. However, the numerical analysis of relevant inverse problems / optimal control remains under-explored. In this talk, I will describe recent results in the last aspect, illustrate with the inverse conductivity problem of recovering the conductivity from the distributed observation.

KEY WORDS: inverse problems, numerical analysis, time-fractional diffusion

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The time-fractional Cahn–Hilliard equation: analysis and approximation

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We consider a time-fractional Cahn–Hilliard equation where the conventional first order time derivative is replaced by a Caputo fractional derivative of order $\alpha \in (0, 1)$. Based on an *a priori* bound for the exact solution, global existence of solutions is proved and detailed regularity results are included. A finite element method is then analyzed in a spatially discrete case and in a completely discrete case based on a convolution quadrature in time generated by the backward Euler method. Error bounds of optimal order are obtained for solutions with smooth and nonsmooth initial data, extending thereby earlier studies on the classical Cahn–Hilliard equation. Further, by proving a new result concerning the positivity of a discrete time-fractional integral operator, it is shown that the proposed numerical scheme inherits a discrete energy dissipation law at the discrete level. Numerical examples are presented to illustrate the theoretical results.

KEY WORDS: time-fractional Cahn–Hilliard equation, energy dissipation, global solution, finite element method, convolution quadrature, nonsmooth initial data, error estimate

Barrier functions in the error analysis for fractional-derivative parabolic problems on quasi-graded meshes

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An initial-boundary value problem with a Caputo time derivative of fractional order $\alpha \in (0, 1)$ is considered, solutions of which typically exhibit a singular behaviour at an initial time. For this problem, building on some ideas from [1], we give a simple and general numerical-stability analysis using barrier functions, which yields sharp pointwise-in-time error bounds on quasi-graded temporal meshes with arbitrary degree of grading. This approach is employed in the error analysis of the L1 and Alikhanov L2-1_σ fractional-derivative operators [2], as well as an L2-type discretization of order $3 - \alpha$ in time [3]. This methodology is also generalized for semilinear fractional parabolic equations [4]. In particular, our error bounds accurately predict that milder (compared to the optimal) grading yields optimal convergence rates in positive time. The theoretical findings are illustrated by numerical experiments.

The simplicity of our approach is due to the usage of versatile barrier functions, which can be used in the analysis of any discrete fractional-derivative operator that satisfies the discrete maximum principle [2] (or, more generally, is associated with an inverse-monotone matrix [3]). In the semilinear case, a similar approach is employed in the form of the method of upper and lower solutions, a very elegant technique frequently used in the analysis of semilinear parabolic and elliptic equations, which we generalize to discretizations of semilinear fractional-parabolic equations [4].

KEY WORDS: fractional-order parabolic equation, semilinear, L1 scheme, Alikhanov scheme, L2 scheme, graded temporal mesh, arbitrary degree of grading, pointwise-in-time error bounds

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An Overview of Numerical Solution Methods for Spectral Fractional Elliptic Problems

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The purpose of the talk is to present a review of recent methods and some new results in approximate solution of the problem $\mathcal{A}^\alpha u = f$. Here $\mathcal{A}$ is a self-adjoint and coercive elliptic operator defined on a dense subset of $L^2(\Omega)$, with Ω a bounded Lipschitz domain and $0 < \alpha < 1$. We discuss the discretization $\mathcal{A}_h$ of $\mathcal{A}$ by finite difference or finite element methods on a mesh with mesh-size h so that produces the system $\mathcal{A}_h^\alpha u_h = f_h$ for $u_h \in \mathbb{R}^N$. The fractional power of the operator $\mathcal{A}$ and the matrix $\mathcal{A}_h$ are defined through their spectra.

