Efficient and Accurate Numerical Methods Using the Accelerated SDC for Solving Fractional Differential Equations

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Outline

Fractional ODEs

- Consider

$$ {}_0^C D_t^{\alpha} u(t) = F(t, u(t)),\ t \in [0, T],\ u(0) = u_0, \tag{1} $$

where ${}_0^C D_t^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0, 1]$ defined by

$$ {}_0^C D_t^{\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(s)}{(t-s)^{\alpha}} ds. $$

- Here $u(t), u_0 \in \mathbb{R}^N$ and $F : \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$.
- Assume that F is continuous, bounded and fulfills a Lipschitz condition with respect to the second variable such that the problem (1) is well-posed, i.e., for the problem (1) there exists a unique solution $u(t) : [0, T] \to \mathbb{R}^N$ for $T > 0$. [Diethelm and Ford, 2002]

Existing works

- Analytic solutions of some FDEs are typically obtained by using special functions (e.g., Wright functions) for simple linear problems.
- Finite difference methods: [Sun and Wu, 2006; Lin, Li and Xu, 2011; Cao and Xu, 2013; Gao, Sun and Zhang, 2014; ......]
- Two essential issues:
  - Nonlocality
  - Low regularity
- Spectral methods: [Li and Xu, 2011; Zayernouri and Karniadakis, 2013; Chen, Wang and Shen, 2016]
- Multi-domain methods: Nonuniform mesh [Zheng and Wang et al; Zhang and Sun, 2014; Stynes, Oriordan and Gracia, 2017; ......]
- Long time simulation
  - Finite difference methods [Zeng et al, 2018, 2019; Lubich et al, 1986, 1996]

Outline

SDC for fractional ODEs

- SDC for fractional ODEs [Lv, Azaiez and Xu, 2018] with convergence rate $O(\Delta T^{(2-\alpha)(k+1)})$ (or $O(\Delta T^{(2-\alpha)+k})$) for the uniform mesh (or the Gauss-Lobatto mesh);
- Kernel compression method [Baffet, 2019] with convergence rate $O(\Delta T^{\min(p+1+\alpha, \alpha(k+1)+\delta)}), \delta = 1$ or 2.

SDC

- Applying the fractional integral to (1) yields the analytic solution
$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{F(\tau, u(\tau))}{(t-\tau)^{1-\alpha}} d\tau, \quad t \in [0, T]. \tag{2}$$
- By considering the generic interval $[a, b]$, the above equation reduces to
$$u(t) = u_a(t) + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{F(\tau, u(\tau))}{(t-\tau)^{1-\alpha}} d\tau, \tag{3}$$
where $u_a(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^a \frac{F(\tau,u(\tau))}{(t-\tau)^{1-\alpha}} d\tau$ represents the history term.
- Assume we have an approximation $u^0(t)$ to (3), we then define the error $\delta(t) := u(t) - u^0(t)$, we have
$$\delta(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1}[F(\tau, u^0(\tau) + \delta(\tau)) - F(\tau, u^0(\tau))] d\tau + \epsilon(t, u^0(t)),$$
where the residual function is given as follows:
$$\epsilon(t, u^0(t)) = u_a(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} F(\tau, u^0(\tau)) d\tau - u^0(t).$$

Approximation of the residual function

- Let $F(t, u^k(t))$ be approximated by Gauss-Legendre-Lobotto interpolation

$$F_p(t, u^k(t)) = \mathbb{I}_p F(t, u^k(t)) = \sum_{m=0}^{p} F_m^k \hat{h}_m(t),$$

where $\hat{h}_j(t)$ is the Lagrange interpolation polynomial based on the $p+1$ Legendre-Gauss-Lobatto points in the interval $[a, b]$.

- Then the residual function $\epsilon(s, u^k(s))$ is approximated as follows:

$$\vec{\epsilon}^k = \vec{u}_a + \Delta T^\alpha \boldsymbol{A}\vec{F}^k - \vec{u}^k, \tag{4}$$

where $\vec{\epsilon}^k = [\epsilon_0^k, \epsilon_1^k, \cdots, \epsilon_p^k]^T$, $\vec{u}_a = [u_a(s_0), u_a(s_1), \cdots, u_a(s_p)]^T$, $\vec{F}^k = [F_0^k, F_1^k, \cdots, F_p^k]^T$, $\vec{u}^k = [u_0^k, u_1^k, \cdots, u_p^k]^T$, $\boldsymbol{A} = I_N \otimes A$ is a $N(p+1) \times N(p+1)$ block diagonal matrix, A is the *fractional spectral integration matrix* given by

$$A_{ij} = \frac{1}{2^\alpha \Gamma(\alpha)} \int_{-1}^{r_i} (r_i - r)^{\alpha-1} h_j(r) dr.$$

Linear Problem

- For simplicity, we consider the linear problem, i.e., $F(t,u(t)) = Lu(t) + f(t)$. Then the correction equation becomes

$$\delta(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} L\delta(\tau)d\tau + \epsilon(t).$$

- For the k-th correction step, by discretizing the above equation with the backward Euler scheme and noting that $\delta_0^k = \epsilon_0^k = 0$, we obtain

$$\delta_i^k = \frac{1}{\Gamma(\alpha+1)} \sum_{l=1}^{i} L\tilde{s}_{i,l}^{\alpha}\, \delta_l^k + \epsilon_i^k,\ i = 1, \ldots, p, \tag{5}$$

where $\tilde{s}_{i,l}^{\alpha} = (s_i - s_{l-1})^{\alpha} - (s_i - s_l)^{\alpha}$, $i \geq l \geq 1$.

