# Unconditional energy dissipation law and optimal error estimate of fast L1 schemes for a time-fractional Cahn-Hilliard problem

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# Outline


1. Fractional PDE
2. Unconditional error estimate of the fast L1 FEM scheme
   - The fully discrete fast L1-FEM
   - The boundness of the computed solution in $L^{\infty}$-norm
   - Unconditional error analysis of the fast L1 FEM scheme
3. Unconditional energy stability results
4. Numerical experiments

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# Fractional PDE

Time-fractional Cahn-Hilliard equations (TFCHE):

$$Lu := D_t^{\alpha} u - \kappa\Delta(-\epsilon^2\Delta u + f(u)) = 0 \tag{1}$$

for $(x, t) \in Q := \Omega \times (0, T]$, with

$$u(x, 0) = u_0(x) \quad \text{for } x \in \Omega,$$
$$\partial_{\upsilon} u\big|_{\partial\Omega} = \partial_{\upsilon}(\epsilon^2\Delta u - f(u))\big|_{\partial\Omega} = 0 \ \text{ for } 0 < t \leq T,$$

where $\alpha \in (0, 1)$, $u_0 \in C(\overline{\Omega})$, and $f(u)$ is the derivative of the double well potential $F(u) = \frac{1}{4}(u^2 - 1)^2$. Here the spatial domain $\Omega \subset \mathbb{R}^d$ (where $d \in \{1, 2, 3\}$) is bounded, with a Lipschitz continuous boundary $\partial\Omega$.

$D_t^{\alpha}$ denotes the Caputo fractional derivative defined by

$$D_t^{\alpha} u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t - s)^{-\alpha} \frac{\partial u(x, s)}{\partial s} \, ds.$$

The previous works:

- Linear schemes
  - T. Tang, H. J. Yu, and T. Zhou, SIAM J. Sci. Comput., 41(6): A3757-A3778, 2019.

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    L1 scheme +uniform meshes+ stabilization : $\tilde{E}[u^n] \leq \tilde{E}[u^{n-1}]$.

- Nonlinear schemes
  - M. Al-Maskari, and S. Karaa, IMA J. Numer. Anal., 42(2):1831-1865, 2022.

$$\boxed{\|\partial_t^l u(x,t)\| \leq C t^{\alpha-l} \text{ and CQ generated by BE method} : O(\tau).}$$

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    The nonuniform L1 type schemes : $O(\tau^{2-\alpha})$ + Energy stability results.

  - H. L. Liao, N. Liu, and X. Zhao, arXiv:2210.12514, 2022.

    The nonuniform BDF2 scheme : $O(\tau^2)$ + Energy stability results.

- Other theoretical works
  - C. Y. Quan, T. Tang, and J. Yang, CSIAM-AM, 1:478-490, 2020.

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## Nonuniform meshes in time

**M-conv.** Let $r$ represents the temporal mesh grading constant. There exists a constant $C_r > 0$, independent of $k$, such that $\tau_1 = C_r\tau^{r\alpha}$, $\tau_k \leq C_r\tau \min\{1, t_k^{1-1/r}\}$, $t_k \leq C_r t_{k-1}$, and $\tau_k \leq \rho_k\tau_{k-1}$ for $2 \leq k \leq N$.

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## The sum-of-exponentials technique

$$\left|\frac{t^{-\alpha}}{\Gamma(1-\alpha)} - \sum_{j=1}^{N_q}\omega_j e^{-s_j t}\right| \le \varepsilon,$$

where

$$N_q = \mathcal{O}\left(\log\frac{1}{\varepsilon}\left(\log\log\frac{1}{\varepsilon} + \log\frac{T}{\Delta t}\right) + \log\frac{1}{\Delta t}\left(\log\log\frac{1}{\varepsilon} + \log\frac{1}{\Delta t}\right)\right).$$

## Fast L1 discretisation in time

The Caputo fractional derivative is approximated by the fast L1 scheme

$$D_t^{\alpha} v(x, t_n) \approx D_F^{\alpha} v^n := \underbrace{a_0^{(n)} \nabla_\tau v^n}_{\text{The local part}} + \underbrace{\sum_{j=1}^{N_q} \omega_j e^{-s_j \tau_n} \mathbb{H}_j(t_{n-1})}_{\text{The history part}} \quad \text{for } n = 1, 2, \ldots, N, \quad (2)$$

where $\mathbb{H}_j(t_k)$ is defined by

$$\mathbb{H}_j(t_0) = 0, \; \mathbb{H}_j(t_k) = e^{-s_j \tau_k} \mathbb{H}_j(t_{k-1}) + \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} e^{-s_j(\tau_k - s)} \nabla_\tau v^k \, ds$$