For solving the system $\mathcal{A}_h^\alpha u_h = f_h$ we show that the methods proposed by Bonito and Pasciak, [1], Nochetto, Otárola, and Salgado, [3], and Vabishchevich, [4] are all based on rational approximations $r(z)$ of z^α on $(0, 1]$. Namely, instead of $u_h = \mathcal{A}_h^{-\alpha} f_h$ these methods produce $w_h = r(\mathcal{A}_h^{-1}) f_h$, with $r(z) \in \mathcal{R}_k = \{P_k(z)/Q_k(z)\}$, $P_k(z)$, $Q_k(z)$ polynomials of degree k. We discuss and computationally show that the best uniform rational approximation

$$r(z) = r_{\alpha,k}(z) := \arg\min_{s(z)\in\mathcal{R}_k} \|s(z) - z^\alpha\|_{L^\infty[0,1]},$$

introduced in [2], gives a new method, which is as good as the above cited methods and in many cases significantly outperforms them in terms of efficiency, parallelization, and robustness. Finally, we discuss issues related to computing $r_{\alpha,k}(z)$ and its application to solve $\mathcal{A}_h^\alpha u_h = f_h$.

KEY WORDS: spectral fractional Laplacian, finite elements approximation, efficient solution method

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Computation of Nonlocal Minimal Graphs

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We propose and analyze a finite element discretization for fractional Plateau and the prescribed fractional mean curvature problems on bounded domains subject to exterior data being a subgraph. The problem of order s can be reinterpreted as a Dirichlet problem for a nonlocal, nonlinear, degenerate operator of order $s+1/2$. We prove that our numerical scheme converges in $W_1^{2r}(\Omega)$ for all $r < s$, where $W_1^{2s}(\Omega)$ is closely related to the natural energy space. Moreover, we introduce a geometric notion of error that, for any pair of H^1 functions, in the limit $s \to 1/2$ recovers a weighted L^2-discrepancy between the normal vectors to their graphs. We derive error bounds with respect to this novel geometric quantity as well. We also present a wide variety of numerical experiments that illustrate qualitative and quantitative features of fractional minimal graphs and the associated discrete problems.

KEY WORDS: nonlocal minimal surfaces, finite elements, fractional diffusion.

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¶This work was supported in part by NSF grant DMS-1411808 and the Patrick and Marguerite Sung Fellowship at University of Maryland.

Positive definiteness of certain quadratic forms from discrete convolution operators

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The positive definiteness of real quadratic forms with convolution structures plays an important role in stability analysis for time-stepping schemes for nonlocal operators. In this work, we present a novel analysis tool to handle discrete convolution kernels resulting from variable-step approximations for convolution operators. More precisely, for a class of discrete convolution kernels relevant to variable-step time discretizations, we show that the associated quadratic form is positive definite under some easy-to-check algebraic conditions. Our proof is based on an elementary constructing strategy using the properties of discrete orthogonal convolution kernels and complementary convolution kernels. To the best of our knowledge, this is the first general result on simple algebraic conditions for the positive definiteness of variable-step discrete convolution kernels. Using the unified theory, the stability for some simple non-uniform time-stepping schemes can be obtained in a straightforward way.

KEY WORDS: discrete convolution kernels, positive definiteness, variable-step time-stepping, orthogonal convolution kernels, complementary convolution kernels

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¶This work was supported in part by the National Natural Science Foundation of China under grants 12071216, 11731006, 11822111, 11688101 and K20911001, Science Challenge Project (TZ2018001), National Key Basic Research Program (No. 2018YFB0704304), and Youth Innovation promotion Association of CAS.

Efficient shifted fractional trapezoidal rule for subdiffusion problems with nonsmooth solutions on uniform meshes

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Robust but simple correction techniques for a class of second-order time stepping methods based on the shifted fractional trapezoidal rule (SFTR) are developed to resolve the initial singularity of subdiffusion problems. The stability analysis and sharp error estimates in terms of the smoothness of the initial data and source term are presented. As a byproduct in numerical tests, we find amazingly that the Crank-Nicolson scheme ($\theta = \frac{1}{2}$) without initial corrections can restore the optimal convergence rate for the subdiffusion problems with smooth initial data. To deal with the long-time memory of the fractional operator, we further consider two fast algorithms for the SFTR. Numerical tests are performed to verify the sharpness of the theoretical results and the accuracy of initial corrections.