- Writing the above equation into the matrix form yields

$$(\boldsymbol{I} - \Delta T^{\alpha} \widetilde{\boldsymbol{A}}\boldsymbol{L})\vec{\delta}^k = \vec{\epsilon}^k, \tag{6}$$

where $\vec{\delta}^k = [\delta_0^k, \delta_1^k, \cdots, \delta_p^k]^T$, $\boldsymbol{L} = diag(L) \otimes I_{p+1}$, $\widetilde{\boldsymbol{A}} = I_N \otimes \widetilde{A}$.

Matrix $\Delta T^{\alpha}\widetilde{A}$

- The matrix $\Delta T^{\alpha}\widetilde{A}$ is given by

$$\Delta T^{\alpha}\widetilde{A} = \frac{1}{\Gamma(\alpha+1)}\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \tilde{s}^{\alpha}_{1,1} & 0 & \cdots & 0 & 0 \\ 0 & \tilde{s}^{\alpha}_{2,1} & \tilde{s}^{\alpha}_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \tilde{s}^{\alpha}_{p-1,1} & \tilde{s}^{\alpha}_{p-1,2} & \cdots & \tilde{s}^{\alpha}_{p-1,p-1} & 0 \\ 0 & \tilde{s}^{\alpha}_{p,1} & \tilde{s}^{\alpha}_{p,2} & \cdots & \tilde{s}^{\alpha}_{p,p-1} & \tilde{s}^{\alpha}_{p,p} \end{pmatrix}.$$

- Similarly, for the explicit Euler scheme, the matrix $\Delta T^{\alpha}\widetilde{A}$ takes the form

$$\Delta T^{\alpha}\widetilde{A} = \frac{1}{\Gamma(\alpha+1)}\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ \tilde{s}^{\alpha}_{1,1} & 0 & \cdots & 0 & 0 & 0 \\ \tilde{s}^{\alpha}_{2,1} & \tilde{s}^{\alpha}_{2,2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \tilde{s}^{\alpha}_{p-1,1} & \tilde{s}^{\alpha}_{p-1,2} & \cdots & \tilde{s}^{\alpha}_{p-1,p-1} & 0 & 0 \\ \tilde{s}^{\alpha}_{p,1} & \tilde{s}^{\alpha}_{p,2} & \cdots & \tilde{s}^{\alpha}_{p,p-1} & \tilde{s}^{\alpha}_{p,p} & 0 \end{pmatrix}.$$

Neumann series expansion for $\vec{\delta}$

- Assuming we have the provisional solution $\vec{u}^k$, we have

$$\vec{u}^{k+1} = \vec{u}^k + \vec{\delta}^k = \vec{u}^k + (\boldsymbol{I} - \Delta T^{\alpha} \widetilde{\boldsymbol{A}}\boldsymbol{L})^{-1} \vec{\epsilon}^k.$$

- By the above equation and (4), i.e., eqn for $\vec{\epsilon}^k$, we derive

$$\begin{aligned} \vec{u}^{k+1} &= \vec{u}^k + (\boldsymbol{I} - \Delta T^{\alpha} \widetilde{\boldsymbol{A}}\boldsymbol{L})^{-1} \left(\vec{u}_a + \Delta T^{\alpha} \boldsymbol{A}(\boldsymbol{L}\vec{u}^k + \vec{f}) - \vec{u}^k \right) \\ &= \cdots\cdots \\ &= (\boldsymbol{I} - \Delta T^{\alpha} \widetilde{\boldsymbol{A}}\boldsymbol{L})^{-1} \vec{u}_a + \boldsymbol{C}\vec{u}^k + (\boldsymbol{I} - \Delta T^{\alpha} \widetilde{\boldsymbol{A}}\boldsymbol{L})^{-1} \Delta T^{\alpha} \boldsymbol{A}\vec{f}, \end{aligned}$$

  where $\boldsymbol{C}$ is the so called “correction matrix” given by

$$\boldsymbol{C} = (\boldsymbol{I} - \Delta T^{\alpha} \widetilde{\boldsymbol{A}}\boldsymbol{L})^{-1} \Delta T^{\alpha} (\boldsymbol{A} - \widetilde{\boldsymbol{A}})\boldsymbol{L}. \tag{7}$$