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for $k \geq 1, 1 \leq j \leq N_q$. The fast L1 scheme (2) can be rewritten as:

$$D_F^{\alpha} v^n := \sum_{k=1}^{n} A_{n-k}^{(n)} \nabla_\tau v^k,$$

where

$$A_0^{(n)} := a_0^{(n)} \quad \text{and} \quad A_{n-k}^{(n)} := \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} \sum_{j=1}^{N_q} \omega_j e^{-s_j(\tau_n - s)} \, ds \quad \text{for } 1 \leq k \leq n-1.$$

## Lemma 1

*Assume* $\|\partial_t^l v(x,t)\| \leq C(1+t^{\alpha-l})$ *for* $l=0,1,2$. *Then there exists a constant* $C_T$ *satisfying*

$$\left\|D_t^\alpha v(\cdot,t_n) - D_F^\alpha v^n\right\| \leq C_T(t_n^{-\alpha}\tau^{-\min\{2-\alpha,r\alpha\}} + \varepsilon)$$

*and*

$$\|v^n - v^{n-1}\| \leq C_T\tau^{\min\{1,r\alpha\}}$$

*for* $n=1,2,\ldots,N$.

The equivalent formulation:

$$\begin{cases} D_t^{\alpha} u - \kappa \Delta w = 0 \ \forall (x,t) \in Q, \\ w + \epsilon^2 \Delta u - f(u) = 0 \ \ \forall \ (x,t) \in Q, \\ u(x,0) = u_0(x) \ \text{ for } \ x \in \Omega, \\ \partial_{\upsilon} u|_{\partial\Omega} = \partial_{\upsilon} w|_{\partial\Omega} = 0 \ \text{ for } 0 < t \leq T. \end{cases} \tag{3}$$

The time-discrete system:

$$\begin{cases} D_F^{\alpha} U^n - \kappa \Delta W^n = 0 \ \forall (x,t) \in Q, \\ W^n + \epsilon^2 \Delta U^n - f(U^{n-1}) - S(U^n - U^{n-1}) = 0 \quad \forall \ (x,t) \in Q, \\ U^0(x) = u_0(x) \quad \text{for} \quad x \in \Omega, \\ \partial_\upsilon U^n|_{\partial\Omega} = \partial_\upsilon W^n|_{\partial\Omega} = 0 \quad \text{for } 0 < t \leq T, \end{cases} \tag{4}$$

where $S \geq 0$ is a stabilization constant.

## FEM discretisation in space

Let $M$ be a positive integer. Partition $\Omega$ by a quasiuniform mesh of $M$ elements $\{K_m : m = 1, \ldots, M\}$. Set

$$h_m = \text{diam}(K_m) \text{ for each } m \text{ and } h = \max_{1 \le m \le M} \{h_m\}.$$

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Define the finite element spaces on spatial mesh by

$$V_h := \left\{ v_h \in C(\bar{\Omega}) \cap H^2(\Omega) : v_h|_{K_m} \in Q_1(K_m) \text{ on each } K_m \in \mathcal{T}_h \text{ and } \int_\Omega v_h \, dx = 0. \right\}.$$

## Three operators

Define the *Ritz projector* $R_h : H^1(\Omega) \to V_h$ by

$$(\nabla R_h w, \nabla v_h) = (\nabla w, \nabla v_h) \quad \forall\ v_h \in V_h.$$

It is well known that

$$\|w - R_h w\| + h\|w - R_h w\|_1 \leq Ch^{k+1}|w|_{k+1} \quad \forall\ w \in H^{k+1}(\Omega) \cap H^1(\Omega). \tag{5}$$

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Define the *discrete Laplacian* $\Delta_h : V_h \to V_h$ by

$$\boxed{(\Delta_h v, w) = -(\nabla v, \nabla w) \quad \forall\ v, w \in V_h}. \tag{6}$$

C. M. Elliott and S. Larsson, Math. Comp., 58(198):603-630, S33-S36, 1992.

$$\boxed{\Delta_h R_h v = P_h \Delta v \quad \forall\ v \in H^2(\Omega)}. \tag{7}$$