KEY WORDS: subdiffusion problems, sharp error estimates, nonsmooth solutions, SFTR

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¶The work of the second author was supported in part by grants NSFC 12061053, 11661058. The work of the third author was supported in part by the grant NSFC 11761053. The work of the fourth author was supported in part by grants NSFC 11871092 and NSAF U1930402.

Uniqueness and numerical schemes for an inverse moving source problem for (time-fractional) evolution equations

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This talk is concerned with an inverse moving source problem on determining profile functions in (time-fractional) evolution equations with an order $\alpha \in (0, 2]$ of the time derivative. Supposing that the sources move along given straight lines, we employ the vanishing property of homogeneous problems to prove the uniqueness in recovering $\lceil \alpha \rceil$ moving profiles by partial interior observation data, where $\lceil \cdot \rceil$ denotes the ceiling function. Remarkably, one can uniquely determine 2 moving profiles simultaneously in the case of $1 < \alpha \leq 2$. Numerically, we adopt a minimization procedure with regularization to construct iterative thresholding schemes for the reduced backward problems on recovering one or two unknown initial value(s). Moreover, an elliptic approach is proposed to solve a convection equation in the case of two profiles.

KEY WORDS: Inverse moving source problem, unique determination, vanishing property, iterative thresholding scheme

¶This work was supported in part by JSPS KAKENHI Grant Numbers JP20K14355, JP20H00117 and RUDN University Program 5-100.

Towards Perfectly Matched Layers for space-fractional PDEs

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A simple pseudospectral method for the computation of the space fractional equation with Perfectly Matched Layers (PML) is proposed. Within this approach, basic and widely used FFT-based solvers can be adapted without much effort to compute Initial Boundary Value Problems (IBVP) for well-posed fractional equations with absorbing boundary layers. We analyze mathematically the method, and propose some illustrating numerical experiments [1, 2].

This is a joint work with Prof. Xavier Antoine (Universite de Lorraine, IECL) and Yong Zhang (Tianjin University).

KEY WORDS: Fractional PDE, Perfectly Matched Layers, Pseudospectral methods

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A gray level indicator in a fractional-order nonlinear diffusion equation for multiplicative noise removal

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Among various methods that have been introduced for image denoising, methods based on partial differential equations (PDEs) have been drawn so much attention. Most of the existing models for multiplicative noise removal use integer-order derivatives, which due to their local property, may lead to producing blur edges and would not preserve some image details. A fractional-order based method applying for preserving image details achieves a good trade-off between eliminating speckle artifacts and restraining staircase effect in texture images. By inspiration of equation presented by Yao et al. [1], we propose a fractional-order diffusion equation to denoise texture images corrupted by some multiplicative noises. According to the noisy image variance definition, the standard deviation of a noisy image depends on the grayscale of the original image and standard deviation of the multiplicative noise. By considering this fact, we introduce a gray level indicator for the proposed fractional PDE. The applied fractional derivative in the proposed equation has nonlocal and also long-range dependency properties. Therefore, it can preserve texture image features. The proposed gray level indicator controls anomalous diffusion in distinct areas by identifying the gray information of the image and causes more details of the image to be preserved. To solve the proposed equation, an algorithm based on a finite difference scheme has been constructed by considering the definition of the Grünwald-Letnikov fractional-order derivative. We use the peak signal to noise ratio (PSNR) to evaluate the quality of the restored image. The higher the PSNR value, the closer the denoised image is to the original image. Experimental results show that the PSNR values corresponding to the proposed equation are higher than Yao et al. results, also the minimum value of PSNR's is more elevated. Thus the proposed equation has a better operation in texture images noise removal and preservation of texture images details than former equations.

KEY WORDS: Multiplicative noise removal, fractional-order diffusion equation, texture images, gray level indicator, fractional-order derivative

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Time-fractional Fokker-Planck equations with general forcing

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In this talk, we discuss briefly the well-posedness and regularity of the continuous solution of the time-fractional Fokker-Planck equation: for $0 < \alpha \leq 1$,

$$\partial_t u + \nabla \cdot (\partial_t^{1-\alpha} \kappa_\alpha \nabla u + F \partial_t^{1-\alpha} u) = f, \quad \text{on} \ \ \Omega \times (0, T]$$

with initial condition $u(x, 0) = u_0(x)$, and Ω is a convex polyhedral domain in R^d $(d \geq 1)$. The diffusivity coefficient κ_α is a function of x, and the driving force F is a vector function of x and t. The fractional derivative is taken in the Riemann–Liouville sense.