- Consequently, by replacing $k+1$ by k, we have

$$\vec{u}^k = (\boldsymbol{I} - \Delta T^{\alpha} \widetilde{\boldsymbol{A}}\boldsymbol{L})^{-1} \vec{u}_a + \boldsymbol{C}\vec{u}^{k-1} + (\boldsymbol{I} - \Delta T^{\alpha} \widetilde{\boldsymbol{A}}\boldsymbol{L})^{-1} \Delta T^{\alpha} \boldsymbol{A}\vec{f}.$$

Neumann series expansion for $\vec{\delta}$

- Subtracting the above equation from (11) yields the recursive relationship

$$\vec{\delta}^k = \boldsymbol{C}\vec{\delta}^{k-1} = \boldsymbol{C}^k\vec{\delta}^0.$$

- Consequently, we have the solution after k corrections given by the Neumann series expansion:

$$\vec{u}^k = \vec{u}^0 + \sum_{i=0}^{k-1} \boldsymbol{C}^i\vec{\delta}^0. \tag{8}$$

This means

$$\vec{\delta} = \sum_{i=0}^{k-1} \boldsymbol{C}^i\vec{\delta}^0.$$

Neumann series expansion for $\vec{\delta}$

- Discretizing error equation (5) at the Gauss-type collocation points

$$(\boldsymbol{I} - \Delta T^{\alpha} \boldsymbol{A}\boldsymbol{L})\vec{\delta} = \vec{\epsilon},$$

- The SDC procedure is to iteratively approximate the above system with the low order approximations $\vec{\delta}^{k}$ for $k = 1, 2, \ldots$. Applying the low-order preconditioner $(\boldsymbol{I} - \Delta T^{\alpha} \widetilde{\boldsymbol{A}}\boldsymbol{L})^{-1}$ to the above system, we obtain

$$(\boldsymbol{I} - \Delta T^{\alpha} \widetilde{\boldsymbol{A}}\boldsymbol{L})^{-1}(\boldsymbol{I} - \Delta T^{\alpha} \boldsymbol{A}\boldsymbol{L})\vec{\delta} = (\boldsymbol{I} - \Delta T^{\alpha} \widetilde{\boldsymbol{A}}\boldsymbol{L})^{-1}\vec{\epsilon} = \vec{\delta}^{0}. \tag{9}$$

  Let $\boldsymbol{C}$ be given by (7), then, we have

$$(\boldsymbol{I} - \boldsymbol{C})\vec{\delta} = \vec{\delta}^{0}. \tag{10}$$

- The solution to the above is given by the Neumann series expansion

$$\vec{\delta} = \vec{\delta}^{0} + \boldsymbol{C}\vec{\delta} = \vec{\delta}^{0} + \boldsymbol{C}(\vec{\delta}^{0} + \boldsymbol{C}\vec{\delta}) = \vec{\delta}^{0} + \boldsymbol{C}\vec{\delta}^{0} + \boldsymbol{C}^{2}\vec{\delta} = \vec{\delta}^{0} + \boldsymbol{C}\vec{\delta}^{0} + \boldsymbol{C}^{2}\vec{\delta}^{0} + \cdots,$$

  which is equivalent to equation (8).

Error estimate for the viewpoint of spectrum of C

- We have the following convergence result:

Theorem

*For linear fractional ODEs, the SDC iteration is convergent if and only if the spectral radius of the correction matrix **C** given in (7) is less than 1, i.e., $\rho(\mathbf{C}) < 1$.*

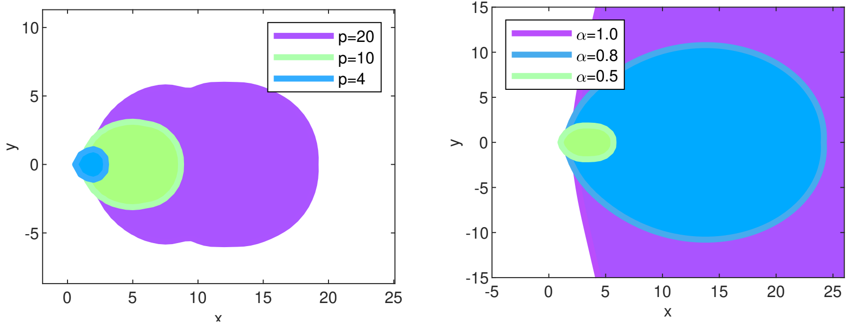
- The convergence rate for the SDC iteration is
  - $(\rho(\mathbf{C}))^k = \left(\rho\left((\mathbf{I} - \Delta T^\alpha \widetilde{\mathbf{A}}\mathbf{L})^{-1} \Delta T^\alpha (\mathbf{A} - \widetilde{\mathbf{A}})\mathbf{L}\right)\right)^k.$
- Recall that the convergence rates is
  - $O(\Delta T^{(2-\alpha)(k+1)})$ (or $O(\Delta T^{(2-\alpha)+k})$) [Lv, Azaiez and Xu, 2018] or
  - $O(\Delta T^{\min(p+1+\alpha, \alpha(k+1)+\delta)}), \delta = 1$ or 2 [Baffet, 2019].