Define the operator $T_h$ by $T_h := (-\Delta_h)^{-1}$, and we have

$$(T_h v, g) = (\nabla T_h v, \nabla T_h g) \ \text{ for any } v, g \in L_2(\Omega). \tag{8}$$

The fully discrete fast L1-FEM:

$$\begin{cases} D_F^{\alpha} U_h^n - \kappa \Delta_h W_h^n = 0, \\ W_h^n + \epsilon^2 \Delta_h U_h^n - P_h \left[ f(U_h^{n-1}) + S(U_h^n - U_h^{n-1}) \right] = 0, \\ U_h^0 := R_h u_0, \\ \partial_\upsilon U_h^n|_{\partial\Omega} = \partial_\upsilon W_h^n|_{\partial\Omega} = 0, \end{cases} \tag{9}$$

for $n = 1, \ldots, N$.

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- Error estimate:

$$
\begin{aligned}
\|(u^n)^3 - (U_h^n)^3\| &= \| \left[(u^n)^2 + u^n U_h^n + (U_h^n)^2\right] (u^n - U_h^n)\| \\
&\le \left[\|u^n\|_\infty^2 + \|u^n\|_\infty \|U_h^n\|_\infty + \|U_h^n\|_\infty^2\right] \|u^n - U_h^n\|
\end{aligned}
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&\le \left[\|u^n\|_\infty^2+\|u^n\|_\infty\|U_h^n\|_\infty+\|U_h^n\|_\infty^2\right]\|u^n-U_h^n\|
\end{aligned}
$$

Time-step restriction:

$$
\begin{aligned}
\|U_h^n\|_{L^\infty} &\le \|R_hu^n\|_{L^\infty}+\|R_hu^n-U_h^n\|_{L^\infty} \\
&\le \|R_hu^n\|_{L^\infty}+Ch^{-d/2}\|R_hu^n-U_h^n\|_{L^2} \\
&\le C\|u^n\|_2+Ch^{-d/2}(\tau^{\min\{1,r\alpha\}}+\varepsilon+h^2)
\end{aligned}
$$

Without certain time-step restrictions:

$$
\begin{aligned}
\|U_h^n\|_{L^\infty} &\leq \|R_h U^n - U_h^n\|_{L^\infty} + \|R_h U^n\|_{L^\infty}, \\
&\leq C_\Omega h^{-d/2} \underbrace{\|R_h U^n - U_h^n\|}_{\text{The error in space}} + C_\Omega \|U^n\|_2, \\
&\leq C_\Omega h^{-d/2} h^{\frac{7}{4}} + C_\Omega (1 + C_1) \\
&\leq K_1.
\end{aligned}
$$

## Error equation in time

Denote

$$e_u^n := u^n - U^n \text{ and } e_w^n := w^n - W^n.$$

From (3) and (4), one has

$$\begin{cases} D_F^\alpha e_u^n - \kappa \Delta e_w^n = \mathbb{P}^n, \\ e_w^n + \epsilon^2 \Delta e_u^n = \mathbb{Q}^n \\ e_u^0 = 0, \\ \partial_\upsilon e_u^n|_{\partial\Omega} = \partial_\upsilon e_w^n|_{\partial\Omega} = 0, \end{cases} \tag{10}$$

where $\mathbb{P}^n$ and $\mathbb{Q}^n$ are defined by

$$\mathbb{P}^n = D_F^\alpha u^n - D_t^\alpha u^n,$$
$$\mathbb{Q}^n = (u^n)^3 - u^n - (U^{n-1})^3 + U^{n-1} - S(U^n - U^{n-1}).$$

The error equation of the time-discrete system:

$$\boxed{D_F^{\alpha} e_u^n + \kappa\epsilon^2\Delta^2 e_u^n = \mathbb{P}^n + \kappa\Delta\left(\phi_{u,1}^n + (\phi_{u,2}^n - 1 - S)e_u^{n-1} + Se_u^n\right),} \tag{11}$$

where $\phi_{u,1}^n$ and $\phi_{u,2}^n$ are defined by

$$\phi_{u,1}^n := (u^n)^2 + u^n u^{n-1} + (u^{n-1})^2(u^n - u^{n-1})$$

and

$$\phi_{u,2}^n := (u^{n-1})^2 + u^{n-1}U^{n-1} + (U^{n-1})^2.$$

## Lemma 2 (The robust discrete fractional Grönwall inequality)

*Let $\lambda_s$ be nonnegative constants with $0 < \sum_{s=1}^{n} \lambda_s \le \Lambda$ for $n \ge 1$, where $\Lambda$ is a positive constant independent of $n$. Suppose that the nonnegative sequences $\{\xi^n, \eta^n : n \ge 1\}$ are bounded and the nonnegative grid function $\{ v^n \mid n \ge 0\}$ satisfies*

$$D_F^{\alpha}(v^n)^2 \le \sum_{s=1}^{n} \lambda_{n-s}(v^s)^2 + \xi^n v^n + (\eta^n)^2 \quad \text{for} \quad n \ge 1. \tag{12}$$