For the numerical solutions, the finite element method in space, and backward Euler and $L1$ schemes in time will be introduced. We discuss the stability and the convergence results of these methods. The singular behavior of the continuous solution near the origin is compensated by employing a time-graded mesh. To confirm the achieved theoretical convergence results numerically, some examples will be delivered.

KEY WORDS: Fractional Fokker-Planck equations, well-posedness, regularity, finite element method, backward Euler and $L1$ methods, graded meshes.

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Galerkin finite element method for nonlinear boundary value problems with Riemann-Liouville and Caputo fractional derivatives

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The collocation method for boundary value problems with Caputo derivative has been studied in [2]. In this paper, we investigate the Galerkin Lagrange finite element method for a class of semi-linear FDEs of Riemann-Liouville and Caputo types. A weak formulation of the problems is introduced in some suitable function spaces constructed by considering the fractional Sobolev and Musielak-Orlicz spaces duo to the presence of fractional and the nonlinear terms [1, 3, 4]. The existence and uniqueness issue of the weak solution together with the regularity is discussed. The weak formulation is discretized by the Galerkin method with piecewise linear polynomials basis functions. Finding an error bound in $H^{\frac{s}{2}}$-norm is considered for the Riemann-Liouville and Caputo fractional differential equations. Different examples with the varieties of the nonlinear terms have been examined and the absolute errors are given in L^2 and $H^{\frac{s}{2}}$-norms.

KEY WORDS: Fractional differential operators, Caputo derivative, Riemann-Liouville derivative, variational formulation, nonlinear operator, Galerkin method.

AMS SUBJECT CLASSIFICATIONS: 26A33, 65R20, 65J15, 65L60.

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A Numerical Study for the Generalized Fractional Benjamin-Bona-Mahony Equation

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In this study, we consider generalized fractional Benjamin-Bona-Mahony (gfBBM) equation when $0 < \alpha < 1$. The local well-posedness of solutions to the Cauchy problem is proved in the Sobolev space $H^r(\mathbb{R})$, $r > \frac{3}{2} - \alpha$, by using energy estimates. We give the existence and non-existence properties of the solitary wave solutions for the gfBBM equation. We then generate the solitary wave solutions numerically by using Petviashvili iteration method as the exact solution of the equation is not known. We also investigate the evolution of the generated solutions by using the Fourier spectral method.

KEY WORDS: Generalized Fractional Benjamin-Bona-Mahony Equation, Local Well-posedness, Solitary Waves, Petviashvili Method

Time fractional gradient flows: Theory and numerics

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We consider a so-called fractional gradient flow: an evolution equation aimed at the minimization of a convex and l.s.c. energy, but where the evolution has memory effects. This memory is characterized by the fact that the negative of the (sub)gradient of the energy equals the so-called Caputo derivative of the state.

We introduce a notion of "energy solutions" for which we refine the proofs of existence, uniqueness, and certain regularizing effects provided in [1]. This is done by generalizing, to non-uniform time steps the "deconvolution" schemes of [1], and developing a sort of "fractional minimizing movements" scheme.

We provide an a priori error estimate that seems optimal in light of the regularizing effects proved above. We also develop an a posteriori error estimate, in the spirit of [2] and show its reliability.

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¶This work was supported in part by the National Science Foundation of the USA under grant DMS-1720213.

Inverse source problem for time-fractional diffusion equation

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We consider a class of linear inverse source problem for time-fractional diffusion system. Based on the regularity result of the solution to the direct problem, we establish the solvability of this kind of inverse problem as well as the conditional stability in suitable function space with a weak norm. By a variational identity connecting the unknown source and the measurement data, the conjugate gradient method is also introduced to construct the inversion algorithm under the framework of regularizing scheme. We show the validity of the proposed scheme by several numerical examples.

