Numerical stability of Grünwald-Letnikov method for fractional delay differential equations

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Outline

1. Numerical stability and stability region
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   - F-ODEs
     - Mittag-Leffler stability
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     - Numerical Mittag-Leffler stability

2. Numerical stability and stability region for F-DDEs
   - F-DDEs
   - Numerical stability region
   - Numerical Mittag-Leffler stability

3. Numerical experiments

Outline

Test equation for ODEs: $y'(t) = \lambda y, \quad \lambda \in \mathbb{C}$.

- Stability and stability region:

$$S_* = \{\lambda \in \mathbb{C} | y(t) \to 0\} \text{ as } t \to +\infty$$
$$= \{\lambda \in \mathbb{C} | Re(\lambda) < 0\}.$$

Exponential stable: If $\lambda \in S_*$, then $|y(t)| \leq Ce^{Re(\lambda)}t$.

- Numerical stability and stability region: [1]

$$S_*^h = \{z = \lambda h \in \mathbb{C} | y_n \to 0\} \text{ as } t_n \to +\infty$$

Exponential stable: If $z = \lambda \in S_*^h$, then $|y_n| \leq Ce^{Re(\lambda)} t_n$.

- *A*-stable: If $S_*^h \supseteq S_* = \mathbb{C}^-$

[1] Numerical solutions y_n approximate $y(t_n)$ at $t_n = nh$. Linear multistep methods or Runge-Kutta methods. Back ward Euler: $y_n = y_{n-1} + \lambda h y_n$.

Outline

Mittag-Leffler stability

Definition

The trivial solution of F-ODEs $\mathcal{D}_t^{\alpha} y(t) = f(t, y)$ is said to be Mittag-Leffler stable if there exist positive constants β, δ and M independent of t such that

$$\sup_{t \geq 0} t^{\beta} \|y(t)\| \leq M \text{ for any } \|y_0\| \leq \delta. \tag{1}$$

- **Remark:** The inequality (1) can be replaced by

$$\|y(t)\| \leq V(y_0) E_{\alpha}(-L t^{\alpha}), \quad t > 0,$$

where $L > 0$ and the function $V(y)$ is locally Lipschitz continuous and satisfies that $V(0) = 0$ and $V(y) \geq 0$.

F-ODEs with or without small perturbation

Consider the F-ODEs

$$\mathcal{D}_t^{\alpha} y(t) = Ay + f(t, y), \quad t > 0 \tag{2}$$

for $y \in \mathbb{R}^d$ satisfying the initial value $y(0) = y_0$, where $A \in \mathbb{R}^{d\times d}$, $f : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ is continuous.

Lemma (Matignon, 1996)

Consider the F-ODEs (2) *with* $f \equiv 0$*. Then it holds that*
(i) The solution to (2) *is asymptotically stable if and only iff that*

$$\lambda_A \in \Lambda_\alpha^s := \left\{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| > \frac{\alpha\pi}{2} \right\}. \tag{3}$$

(ii) If all eigenvalues $\lambda_A \in \Lambda_\alpha^s$*, the solution to* (2) *is Mittag-Leffler stable, i.e.,* $\|y(t)\| = O(t^{-\alpha})$ *as* $t \to \infty$*.*

F-ODEs with or without small perturbation

Lemma (Cong et.al., 2018, 2019)

Assume that $\lambda_A \in \Lambda_\alpha^s$. (a) Assume $f(t,y)$ satisfies that

$$f(t,0)=0, \qquad \|f(t,x)-f(t,y)\| \leq L(t)\|x-y\|, \tag{4}$$

where $L(t):[0,\infty)\to\mathbb{R}_+$ is a continuous Lipschitz function and satisfies one of the three conditions:

$$\begin{aligned} &(i) \;\; q_1 := \sup_{t\geq 0}\int_0^t (t-s)^{\alpha-1}\|E_{\alpha,\alpha}((t-s)^\alpha A)\|L(s)ds < 1, \\ &(ii) \;\; \sup_{t\geq 0} L(t) < q_2 := \frac{1}{2\int_0^t t^{\alpha-1}\|E_{\alpha,\alpha}(t^\alpha A)\|dt}, \\ &(iii) \;\; \lim_{t\to\infty} L(t) = 0. \end{aligned} \tag{5}$$

Then the trivial solutions to F-ODEs (2) *is asymptotical stable.*

Nonlinear F-ODEs

Lemma (Vergara and Zacher, 2015.)