*If the nonuniform grid satisfies the maximum time-step criterion $\tau \le \left[3\Gamma(2-\alpha)\Lambda\right]^{-1/\alpha}$, then*

$$v^n \le 2E_\alpha\left(3\Lambda t_n^{\alpha}\right)\left[v^0 + \max_{1\le k\le n}\sum_{j=1}^{k} P_{k-j}^{(k)}(\xi^j + \eta^j) + \max_{1\le j\le n}\{\eta^j\}\right] \quad \text{for} \quad 1 \le n \le N. \tag{13}$$

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H. Chen and M. Stynes, IMA J. Numer. Anal., 41(2):974-997, 2021.

$$\sum_{j=1}^{n} P_{n-j}^{(n)} j^{r(\gamma-\alpha)} \le \frac{3\Gamma(1+\gamma-\alpha)}{2\Gamma(1+\gamma)} T^\alpha \left(\frac{t_n}{T}\right)^\gamma N^{r(\gamma-\alpha)}$$

## The boundless of $U^n$, $D_F^\alpha U^n$, and $W^n$

### Lemma 3

*The time discrete system* (4) *has a unique solution* $U^n$. *For* $0 \le n \le N$, *if* $\tau \le \tau_1^*$, *one has*

$$\|e_u^n\|_2 \le C_1^*(\tau^{\min\{1,r\alpha\}} + \varepsilon), \tag{14}$$

$$\|U^n\|_2 \le 1 + C_1. \tag{15}$$

*Furthermore, if* $1 \le r \le 1/\alpha$, *one has*

$$\|D_F^\alpha U^n\|_2 \le C_2^* \quad \text{for } 1 \le n \le N. \tag{16}$$

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### Lemma 4

*The solution* $W^n$ *of the time discrete system* (4) *satisfies*

$$\|W^n\|_2 \le C_3^* \quad \text{for } \ 1 \le n \le N. \tag{17}$$

## Error equation in space

Denote

$$U^n - U_h^n = (R_h U^n - U_h^n) - (R_h U^n - U^n) := \vartheta_u^n - \rho_u^n,$$

$$W^n - W_h^n = (R_h W^n - W_h^n) - (R_h W^n - W^n) := \vartheta_w^n - \rho_w^n.$$

From (4) and (9), one has

$$\begin{aligned} D_F^\alpha \vartheta_u^n - \kappa \Delta_h \vartheta_w^n &= [R_h(D_F^\alpha U^n) - \kappa \Delta_h R_h W^n] - [D_F^\alpha U_h^n - \kappa \Delta_h W_h^n] \\ &= (R_h - P_h) D_F^\alpha U^n + P_h [D_F^\alpha U^n - \kappa \Delta W^n] \\ &= P_h D_F^\alpha \rho_u^n, \end{aligned} \tag{18}$$

and

$$
\begin{aligned}
\vartheta_w^n + \epsilon^2 \Delta_h \vartheta_u^n &= \left[ R_h W^n + \epsilon^2 \Delta_h R_h U^n \right] - \left[ W_h^n + \epsilon^2 \Delta_h U_h^n \right] \\
&= (R_h - P_h) W^n + P_h \left[ W^n + \epsilon^2 \Delta U^n \right] \\
&\quad - P_h \left[ (U_h^{n-1})^3 - U_h^{n-1} + S(U_h^n - U_h^{n-1}) \right] \\
&= P_h \left[ \rho_w^n + (\psi_u^n - 1 - S)(\vartheta_u^{n-1} - \rho_u^{n-1}) + S(\vartheta_u^n - \rho_u^n) \right], \qquad (19)
\end{aligned}
$$

where $\psi_u^n$ is defined by

$$
\psi_u^n := (U^{n-1})^2 + (U_h^{n-1})^2 + U^{n-1} U_h^{n-1}.
$$

and

$$\vartheta_w^n + \epsilon^2 \Delta_h \vartheta_u^n = \left[R_h W^n + \epsilon^2 \Delta_h R_h U^n\right] - \left[W_h^n + \epsilon^2 \Delta_h U_h^n\right]$$