KEY WORDS: Time-fractional derivative, uniqueness, conditional stability, numerics

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¶This work was supported in part by the National Natural Science Foundation of China under grant 11971104, 11871149.

A hidden-memory variable-order fractional diffusion PDE: modeling, analysis, and approximation

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Variable-order space-time fractional diffusion equations, in which the variation of the fractional orders determined by the fractal dimension of the media via the Hurst index characterizes the structure change of porous materials, provide a competitive means to describe anomalously diffusive transport of particles through deformable heterogeneous materials. We consider a hidden-memory variable-order fractional diffusion PDE, which provides a physically more relevant variable-order fractional diffusion equation modeling. However, due to the impact of the hidden memory, the problem provides more obstacles in terms of analysis and approximation, which we will address in this talk.

KEY WORDS: Hidden-memory variable-order diffusion equation, analysis, approximation, modeling

Well-posedness and numerical approximation of a fractional diffusion equation with a nonlinear variable order

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We prove well-posedness and regularity of solutions to a fractional diffusion porous media equation with a variable fractional order that may depend on the unknown solution. We present a linearly implicit time-stepping method to linearize and discretize the equation in time, and present rigorous analysis for the convergence of numerical solutions based on proved regularity results.

KEY WORDS: fractional diffusion equation, variable order, nonlinear, well-posedness, regularity, numerical approximation, convolution quadrature, convergence

Efficient Semi-analytic Methods for Integral Fractional Laplacian in Multiple Dimensions

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PDEs involving integral fractional Laplacian in multiple dimensions pose significant numerical challenges due to the nonlocality and singularity of the operator. In this talk, we shall report our recent attempts towards fast and accurate semi-analytic computation of the underlying fractional stiffness matrix. We show that for the rectangular or L-shaped domains, each entry of FEM stiffness matrix associated with the tensorial rectangular elements can be expressed explicitly by some one-dimensional integrals, which can be evaluated accurately. In particular, for the uniform meshes, the matrix is Toepliz or block-Toepliz. The key is to implementing the FEM in the Fourier transformed space. For fractional Laplacian in $\mathbb{R}^d$, we demonstrate that spectral methods using the generalised Hermite functions with their adjoint can lead to diagonal stiffness matrix. We also demonstrate that the bi-orthogonal Fourier-like mapped Chebyshev basis under the Dunford-Taylor formulation of the fractional Laplacian operator. This talk is largely based on some recent works listed below.

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Uniqueness and stability for inverse problems for fractional partial differential equations on the basis of the forward analysis

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We discuss several kinds of inverse problems of determining spatially and temporally functions in time-fractional partial differential equations.

One inverse problem is as follows: We consider

$$\begin{cases} \partial_t^{1/2} y(x,t) = \partial_x^2 y + p(x)y + R(x,t)f(x), \quad 0 < x < 1,\, 0 < t < T, \\ y(x,0) = 0, \quad 0 < x < 1. \end{cases}$$

Let $0 < t_0 < T$, and $p \in C^2[0,T]$, smooth R be fixed.
Then we consider
Inverse source problem.
Determine $f(x)$, $0 < x < 1$ *by* $y(x,t_0)$ *for* $0 < x < 1$ *and* $y(0,t), \partial_x y(0,t)$ *for* $0 < t < T$.

For proving the uniqueness, we need some extra regularity for $y(x,t)$ and we should establish such regularity in terms of data, which is a topic for the analysis of the corresponding forward problem.

Recently the author has established a convenient framework [1] for the forward problems for the time-fractional partial differential equations. We apply it to discuss inverse problems.

We can expect that such results for the forward problems as well as the inverse problems should be helpful for constructing numerical methods.

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¶This work was partly supported by Grant-in-Aid for Scientific Research (S) 15H05740 of Japan Society for the Promotion of Science and by The National Natural Science Foundation of China (no. 11771270, 91730303).