Let $y(t) \in H^1_{1,loc}(\mathbb{R}^+)$ be the solution of F-ODE:

$$\mathcal{D}_t^\alpha y(t) = -\lambda y(t)^\gamma \ \text{ for } t > 0, \ \text{ with } y(0) = y_0 > 0,$$

where λ and γ are positive parameters. Then there exist positive constants C_1, C_2, which are independent of t, such that

$$\frac{C_1}{1 + t^{\alpha/\gamma}} \le y(t) \le \frac{C_2}{1 + t^{\alpha/\gamma}} \ \text{ for } t \ge 0.$$

Convolution quadrature (CQ): C. Lubich, 1980s.

Numerical approximating of RL fractional integral

$$\mathcal{J}_t^\alpha y(t) = \int_0^t k_\alpha(s) y(t-s) ds = \int_0^t \left(\frac{1}{2\pi i} \int_{\mathcal{C}} e^{s\lambda} K_\alpha(\lambda) \right) y(t-s) d\lambda ds$$
$$= \frac{1}{2\pi i} \int_{\mathcal{C}} \left(\int_0^t e^{s\lambda} y(t-s) \right) K_\alpha(\lambda) ds d\lambda,$$

where $K_\alpha(\lambda) = \lambda^{-\alpha}$ is the Laplace transform of kernel $k_\alpha(t)$.

- **Key Observation:**

$u(t) := \int_0^t e^{s\lambda} y(t-s) ds$ solves the ODE

$$u'(t) = \lambda u(t) + y(t), t > 0$$

with the initial value $u(0) = 0$.

Convolution quadrature (CQ): C. Lubich, 1980s.

Applying k-step LMMs $\rho(z) = \sum_{j=0}^{k} \alpha_j z^j$ and $\sigma(z) = \sum_{j=0}^{k} \beta_j z^j$ and putting $F_u(z) = \sum_{j=0}^{\infty} u_j z^j$ and $F_y(z) = \sum_{j=0}^{\infty} y_j z^j$,

$$u_n = \left[\left(\frac{\delta(z)}{h} - \lambda \right)^{-1} F_y(z) \right]_n,$$

$$\delta(z) = \frac{z^k \rho(z^{-1})}{z^k \sigma(z^{-1})} = \frac{\alpha_0 z^k + \cdots + \alpha_{k-1} z + \alpha_k}{\beta_0 z^k + \cdots + \beta_{k-1} z + \beta_k}.$$

Convolution quadrature (CQ): C. Lubich, 1980s.

Applying the Cauchy integral formula, we arrive at

$$\begin{aligned}
\mathcal{J}_{t_n}^{\alpha} y(t_n) &= \frac{1}{2\pi i}\int_{\mathcal{C}} \left(\int_0^{t_n} e^{t_n\lambda} y(t_n - s)ds \right) K_\alpha(\lambda) d\lambda \\
&\approx \frac{1}{2\pi i}\int_{\mathcal{C}} \left[\left(\frac{\delta(z)}{h} - \lambda \right)^{-1} F_y(z) \right]_n K_\alpha(\lambda) d\lambda \\
&= \left[K_\alpha\left(\frac{\delta(z)}{h}\right) F_y(z) \right]_n = \left[\left(\frac{\delta(z)}{h}\right)^{-\alpha} F_y(z) \right]_n := h^\alpha \sum_{j=0}^{n} \omega_{n-j} y_j
\end{aligned}$$

for $n \geq 1$, where we have written

$$F_\omega(z) := (\delta(z))^{-\alpha} = \sum_{j=0}^{\infty} \omega_j z^j. \tag{6}$$

Convolution quadrature (CQ): C. Lubich, 1980s.