$$= (R_h - P_h) W^n + P_h \left[W^n + \epsilon^2 \Delta U^n\right]$$

$$- P_h \left[(U_h^{n-1})^3 - U_h^{n-1} + S(U_h^n - U_h^{n-1})\right]$$

$$= P_h \left[\rho_w^n + (\psi_u^n - 1 - S)(\vartheta_u^{n-1} - \rho_u^{n-1}) + S(\vartheta_u^n - \rho_u^n)\right], \qquad (19)$$

where $\psi_u^n$ is defined by

$$\psi_u^n := (U^{n-1})^2 + (U_h^{n-1})^2 + U^{n-1} U_h^{n-1}.$$

Applying (18) and (19) yields

$$D_F^\alpha \vartheta_u^n + \kappa \epsilon^2 \Delta_h^2 \vartheta_u^n = D_F^\alpha \rho_u^n + \kappa \Delta_h P_h \left[\rho_w^n + (\psi_u^n - 1 - S)(\vartheta_u^{n-1} - \rho_u^{n-1}) + S(\vartheta_u^n - \rho_u^n)\right]. \qquad (20)$$

## The boundless of the numerical solution $U_h^n$

### Theorem 5

*Assume* $\tau \le \tau_2^*$ *and* $h \le h_1^*$. *Let* $U^n$ *and* $U_h^n$ *be the solutions of* (4) *and* (9), *respectively. Then for* $n = 0, 1, \ldots, N$, *one has*

$$\|R_h U^n - U_h^n\| \le h^{\frac{7}{4}}, \tag{21}$$

*and*

$$\|U_h^n\|_{L^\infty} \le K_1. \tag{22}$$

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## Error equation of the fully discrete scheme

Denote

$$u^n - U_h^n = R_h u^n - U_h^n - (R_h u^n - u^n) := \eta_u^n - \varrho_u^n,$$
$$w^n - W_h^n = R_h w^n - W_h^n - (R_h w^n - w^n) := \eta_w^n - \varrho_w^n.$$

From (3) and (9), we get

$$D_F^\alpha \eta_u^n + \kappa\epsilon^2 \Delta_h^2 \eta_u^n = P_h(D_t^\alpha \varrho_u^n - R_h\varphi^n) + \Delta_h P_h \left[ \varrho_w^n + \phi_{u,1}^n + \Phi_u^n(\eta_u^{n-1} - \varrho_u^{n-1}) + S(\eta_u^n - \varrho_u^n) \right], \quad (23)$$

where $\Phi_u^n$ is defined by

$$\Phi_u^n = (u^{n-1})^2 + u^{n-1} U_h^{n-1} + (U_h^{n-1})^2 - 1 - S.$$

The boundless of $\Phi_u^n$:

$$\|\Phi_u^n\|_\infty \le \|u^{n-1}\|_\infty^2 + \|u^{n-1}\|_\infty \|U_h^{n-1}\|_\infty + \|U_h^{n-1}\|_\infty^2 + 1 + S$$
$$\le C_\Omega(C_1^2 + C_1 K_1 + K_1^2) + 1 + S := C_4, \qquad (24)$$

where $\|u^k\|_{L^\infty} \le C_1$ and $\|U_h^k\|_{L^\infty} \le K_1$ are used.

Set

$$C_4 := C_\Omega(C_1^2 + C_1 K_1 + K_1^2) + 1 + S \quad \text{and} \quad \Lambda_3^* := \frac{2\kappa(C_4^2 + S^2)}{\epsilon^2}.$$

## Theorem 6 (Error estimate for the fast L1 FEM)

*Assume* $\tau \le \min\left\{\tau_2^*, \left[3\Gamma(2-\alpha)\Lambda_3^*\right]^{-1/\alpha}\right\}$ *and* $h \le h_1^*$. *Let* $u^n$ *and* $U_h^n$ *be the solutions of* (3) *and* (9), *respectively. Then for* $n = 1, 2, \ldots, N$, *one has*

$$\|u^n - U_h^n\| \le \Theta_n(\tau, r) + C_R h^2, \tag{25}$$

*where*

$$\Theta_n(\tau, r) := 2E_\alpha\left(3\Lambda_3^* t_n^\alpha\right)\Bigg[\left(3C_T\Gamma(1-\alpha) + \frac{3\sqrt{\kappa}C_\Omega C_T C_1^2}{2\epsilon}(3\Gamma(1-\alpha)t_n^\alpha + 2)\right)\tau^{\min\{1, r\alpha\}}$$
$$+3C_T\Gamma(1-\alpha)t_n^\alpha\varepsilon + C_R C_1\left(3\Gamma(1-\alpha)t_n^\alpha + \frac{\sqrt{\kappa}(1 + C_4 + S)}{2\epsilon}(3\Gamma(1-\alpha)t_n^\alpha + 2)\right)h^2\Bigg].$$