Convergence region

- Consider the following scalar problem:

$$ {}_0^C D_t^{\alpha} u(t) = \lambda\, u(t), t \in [0, T]. $$

- Let $T = 2$ and use implicit scheme



$\rho(\boldsymbol{C}(\lambda)) = 1,\ \alpha = 0.6$ $\quad$ $\rho(\boldsymbol{C}(\lambda)) = 1,\ p = 10$

Figure: Contours of $\rho(\boldsymbol{C}(\lambda)) = 1$.

Spectral radius

- The spectral radius $\rho(\boldsymbol{C}(\lambda))$ for different values of p and α; $\lambda = -10000$.

p \ α	1.0	0.9	0.8	0.7	0.6	0.5	0.3
7	0.8595	0.8007	0.7398	0.6758	0.6076	0.5338	0.3621
11	0.9677	0.8971	0.8255	0.7516	0.6738	0.5906	0.3992
15	1.0221	0.9454	0.8684	0.7894	0.7069	0.6189	0.4178
19	1.0544	0.9741	0.8938	0.8120	0.7266	0.6360	0.4290
25	1.0838	1.0003	0.9172	0.8327	0.7449	0.6517	0.4395

Computation of the history term u_a

- Assume the current interval is $[t_j, t_{j+1}]$, i.e., $a = t_j,\ b = t_{j+1}$, we can approximate $u_a(t)$ by the following:

$$u_a(t) \approx u_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_j} \frac{F_p(\tau, u(\tau))}{(t-\tau)^{1-\alpha}} d\tau = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{F_p(\tau, u(\tau))}{(t-\tau)^{1-\alpha}} d\tau$$
$$= u_0 + \frac{1}{\Gamma(\alpha)} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} (t-\tau)^{\alpha-1} \sum_{m=0}^{p} F_m^k \hat{h}_m(\tau) d\tau$$
$$= u_0 + \frac{\Delta T}{2\Gamma(\alpha)} \sum_{l=0}^{j-1} \int_{-1}^{1} (t - t_l - \frac{\Delta T}{2}(1+r))^{\alpha-1} \sum_{m=0}^{p} F_m^k h_m(r) dr, \tag{11}$$

- We expand F_p in terms of Jacobi polynomials

$$\sum_{m=0}^{p} F_m^k h_m(\tau(r)) = \sum_{n=0}^{p} \hat{F}_n^k J_n^{\tilde{a},\tilde{b}}(r),$$

and obtain $\{\hat{F}_n^k\}_{n=0}^{p}$ by using the forward discrete transform.

Computation of the history term u_a

- Then (11) becomes to

$$u_a(t) = u_0 + \frac{\Delta T}{2\Gamma(\alpha)} \sum_{l=0}^{j-1} \sum_{n=0}^{p} \left[\int_{-1}^{1} (t - t_l - \frac{\Delta T}{2}(1+r))^{\alpha-1} J_n^{\tilde{a},\tilde{b}}(r) dr \right] \hat{F}_n^k.$$

- The integrals in the above equations can be computed by a high accurate hybrid approach originally developed in [Chen, Xu and Heshthaven, 2015]. In particular, we use the three-term-recurrence relation when $j-l$ is small while use the Gauss quadrature when $j-l$ is large.

Error estimate

- Now we show the error estimate with respect to the degree of polynomial p for the solution of the SDC method at each subinterval.
- Let $u_p(t)$ be the limit approximation solution of the SDC procedure. Then, $u_p(t)$ satisfies:

$$u_p(t) = u_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \mathbb{I}_p F(\tau) d\tau, \quad t \in [a,b]. \tag{12}$$

- Using the above equation and equation (3), we obtain

$$|u - u_p| = \frac{1}{\Gamma(\alpha)} \left| \int_0^t (t-\tau)^{\alpha-1} (F - \mathbb{I}_p F)(\tau) d\tau \right| \leq$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} d\tau \cdot \|F - \mathbb{I}_p F\|_\infty \leq \frac{\Delta T^\alpha}{\Gamma(1+\alpha)} \|F - \mathbb{I}_p F\|_\infty .$$

Error estimate

- Then, we have the following estimate:

Theorem

Suppose $F(t) \in H^m([a,b])$, *let* $u(t)$ *and* $u_p(t)$ *be the solutions of* (3) *and* (12), *respectively, it holds that*

$$\|u(t) - u_p(t)\|_\infty \leq C\Delta T^\alpha p^{1/2-m}\|F(t)\|_m.$$

- *Remark*: The above estimate indicates that the convergence depends only on the regularity of F with respect to t.

Outline

Order reduction phenomenon

- If $\rho(\boldsymbol{C}(\boldsymbol{L})) > 1$, then the Neumann series expansion (8) is divergent.
- If $\rho(\boldsymbol{C}(\boldsymbol{L})) < 1$ but close to 1, then the SDC iterations still converges but very slowly. This is the so called order reduction phenomenon, which usually happens for stiff problems.