Through a small correction, we obtain the F-LMMs for F-ODEs:

$$y_n = y_0 + h^\alpha \sum_{j=1}^{n} \omega_{n-j} \left(Ay_j + f(t_j, y_j) \right), \quad n \geq 1, \tag{7}$$

where ω_j are still given by (6).

- **Remark:**

The numerical method is now also consistent at $n = 0$ by noting the sum is zero when the upper index is smaller than the lower index, and also fully consistent with the scheme derived by the convolution inverse.

Linear stability analysis, C.Lubich, 1985.

Definition

Strong $A(\beta)$-stability of a LMM defined by a generating polynomial $F_{\overline{\omega}}(z) = F_{\overline{\omega}(\rho,\sigma)}(z) = \delta(z)^{-1}$ for the classical ODE, with order $p \geq 1$:

- $\delta(z)$ is analytic, with no zeros in a neighborhood of the unit disk $|z| \leq 1$ except $z = 1$;
- $|\arg \delta(z)| \leq \pi - \beta$ for $|z| < 1$;
- $\frac{1}{h}\delta\left(\mathrm{e}^{-h}\right) = 1 + O\left(h^{p}\right),$ with $p \geq 1$.

Linear stability analysis and stability region, C.Lubich, 1985.

Applying F-LMMs to the fractional test equation $\mathcal{D}_t^{\alpha} y(t) = \lambda y$ gives

$$y_n = y_0 + \lambda h^{\alpha}[\omega * (y - y_0\delta_d)]_n, \quad n \geq 0,$$

Lemma (Lubich, 1985)

Consider a classical LMM defined by a generating polynomial $F_{\varpi}(z) = \delta(z)^{-1}$ satisfies the stability conditions. Let $\mathcal{S}_h$ and $\mathcal{S}_h^{\alpha}$ be the stability regions of the standard LMM and its corresponding F-LMM defined by $F_{\omega}(z) = (F_{\varpi}(z))^{\alpha} = \delta(z)^{-\alpha}$ respectively. Then it holds that
(i) $\mathcal{S}_h^{\alpha} = \mathbb{C} \setminus \{1/F_{\omega}(z) : |z| \leq 1\}$;
(ii) $\left(\mathbb{C} \setminus \mathcal{S}_h^{\alpha}\right) = (\mathbb{C} \setminus \mathcal{S}_h)^{\alpha}$;
(iii) LMM is A-stable if and only if the F-LMM is A-stable;
(iv) with $\pi - \varphi = \alpha(\pi - \psi)$, LMM is A($\varphi$)-stable if and only if the F-LMM is A(ψ)-stable.

Mittag-Leffler stability without perturbation

Theorem (W and Zou, 2021)

Consider the homogenous liner F-ODE $\mathcal{D}_t^\alpha y(t) = Ay$ (i.e., $f \equiv 0$) and assume that all the eigenvalues of A satisfy that $\lambda_A \in \Lambda_\alpha^s$. Then the numerical solutions obtained from the strong A-stable F-LMMs or $\mathcal{L}1$ scheme are Mittag-Leffler stable, i.e.,

$$\|y_n\| = O(t_n^{-\alpha}) \text{ as } n \to \infty.$$

Mittag-Leffler stability with perturbation

Theorem (W and Zou, 2021)