If $r \ge 1/\alpha$, then one has

$$\|u^n - U_h^n\| \le C\left(\tau + \varepsilon + h^2\right) \quad \text{for } n = 0, 1, \ldots, N.$$

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   - The fully discrete fast L1-FEM
   - The boundness of the computed solution in $L^{\infty}$-norm
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4. Numerical experiments

- The approximation of the modified energy:

$$
\begin{aligned}
f(U_h^{n-1})\nabla_\tau U_h^n &= F(U_h^n) - F(U_h^{n-1}) - \int_0^1 f'(U_h^{n-1} + s\nabla_\tau U_h^n)(1-s)\ ds\ (\nabla_\tau U_h^n)^2 \\
&\geq F(U_h^n) - F(U_h^{n-1}) + \frac{1}{2}(\nabla_\tau U_h^n)^2 \\
&\quad - \int_0^1 3\left((1-s)\|U_h^{n-1}\|_\infty + s\|U_h^n\|_\infty\right)^2 (1-s)\ ds\ (\nabla_\tau U_h^n)^2
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\end{aligned}
$$

- Two assumptions: (D. Li and Z. H. Qiao, J. Sci. Comput., 70(1):301-341, 2017.)

  The Lipschitz assumption:

$$
\max_{u\in R} |f'(u)| \leq L. \tag{26}
$$

  $L^\infty$ bounds on the numerical solution:

$$
\|U_h^n\|_\infty \leq L. \tag{27}
$$

The discrete energy functional $E[U_h^n]$:

$$E[U_h^n] := \frac{\epsilon^2}{2}\|\nabla U_h^n\|^2 + (F(U_h^n), 1) \text{ with } F(U_h^n) := \frac{1}{4}\left((U_h^n)^2 - 1\right)^2 \text{ for } 0 \leq n \leq N.$$

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The modified discrete energy $E_\alpha[U_h^n]$:

$$E_\alpha[U_h^0] := E[U_h^0] \quad \text{and} \quad E_\alpha[U_h^n] := E[U_h^n] + \frac{\kappa}{2}\sum_{j=1}^{n} P_{n-j}^{(n)}\|\nabla W_h^j\|^2 \text{ for } \ 1 \leq n \leq N.$$

## The energy stability result

**Theorem 7 (The energy stability result for the modified energy)**

*Let* $S \geq \frac{3K_1^2}{2} - \frac{1}{2}$. *Assume* $\tau \leq \tau_2^*$ *and* $h \leq h_1^*$, *the fully discrete semi-implicit L1-FEM* (9) *preserves the following discrete energy dissipation law*

$$E_\alpha[U_h^n] \leq E_\alpha[U_h^{n-1}] \quad \textit{for} \ \ 1 \leq n \leq N.$$

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The energy stability property for $E[U_h^n]$:

$$E[U_h^n] \leq E_\alpha[U_h^n] \leq E[U_h^0] \quad \text{for } 1 \leq n \leq N.$$

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# Numerical experiments

## Example 1

To verify the accuracy in time and space, we consider the time-fractional Cahn-Hilliard problem (1) in two-dimensional with $\kappa = 1$, $\epsilon = 1$, $\Omega = (0, 2\pi) \times (0, 2\pi)$, $T = 1$, and $u_0(x, y) = \cos(x)\cos(y)$. In addition, the graded mesh $t_n := T(n/N)^r$ is used in temporal direction.

Taking $r = 1/\alpha$ and $N = M$, the spatial error dominates the result. Predicted rate: $O(\tau)$.