Accelerated SDC

- The numerical solution after k-th SDC iteration can be represented by the Neumann series expansion, i.e.,

$$\vec{u}^k - \vec{u}^0 = \vec{\delta}^0 + \boldsymbol{C}\vec{\delta}^0 + \boldsymbol{C}^2\vec{\delta}^0 + \cdots + \boldsymbol{C}^k\vec{\delta}^0.$$

- This encourages us to search for the optimal solution in the Krylov subspace $\boldsymbol{K}(\boldsymbol{C}, \vec{\delta}^0) = span\{\vec{\delta}^0, \boldsymbol{C}\vec{\delta}^0, \cdots, \boldsymbol{C}^k\vec{\delta}^0\}$ by using the GMRES or other Krylov subspace based iterative methods for the linear system (9).
- We use the GMRES algorithm with restart, denoted by *Re*, to accelerate the convergence of the original SDC method.
- For the nonlinear problem, we use the Newton iteration method by using the implicit scheme based SDC method or semi-implicit scheme based SDC method.

Outline

Numerical test

- Example 1: We begin by considering linear fractional ODEs, namely,

$$F(t, u(t)) = \lambda u(t) + f(t).$$

- We first present accuracy tests for both $\lambda = -1$ (non-stiff case) and $\lambda = -10000$ (stiff case) with a smooth F (with respect to t).
- In particular, let $F(t, u(t)) = \cos(t)$. In this case, we have $f(t) = \cos(t) - \lambda\, {}_0I_t^{\alpha} \cos(t),\ u_0 = 0$ and the exact solution is $u(t) = {}_0I_t^{\alpha} \cos(t)$.

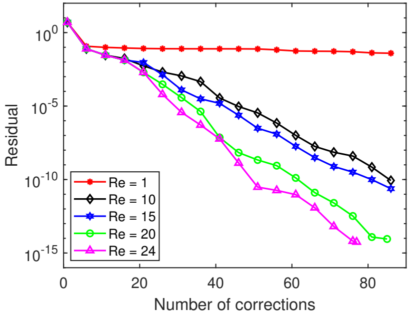
Numerical test: # of iteration

- Number of SDC iterations for the stiff case ($\lambda = -10000$). Here we set the max iteration number to be 1000.

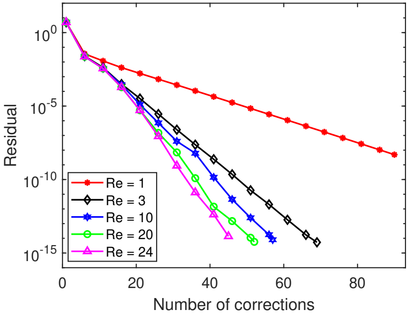
α \ p	7	9	11	13	15	17	19	21
0.7	81	95	107	116	125	133	139	145
0.9	141	200	276	376	518	732	1000	1000

SDC vs. GMRES-SDC

- Convergence of the residual of the implicit GMRES-SDC method with $p = 25$ and different values of α.



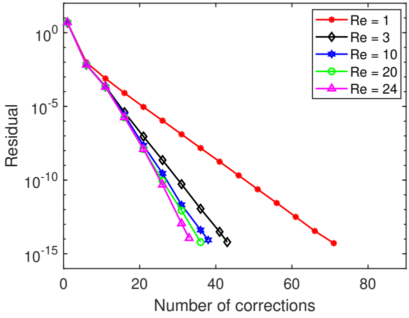
$\alpha = 1.0, p = 25$



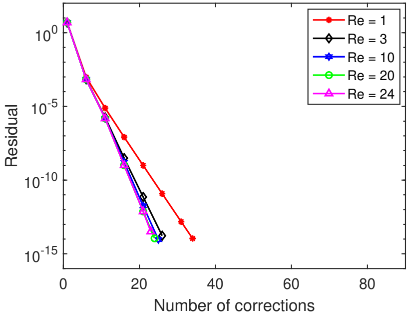
$\alpha = 0.7, p = 25$

SDC vs. GMRES-SDC

- Convergence of the residual of the implicit GMRES-SDC method with $p = 25$ and different values of α.



$\alpha = 0.5, p = 25$



$\alpha = 0.3, p = 25$

Explicit GMRES-SDC

- The L^∞-error and the relative residual of the GMRES-SDC iteration for $\lambda = -10$ by using the *explicit* GMRES-SDC with different pairs of (α, p).