For the F-ODEs model (2), we assume $\lambda_A \in \Lambda_\alpha^s$, and that f is continuous, $f(t,0) = 0$, and further satisfies

$$\|f(t,x(t)) - f(t,y(t))\| \leq L(t)\|x(t) - y(t)\|, \quad \forall\, t \geq 0, \;\; x,y \in \mathbb{R}^d,$$

where $L : [0,\infty) \to \mathbb{R}_+$ is a continuous Lipschitz function. Letting $\mathcal{L}_0 = \sup_{t\geq 0} L(t)$ then there exists constant $h_0 > 0$ such that for any $0 < h < h_0$, the trivial solutions are numerically Mittag-Leffler stable, i.e., $\|y_n\| = O(t_n^{-\alpha})$ as $n \to \infty$, provided that

$$1 - \|D_0\|\mathcal{L}_0 > 0, \quad \frac{1}{1 - \|D_0\|\mathcal{L}_0}\left(\lim_{n\to\infty}\sum_{k=0}^{n-1}\|D_{n-k}\|L(t_k)\right) \leq \rho_0 < 1.$$

Fractional comparison principle

Lemma (Discrete fractional comparison principle, Li and W, 2019.)

Let $\mathcal{D}_h^\alpha$ be the $\mathcal{CM}$-preserving discrete operator. Let $f(\cdot)$ be nondecreasing. Suppose that the sequences $\{u_j\}_{j=0}^\infty, \{y_j\}_{j=0}^\infty, \{v_j\}_{j=0}^\infty$ satisfy $u_0 \le y_0 \le v_0$ and

$$\mathcal{D}_h^\alpha(u_n)+f(u_n) \le 0, \quad \mathcal{D}_h^\alpha(y_n)+f(y_n) = 0, \quad 0 \le \mathcal{D}_h^\alpha(v_n)+f(v_n) \ \ \text{for } n \ge 1.$$

Then $u_n \le y_n \le v_n$ for $n = 0, 1, 2, \ldots$.

In this Lemma, we say that $\{u_n\}$ is a *discrete subsolution for* $\{y_n\}$ and $\{v_n\}$ is a *discrete supersolution for* $\{y_n\}$.

- **Remark:**

Typical examples of $\mathcal{CM}$-preserving schemes: Grünwald-Letnikov method and L1 scheme.

Nonlinear F-ODEs

Theorem (W and Stynes, 2022.)

For the model equation, consider the numerical scheme

$$\mathcal{D}_h^\alpha(y_n) = \frac{1}{h^\alpha}\left(\sum_{k=1}^{n}\omega_{n-k}y_k + \delta_n y_0\right) = -\lambda y_n^\gamma \quad \text{for } n \geq 1, \text{ with } y_0 > 0.$$

where $\lambda, \gamma > 0$ and $\mathcal{D}_h^\alpha$ is $\mathcal{CM}$-preserving. Then there exists $h_0 > 0$ such that $0 < h \leq h_0$, the solution $\{y_n\}$ satisfies

$$\frac{C_5}{1 + t_n^{\alpha/\gamma}} \leq y_n \leq \frac{C_6}{1 + t_n^{\alpha/\gamma}} \quad \text{for } n = 0, 1, 2, \ldots$$

where the positive constants C_5, C_6 are independent of n and h.

Outline

F-DDEs

$$\begin{aligned} &\mathcal{D}_c^{\alpha} y(t) = ay(t) + by(t-\tau), \quad t > 0, \\ &y(t) = \varphi(t), \quad -\tau \le t \le 0, \end{aligned} \tag{8}$$

where $a, b \in \mathbb{R}$, the delay $\tau > 0$ is a fixed constant. We first observe that if we redefine

$$\tilde{a} = a\tau^{\alpha}, \;\; \tilde{b} = b\tau^{\alpha}, \;\; \tilde{t} = t/\tau, \;\; \tilde{y}(\tilde{t}) = y(t), \;\; \tilde{\varphi}(\tilde{t}) = \varphi(t), \tag{9}$$

then the form of (8) stays unchanged with $\tau = 1$. Hence, dropping the tildes, we will study the test model (8) with $\tau = 1$:

$$\begin{aligned} &\mathcal{D}_c^{\alpha} y(t) = ay(t) + by(t-1), \quad t > 0, \\ &y(t) = \varphi(t), \quad -1 \le t \le 0. \end{aligned} \tag{10}$$

F-DDEs

Definition

Let $\alpha \in (0, 1)$ and consider parameters $a, b \in \mathbb{R}$.