Higher order time stepping methods for subdiffusion problems based on weighted and shifted Grünwald-Letnikov formulae with nonsmooth data

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Two higher order time stepping methods for solving subdiffusion problems are studied in this paper. The Caputo time fractional derivatives are approximated by using the weighted and shifted Grünwald-Letnikov formulae introduced in Tian et al. [Math. Comp. 84 (2015), pp. 2703-2727]. After correcting a few starting steps, the proposed time stepping methods have the optimal convergence orders $O(k^2)$ and $O(k^3)$, respectively for any fixed time t for both smooth and nonsmooth data. The error estimates are proved by directly bounding the approximation errors of the kernel functions. Moreover, we also present briefly the applicabilities of our time stepping schemes to various other fractional evolution equations. Finally, some numerical examples are given to show that the numerical results are consistent with the proven theoretical results.

KEY WORDS: Weighted and shifted Grünwald -Letnikov formulae, subdiffusion equation, Caputo derivative, Laplace transform, higher order time stepping schemes

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¶This work was supported in part by the National Natural Science Foundation of China under grant 11801214. The third author (AKP) acknowledges the support from SERB, Govt. India via MATRIX grant No. MTR/201S/000309.

Inverse source problem in (fractional) diffusion equation with sparse data

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We consider the inverse source problem in a two dimensional (fractional) diffusion equation. The unknown source term has a semi-discrete form, namely, it is discretized to a step function on time variable t. We assume that the spatial term and the time mesh are both unknown. The measurements we use are the boundary flux data generated from finite points on the boundary. This is where the terminology 'sparse data' comes from. With Laplace transform and the knowledge in complex analysis, we prove the uniqueness theorem, which says the sparse boundary data can uniquely determine multiple unknowns simultaneously. After that, we try several numerical experiments and the corresponding numerical results are given.

KEY WORDS: inverse source problem, (fractional) diffusion equation, sparse data, uniqueness theorem, multiple unknowns, numerical reconstructions.

Hermite cubic spline collocation method for nonlinear fractional differential equations with variable-order

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In this paper, we develop an Hermite cubic spline collocation method (HCSCM) for solving variable-order nonlinear fractional-order differential equations, which apply C^1-continuous nodal basis functions to approximate problem. The main difficulty we coped with is the low singularity of the solution to the fractional-order differential equations. The main advantage of the HCSCM over the multi-domain collocation method is to diminish the numerical oscillation caused by singularity at the endpoint for the left-sided Riemman-Liouville derivative operator. We also verify the convergence rate is about $O(h^{\min\{4-\alpha,p\}})$ while the interpolating function belong to $C^p(p \geq 1)$, where h is the meshsize and α the order of the fractional derivative. Hence, the HCSCM perform well for the fractional-order problem with $C^p(p \geq 1)$ solution, in practice, even for $C^p(0 < p < 1)$ solution. Many numerical tests performed to confirm the effectiveness of the HCSCM for fractional differential equations which include Helmholtz equations and the fractional Burgers equation of constant-order and variable-order with Riemann-Liouville, Caputo and Patie-Simon sense as well as two-sided cases.

KEY WORDS: Hermite cubic spline collocation method; Fractional calculus; Cubic spline ; Fractional Burgers equation

¶This work was supported in part by the National Natural Science Foundation of China under grant 11661048.

Parallel-in-time high-order BDF schemes for diffusion and subdiffusion equations

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In this talk, I will present a parallel-in-time algorithm for approximately solving parabolic equations. We apply the k-step backward differentiation formula, and then develop an iterative solver by using the waveform relaxation technique. Each resulting iterate represents a periodic-like system, which could be further solved in parallel by using the diagonalization technique. The convergence of the waveform relaxation iteration is theoretically examined by using the generating function method. The argument could be further applied to the time-fractional subdiffusion equation, whose discretization shares common properties of the standard BDF methods, because of the nonlocality of the fractional differential operator. Some illustrative numerical results will be presented to complement the theoretical analysis.

KEY WORDS: parabolic equation, subdiffusion equation, backward differentiation formula, parallel-in-time algorithm, convergence analysis, convolution quadrature

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¶This work is partially supported by a Hong Kong RGC grant (project No. 25300818).