(α, p)	*Re*	Residual	Error	(α, p)	*Re*	Residual	Error
	0	2.3e+09	1.8e+11		0	7.2e+16	4.7e+18
	1	2.6e-03	1.1e-02		1	1.1e-03	4.4e-03
(1.0, 15)	3	9.8e-05	4.0e-04	(0.9, 20)	5	2.7e-05	1.3e-04
	5	7.2e-05	3.9e-04		10	4.0e-07	2.2e-06
	15	1.4e-15	3.1e-11		20	4.6e-15	2.1e-14
	0	5.7e+23	3.5e+25		0	4.5e+59	2.2e+61
	1	3.8e-09	1.6e-08		1	7.2e-04	2.3e-03
(0.8, 40)	4	1.3e-12	6.5e-12	(0.7, 50)	5	1.6e-08	6.5e-08
	10	4.1e-14	2.7e-13		10	1.5e-10	9.3e-10
	40	2.3e-15	2.5e-14		50	6.0e-15	2.9e-14

Low regularity issue

- Example 1: Consider again

$$ {}_{0}^{C}D_t^{\alpha} u(t) = \lambda u(t). $$

- The exact solution is given by the Mittag-Leffler function

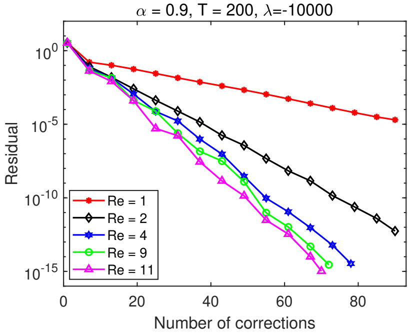
$$ u(t) = E_{\alpha}(-\lambda t^{\alpha}), \text{ where } E_{\alpha}(t) = \sum_{l=0}^{\infty} \frac{t^l}{\Gamma(\alpha l + 1)}. $$

- Low regularity at the origin.
- To resolve this issue, we introduce a geometric mesh near $t = 0$. In particular, we re-divide the first subdomain $[0, \Delta T] = [t_0, t_1]$ into a geometric mesh given by

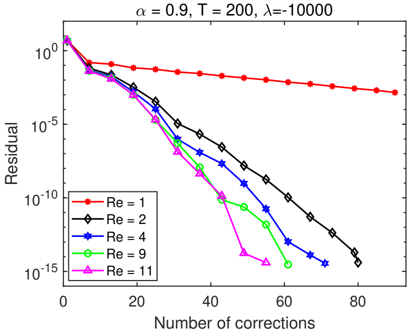
$$ t_{1,0} = t_0, \ t_{1,k} = t_0 + \Delta T \cdot r^{K1-k}, \ k = 1, 2, \ldots, K1. $$

SDC vs GMRES-SDC

- Convergence of the residual of the GMRES-SDC iteration for $\lambda = -10000$ with different values of the restart number *Re* and $\alpha = 0.9$.



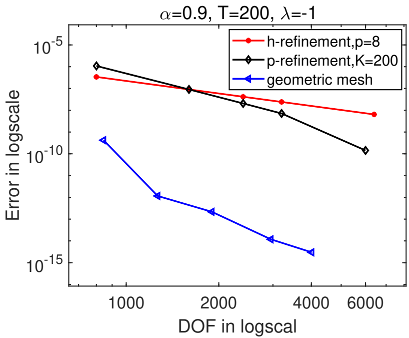
$p = 12$



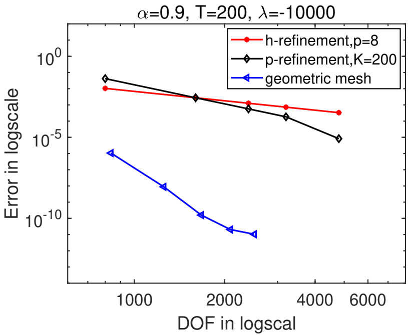
$p = 24$

Error vs DOF

- Comparison of the convergence of the L^∞-error by using $h-p$ refinement and the geometric mesh.

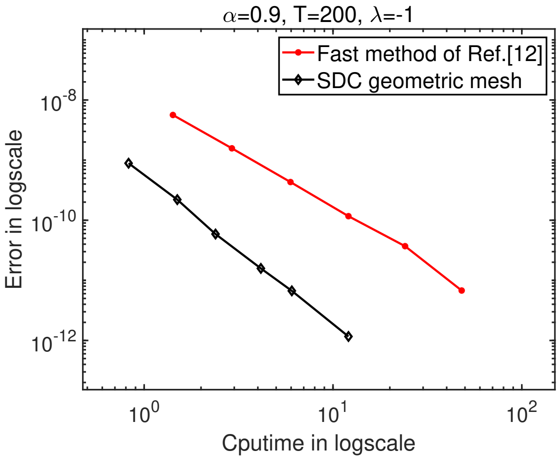


$\lambda = -1$



$\lambda = -10000$

L^∞-error vs. CPU time

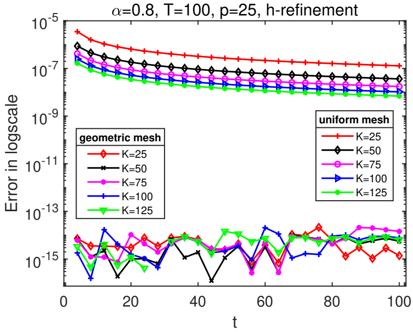


Nonlinear equation

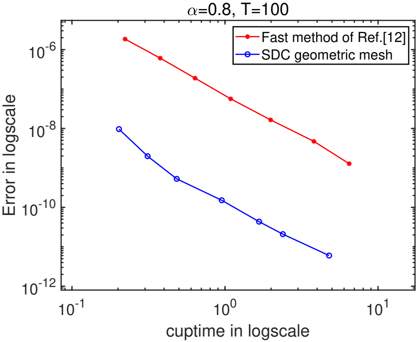
- Example 3: Consider

$$ {}_0^C D_t^\alpha u(t) = -u(t)^3. $$

- Geometric mesh is used to resolve the low regularity issue.