Table 1: $\max_{1\le n\le N} \|u^n - U_h^n\|$ errors and rates of convergence (dominated by temporal error)

| | N=20 | N=40 | N=80 | N=160 |
|---|---|---|---|---|
| $\alpha = 0.4$ | 1.3070E-2 | 7.1577E-3 | 3.7393E-3 | 1.9487E-3 |
| | | 0.8687 | 0.9367 | 0.9402 |
| $\alpha = 0.6$ | 1.5168E-2 | 8.1019E-3 | 4.0908E-3 | 2.0115E-3 |
| | | 0.9046 | 0.9858 | 1.0240 |
| $\alpha = 0.8$ | 1.7323E-2 | 9.6204E-3 | 4.9963E-3 | 2.5035E-6 |
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$$\Downarrow$$

$$\boxed{O(\tau)}$$

Taking $r = (2-\alpha)/\alpha$ and $N = M$, the spatial error dominates the result. Predicted rate: $O(\tau)$.

Table 2: $\max_{1\le n\le N}\|u^n - U_h^n\|$ errors and rates of convergence (dominated by temporal error)

| | N=10 | N=20 | N=40 | $N = 80$ |
|---|---|---|---|---|
| $\alpha = 0.4$ | 7.9016E-3 | 4.4416E-3 | 2.3408E-3 | 1.1972E-3 |
| | | 0.8310 | 0.9240 | 0.9673 |
| $\alpha = 0.6$ | 1.0683E-2 | 5.7319E-3 | 2.9323E-3 | 1.4690E-3 |
| | | 0.8981 | 0.9669 | 0.9971 |
| $\alpha = 0.8$ | 1.5087E-2 | 8.3147E-3 | 4.2919E-3 | 2.1475E-3 |
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$O(\tau)$

Taking $r = 1/\alpha$ and $N = 1000$, the spatial error dominates the result. Predicted rate: $O(h^2)$.

Table 3: $\max_{1\leq n\leq N} \|u^n - U_h^n\|$ errors and rates of convergence (dominated by spatial error)

| | M=8 | M=16 | M=32 | M=64 |
|---|---|---|---|---|
| $\alpha = 0.4$ | 1.7447E-2 | 4.3178E-3 | 1.0758E-3 | 2.6873E-4 |
| | | 2.0146 | 2.0048 | 2.0012 |
| $\alpha = 0.6$ | 1.8195E-2 | 4.4866E-3 | 1.1169E-3 | 2.7891E-4 |
| | | 2.0198 | 2.0061 | 2.0016 |
| $\alpha = 0.8$ | 1.9383E-2 | 4.7551E-3 | 1.1821E-3 | 2.9512E-4 |
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$\Downarrow$

$O(h^2)$

## Example 2

Consider the time-fractional Cahn-Hilliard model (1) with $\kappa = 1$, $\epsilon = 0.05$, $\Omega = (0, 2) \times (0, 2)$. Here, the initial condition

$$u_0(x, y) = 0.1 rand(x, y) - 0.05,$$

where $rand(x, y)$ generates uniform random numbers in the domain $[0, 1]$.

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We use the graded meshes $t_n = T_0(n/N_0)^r$ with $r = 1/\alpha$, $N_0 = 30$, and $T_0 = 0.001$ to handle the weakly singularity near the initial time. The remaining time interval adopts the following time-stepping strategy

$$\tau_{n+1} = \max\left\{\tau_{\min}, \frac{\tau_{\max}}{\sqrt{1+\delta\|\partial_\tau U_h^n\|^2}}\right\} \quad \text{for} \quad n \geq N_0, \tag{28}$$

where $\delta$ is a user chosen constant, $\tau_{\max} = 0.005$, and $\tau_{\min} = 0.001$.

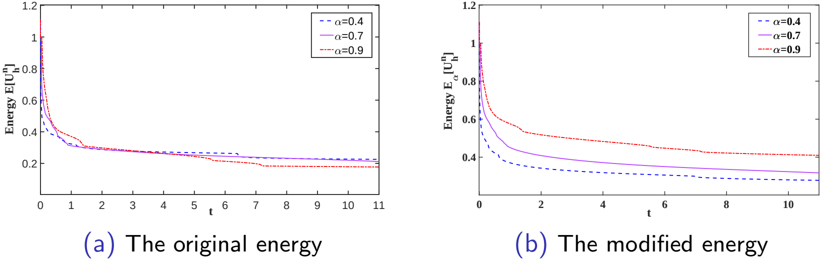
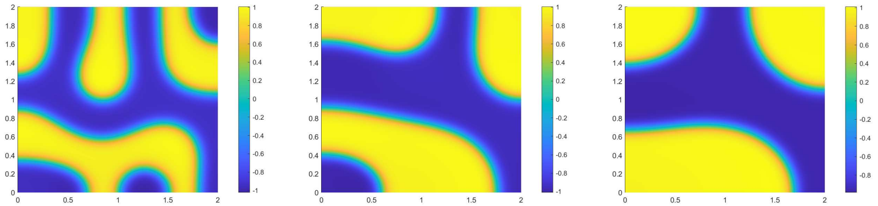
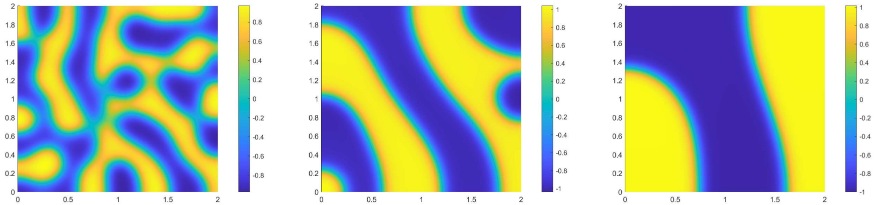