1. The asymptotic stability region in the (a, b)-plane is defined by $\mathcal{S}_* = \{(a, b) : |y(t)| \to 0 \text{ as } t \to +\infty\}$ for all the initial functions ϕ.
2. The solution is called Mittag-Leffler stable if $|y(t)| \leq C_\alpha t^{-\alpha}$ as $t \to +\infty$, where the constant $C_\alpha > 0$ is independent of t.

F-DDEs

Lemma (J. Cermak et.al., 2016, 2017.)

Let $\alpha \in (0,1)$*, and* $a, b \in \mathbb{R}$*. The zero solution of* (10) *is asymptotically stable if and only if* (a, b) *is an interior point of the region* $\mathcal{S}_*$*, bounded by the line* $a + b = 0$ *from above and by the following parametric curve* Γ *from below*

$$\Gamma : a = a(\theta) := \frac{\theta^\alpha \sin\left(\theta + \frac{\alpha\pi}{2}\right)}{\sin(\theta)}, b = b(\theta) := -\frac{\theta^\alpha \sin\left(\frac{\alpha\pi}{2}\right)}{\sin(\theta)}, \tag{11}$$

where $\theta \in ((1-\alpha)\pi, \pi)$*.*

In the asymptotic stability region, the solution is also Mittag-Leffler stable i.e., $|y(t)| \leq C_\alpha t^{-\alpha}$ *as* $t \to \infty$*.*

F-DDEs

The stability region $\mathcal{S}_*$ has a vertex $P = P(a, b)$, where $a = -b = \frac{[((1-\alpha)\pi]^\alpha \sin(\frac{\alpha\pi}{2})}{\sin(\alpha\pi)} > 0$ and $\mathcal{S}_* = \mathcal{R}_1 \cup \mathcal{R}_2$, where

$$\begin{aligned} &(1) \quad \mathcal{R}_1 := \{(a, b) : a \le b < -a, a \le 0\}, \\ &(2) \quad \mathcal{R}_2 := (a, b) : |a| + b < 0 \\ &\qquad \text{and } 1 < \frac{(1-\alpha)\pi/2 + \arccos[(-a/b)\sin(\alpha\pi/2)]}{\left[a\cos(\alpha\pi/2) + \sqrt{b^2 - a^2\sin^2(\alpha\pi/2)}\right]^{1/\alpha}}. \end{aligned} \tag{12}$$

Stable region

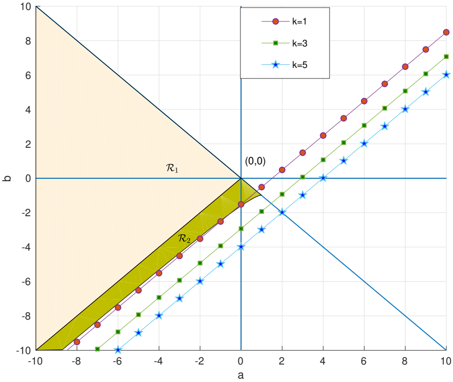

Figure: Position relationship between the lines $a - b = (2k)^{\alpha}$ with $k = 1, 3, 5$ and stability region $\mathcal{S}_*$ for $\tau = 1$ and $\alpha = 0.6$.

Outline

GL scheme for F-DDEs: $(1-z)^{\alpha} = \sum_{j=0}^{\infty} \omega_j z^j$

Let $h = 1/k$ for $k \in \mathbb{N}^+$. The numerical scheme:

$$\mathcal{D}_h^{\alpha} y_n := \frac{1}{h^{\alpha}} \sum_{j=0}^{n} \omega_{n-j}(y_j - y_0) = a y_n + b y_{n-k}, \quad n \geq 1. \tag{13}$$

Definition

1. For any $k \geq 1$, the numerical stability region $\mathcal{S}_k$ is the set of pairs (a, b) such that $|y_n| \to 0$ as $n \to \infty$ for all initial functions φ.
2. The $\tau(0)$-stability region for F-DDEs is defined by

$$\mathcal{S}_{\tau(0)} = \bigcap_{k \geq 1} \mathcal{S}_k. \tag{14}$$

   The numerical method is called $\tau(0)$-stable if $\mathcal{S}_* \subset \mathcal{S}_{\tau(0)}$.
3. If $|y_n| \leq C_{\alpha} t_n^{-\alpha}$ as $n \to \infty$, it is Mittag-Leffler stable.