Errors vs t



Error vs CPU time

Nonlinear system

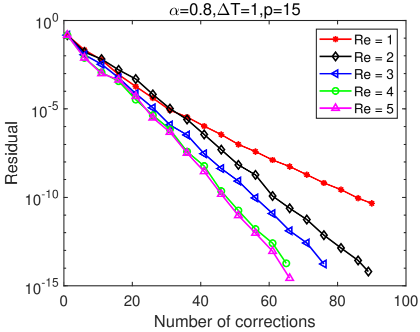
- Example 4: Consider the following fractional nonlinear equation considered in [baffet, 2019]:

$$({}_0^C D_t^\alpha)^2 x(t) - \epsilon(1 - x^2(t))\,{}_0^C D_t^\alpha x(t) + x(t) = 0, \;\; t \in [0, T],$$
$$x(0) = x_0, \;\; {}_0^C D_t^\alpha x(0) = y_0.$$

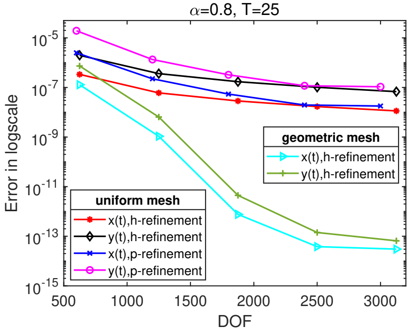
- We rewrite the above equation as a nonlinear fractional ODE system:

$$\begin{aligned}
&{}_0^C D_t^\alpha x(t) = y(t),\\
&{}_0^C D_t^\alpha y(t) = \epsilon(1 - x^2(t))y(t) - x(t),\\
&x(0) = x_0, \; y(0) = y_0.
\end{aligned}$$

Nonlinear system



Explicit GMRES-SDC



Error vs DOF

Fractional Phase Field model

- Consider the following two-dimensional fractional phase field model:

$$\begin{cases} {}_0^C D_t^\alpha u(x,t) + (-\Delta)^\beta(-\varepsilon^2 \Delta u(x,t) + f(u(x,t))) = 0, (x,t) \in \Omega \times (0,T], \\ u(x,0) = u_0(x), x \in \Omega, \end{cases}$$

  with periodic boundary conditions, ε is a positive constant, $x = (x_1, x_2),\ u_0 \in L_\infty(\Omega) \cap H^1_{per}(\Omega)$, and $f(u) = u^3 - u$.

- Weak form: Find $u \in H^{1+\beta}_{per}(\Omega)$, $\forall v \in H^{1+\beta}_{per}(\Omega)$, such that

$$({}_0^C D_t^\alpha u(t), v) + (\varepsilon^2 (-\Delta)^{\frac{(1+\beta)}{2}} u(t), (-\Delta)^{\frac{(1+\beta)}{2}} v) + (f(u(t)), (-\Delta)^\beta v) = 0.$$

Fractional Phase Field model

- Let $\mathbb{X}_N = \text{span}\{e^{ikx_1+ilx_2}, -N \le k, l \le N\}$. The Fourier-Galerkin approximation consists of finding $u_N \in \mathbb{X}_N$, $\forall v \in \mathbb{X}_N$ such that

$$({}_0^C D_t^\alpha u_N(t), v) + (\varepsilon^2(-\Delta)^{\frac{(1+\beta)}{2}} u_N(t), (-\Delta)^{\frac{(1+\beta)}{2}} v) + (f(u_N(t)), (-\Delta)^\beta v) = 0$$

  with initial condition $u_N(0) = \Pi_N u_0$.

- Let $u_N(x,t) = \sum_{k,l=-N/2}^{N/2} \hat{u}_{kl}(t) e^{ikx_1+ilx_2}$, and take $v = e^{ipx_1+iqx_2}, p, q = -N/2, \ldots, N/2$. We arrive at the following fractional ODE system for $k, l = -N/2, \ldots, N/2$

$${}_0^C D_t^\alpha \hat{u}_{kl}(t) + \varepsilon^2(k^2+l^2)^{1+\beta} \hat{u}_{kl}(t) + (k^2+l^2)^\beta \hat{f}_{kl}(t) = 0,$$

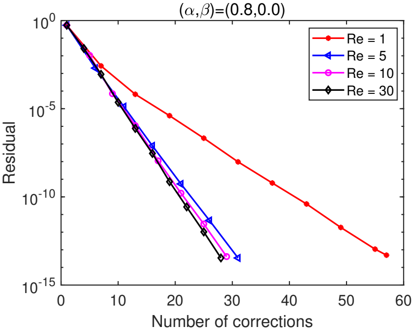
  where $\hat{f}_{kl}(t), k, l = -N/2, \ldots, N/2$, are the Fourier coefficients of $f(u_N)$.