Figure 1: The original energy and the modified energy for Example 2.




(a) The profile of $U_h^n$ with fractional order $\alpha = 0.4$ at $t = 0.1, 1, 11$.



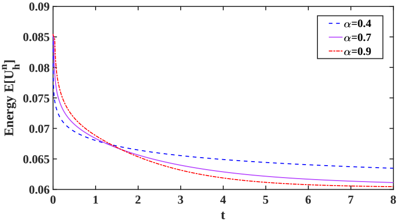

(b) The profile of $U_h^n$ with fractional order $\alpha = 0.9$ at $t = 0.1, 1, 11$.

## Example 3

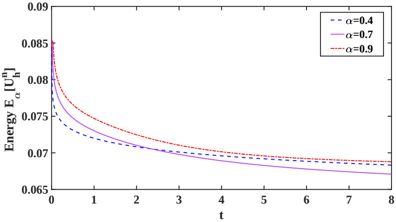
Consider the time-fractional Cahn-Hilliard model (1) with $\kappa = 1$, $\epsilon = 0.02$, $\Omega = (-1, 1) \times (-1, 1)$. The initial condition is chosen as

$$u_0(x, y) = \sum_{i=1}^{2} -\tanh\left(\frac{\sqrt{(x - x_i)^2 + (y - y_i)^2} - 0.36}{\sqrt{2}\epsilon}\right) + 1$$

with $(x_1, y_1) = (-0.4, 0)$ and $(x_2, y_2) = (0.4, 0)$. Actually, this example is often used to describe the coalescence of two kissing bubbles.

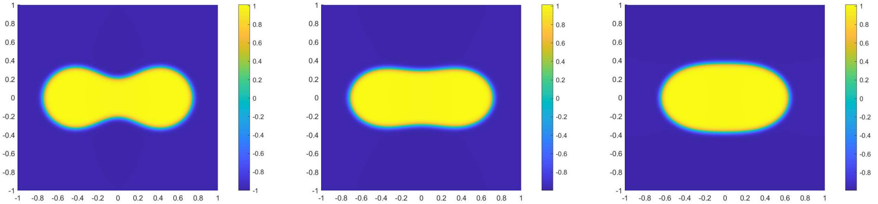



(c) The original energy

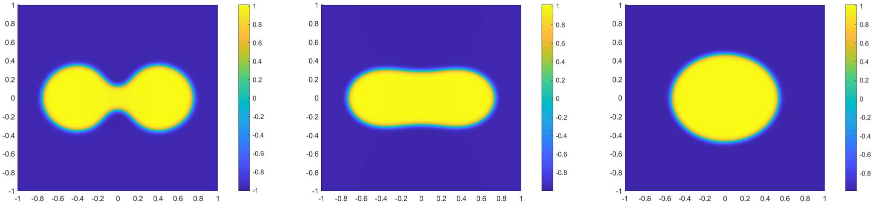



(d) The modified energy

Figure 2: The original energy and the modified energy for Example 3.




(a) The profile of $U_h^n$ with fractional order $\alpha = 0.4$ at $t = 0.1, 1, 8$.




(b) The profile of $U_h^n$ with fractional order $\alpha = 0.9$ at $t = 0.1, 1, 8$.

# Thank You

C. B. Huang, Na. An, and X. J. Yu, Unconditional energy dissipation law and optimal error estimate of fast L1 schemes for a time-fractional Cahn-Hilliard problem,Commun. Nonlinear Sci. Numer. Simul., 124:107300, 2023.