Discrete Laplace transform

Definition

Consider the sequence $f = (f_0, f_1, f_2, ...)$ defined on t_n. Discrete Laplace transform:

$$\mathscr{L}_h\{f\}(s) := h \sum_{j=0}^{\infty} f_j (1 - hs)^j, \quad s \in \mathbb{C}, \tag{15}$$

where the complex number s is taken such that the series is convergent. If the series converges at some $s_* \neq h^{-1}$, then it will also be convergent on the disk $D(h^{-1}, r) := \{s \in \mathbb{C} : |s - h^{-1}| < r\}$ where $r = |s_* - h^{-1}|$.

Discrete Laplace transform

Lemma

Define the weighted discrete convolution $(\cdot * \cdot)_h$ *as*

$$[(f * g)_h]_n := h \sum_{j=0}^{n} f_j g_{n-j}.$$

Let the functions f and g such that $\mathscr{L}_h\{f\}$ *and* $\mathscr{L}_h\{g\}$ *converge on* $D(h^{-1}, r_f)$ *and* $D(h^{-1}, r_g)$*, respectively. Then one has that*

1. (i) $\mathscr{L}_h\{(f * g)_h\}(s) = \mathscr{L}_h\{f\}(s) \cdot \mathscr{L}_h\{g\}(s)$ *on* $D(h^{-1}, r)$*, where* $r = \min\{r_f, r_g\}$.
2. (ii) $\mathscr{L}_h\{\nabla_h f\}(s) = s\mathscr{L}_h\{f\}(s) - f_0$ *on* $D(h^{-1}, r)$.
3. (iii) $\mathscr{L}_h\{\mathcal{D}_h^{\alpha} f\}(s) = s^{\alpha}\mathscr{L}_h\{f\}(s) - s^{\alpha-1} f_0$.
4. (iv) $\mathscr{L}_h\{f_{d_k}\}(z) = (1 - hs)^k \mathscr{L}_h\{f\}(s) + h\sum_{j=1}^{k} f_{-j}(1 - hs)^{k-j}$*, where the k-step delay function given by* $(f_{d_k})_n = f_{n-k}$ *for* $n \geq 0$.

Generating function

Definition

The generating function of a sequence $f = (f_0, f_1, \ldots)$ defined by

$$\mathscr{F}_f(z) = \sum_{n=0}^{\infty} f_n z^n, \quad z \in \mathbb{C}. \tag{16}$$

The discrete Laplace transform is related to the generating function by

$$\mathscr{L}_h\{f\}(s) = h\mathscr{F}_f(1 - hs). \tag{17}$$

If f and g are two scalar sequences with $f_n, g_n \in \mathbb{C}$, we define the discrete convolution $f * g = w, \quad w_n = \sum_{j=0}^{n} f_{n-j} g_j$. It is straightforward to verify that $\mathscr{F}_{f*g}(z) = \mathscr{F}_f(z) \cdot \mathscr{F}_g(z)$.

Expression formula of numerical solution

- **By discrete Laplace transform:**

$$\mathscr{L}_h\{y\}(s) = \left(s^\alpha - a - b(1-hs)^k\right)^{-1} \cdot \left(s^{\alpha-1}y_0 - ahy_0 + bh\sum_{j=1}^{k-1} y_{-j}(1-hs)^{k-j}\right). \tag{18}$$

Introduce

$$Q(s) := s^\alpha - a - b(1-hs)^k \tag{19}$$

with $h > 0$ and $k \in \mathbb{N}^+$, which is defined to be the characteristic polynomial.