Fractional Phase Field model

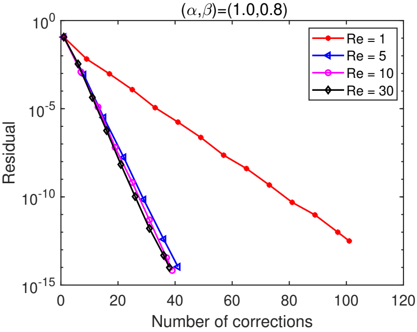
- Example 5: consider the following initial condition given in [Tang, Yu and Zhou, 2019, Section 5.1].

$$u_0(x_1, x_2) = \tanh\left(\frac{1}{\sqrt{2}\varepsilon}\left(\sqrt{x_1^2 + x_2^2} - \frac{1}{4} + \frac{1 - \cos(4\arctan\frac{x_2}{x_1})}{16}\right)\right).$$

Fractional Phase Field model

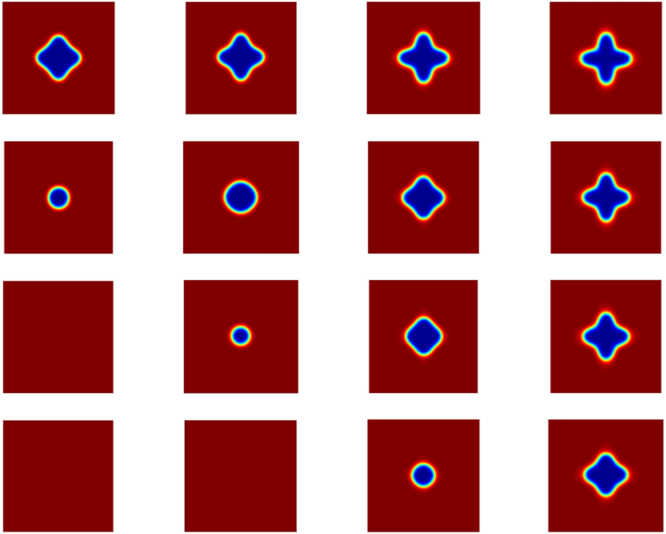


Allen-Cahn Equation



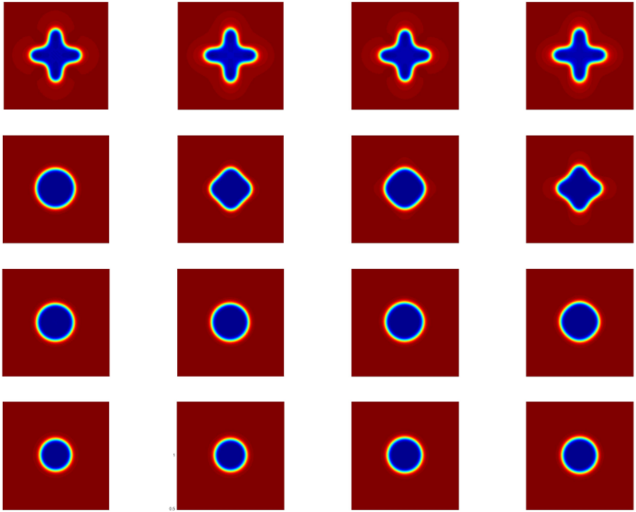
Cahn-Hilliard Equation

Fractional Allen-Cahn Equation



(a)$(\alpha, \beta) = (1.0, 0.0)$ (b)$(\alpha, \beta) = (0.8, 0.0)$ (c)$(\alpha, \beta) = (0.5, 0.0)$ (d)$(\alpha, \beta) = (0.3, 0.0)$

Fractional Cahn-Hilliard Equation



(a)$(\alpha, \beta) = (1.0, 1.0)$ (b)$(\alpha, \beta) = (1.0, 0.8)$ (c)$(\alpha, \beta) = (0.8, 1.0)$ (d)$(\alpha, \beta) = (0.8, 0.8)$

Outline

Conclusion

- We extended the idea of [Huang, Jia and Minion, 2006] for integer case to fractional fractional case and overcame order reduction by accelerating the convergence of the SDC iteration with the GMRES algorithm.
- We numerically analyzed the accelerated SDC method by considering both stiff and non-stiff linear problems showing that the accelerated SDC method is more efficient than the original SDC method.
- We employed the present accelerated SDC method to nonlinear fractional ODEs and fractional phase field models and demonstrated the effectiveness of the accelerated SDC method and the use of the geometric mesh near the origin for problems with singular solutions.
- Furthermore, the use of geometric mesh is advantageous for the long time evolution, since we only need a slight increase in the number of degrees of freedom.

Thank you!

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