Expression formula of numerical solution

- **By generating function:**

$$\mathscr{F}_y(z)$$
$$= \left(1 - h^\alpha \left(a + bz^k\right) \mathscr{F}_\mu(z)\right)^{-1} \left(y_0(1-z)^{-1} + h^\alpha \left(bg(z) - ay_0\right) \mathscr{F}_\mu(z)\right)$$
$$= \left((1-z)^\alpha - h^\alpha \left(a + bz^k\right)\right)^{-1} \left(y_0(1-z)^{\alpha-1} + h^\alpha \left(bg(z) - ay_0\right)\right),$$

where $F_\mu(z) = (1-z)^{-\alpha}$ for the GL scheme and $g(z) = \sum_{\ell=1}^{k-1} y_{\ell-k} z^\ell$.

Introduce

$$P(z) = (1-z)^\alpha - h^\alpha \left(a + bz^k\right).$$

We easily see that

$$P(1-hs) = h^\alpha Q(s). \tag{20}$$

Boundary locus of the numerical stability region

- **Basic idea:**

We investigate the boundary locus of the numerical stability region $\mathcal{S}_k$ by finding the parameters (a, b) such that $Q(s)$ has a root on the circle $\partial D(h^{-1}, h^{-1})$.

Lemma

Let $\alpha \in (0, 1)$, $a, b \in \mathbb{R}$, $\tau > 0$ and $k \in \mathbb{N}^+$.

(i) *If $a + b \geq 0$, then the equation $Q(z) = 0$ has at least one nonnegative real root on the disc $D(h^{-1}, h^{-1})$.*

(ii) *If z is a root of $Q(z)$, then its complex conjugate $\bar{z}$ is also a root of $Q(z)$, i.e., $Q(\bar{z}) = 0$.*

Boundary locus of the numerical stability region

- **Basic idea:**

Take $s \in \partial D(h^{-1}, h^{-1})$, which can be parameterized as $s = 2h^{-1}\cos(\phi)e^{i\phi} = h^{-1}(1 + e^{i2\phi})$, where $\phi \in [-\pi/2, \pi/2]$. Then,

$$\begin{aligned} Q(s) = \tilde{Q}(\phi) &= 2^{\alpha} h^{-\alpha} \cos^{\alpha}(\phi)\left(\cos(\alpha\phi) + i\sin(\alpha\phi)\right) \\ &\quad - a - b(-1)^k \left(\cos(2k\phi) + i\sin(2k\phi)\right). \end{aligned}$$

By the above Lemma (ii), $\tilde{Q}(\phi)$ is an even function. Hence, we need to find parameters (a, b) such that for some $\phi \in [0, \frac{\pi}{2}]$.

$$\begin{cases} 2^{\alpha} h^{-\alpha} \cos^{\alpha}(\phi)\cos(\alpha\phi) = a + b(-1)^k \cos(2k\phi), \\ 2^{\alpha} h^{-\alpha} \cos^{\alpha}(\phi)\sin(\alpha\phi) = b(-1)^k \sin(2k\phi). \end{cases} \tag{21}$$

Boundary locus of the numerical stability region

- **Basic idea:**

Let's discuss the system of equations (21) in three different cases.
Case (I): If $\phi = 0$, then $a + (-1)^k b = (2/h)^\alpha$.
Case (II): If $\phi = \pi/2$, we get the curve $a + b = 0$.
Case (III): Otherwise, $\sin(2k\phi)$ can never be zero. By solving the equation (21) and setting $\theta = k\pi - 2k\phi \in [0, k\pi]$, one has

$$\begin{aligned} a &= \frac{2^\alpha h^{-\alpha} \sin^\alpha\left(\frac{\theta}{2k}\right) \sin(\theta + \alpha(\pi/2 - \theta/2k))}{\sin(\theta)}, \\ b &= -2^\alpha h^{-\alpha} \sin^\alpha\left(\frac{\theta}{2k}\right) \frac{\sin(\alpha\pi/2 - \alpha\theta/(2k))}{\sin(\theta)}. \end{aligned} \tag{22}$$

These curves obtained are essentially the boundary locus!

Boundary locus of the numerical stability region

Theorem

Fix k to be a positive integer. When $k = 1$, the numerical stability region $\mathcal{S}_k$ in the (a, b)-plane lies in the region between $a + b = 0$ and $a - b = 2^\alpha$. When $k \geq 2$, the numerical stability region $\mathcal{S}_k$ in the (a, b)-plane lies between $a + b = 0$ and the curve Γ_0:

$$\Gamma_0 : \quad \begin{cases} a = a_k(\theta) = 2^\alpha k^\alpha \sin^\alpha\left(\dfrac{\theta}{2k}\right) \dfrac{\sin(\theta + \alpha(\pi/2 - \theta/2k))}{\sin(\theta)}, \\ b = b_k(\theta) = -2^\alpha k^\alpha \sin^\alpha\left(\dfrac{\theta}{2k}\right) \dfrac{\sin(\alpha\pi/2 - \alpha\theta/(2k))}{\sin(\theta)}, \end{cases} \tag{23}$$

where $\theta \in \left(\frac{1-\alpha}{1-\alpha/k}\pi, \pi\right)$.

No $\tau(0)$-stability

We define the lower part of the numerically stable region as $\mathcal{R}_2^h$ with lower boundary curve as Γ_0, that is, the original lower boundary curve Γ is replaced by Γ_0.

Theorem

Assume that $\alpha \in (0, 1)$, $\tau = 1$ and $h = 1/k$. For any $k \geq 1$, $\mathcal{R}_1$ always stays in the numerical stability region and there is a portion of $\mathcal{R}_2$ that is outside the numerical stability region $\mathcal{R}_2^h$.
Consequently, the numerical method is never absolutely stable for $\alpha \in (0, 1)$.

No $\tau(0)$-stability

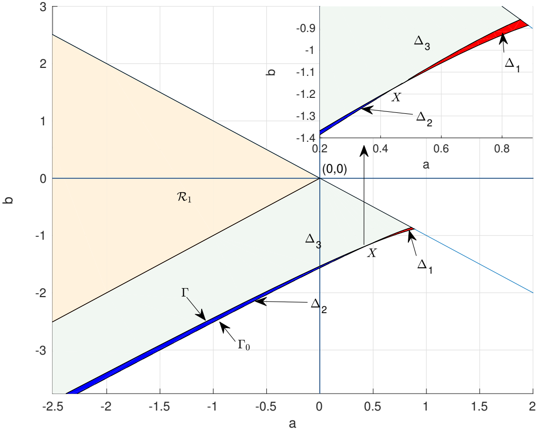

Figure: Comparison of numerical stability region $\mathcal{S}_h$ and continuous stability region $\mathcal{S}_*$ for $\tau = 1$, $h = 0.2$, $k = 5$ and $\alpha = 0.5$.

Outline

Mittag-Leffler numerical stability

Theorem

Let $\alpha \in (0, 1)$*,* $a, b \in \mathbb{R}$*,* $k \in \mathbb{N}^+$ *such that* $h = 1/k$*. Then the numerical solutions for the scheme based on the GL method is Mittag-Leffler stable if* $(a, b) \in \mathcal{S}_k$*. More specifically, when* $(a, b) \in \mathcal{S}_k$*, the numerical solution has the following polynomial decay rate asymptotically*

$$y_n \sim -\frac{y_0}{\Gamma(1-\alpha)\,(a+b)} t_n^{-\alpha} = O(t_n^{-\alpha}) \text{ as } n \to \infty.$$

Linear F-ODE

In the simulation, we take the initial functions $\phi_1(t) = 0.4$, $\phi_2(t) = -0.1t - 0.2$ and $\phi_3(t) = 0.3\sin(6t)$, respectively.

$$p_\alpha(t_n) = -\frac{\ln(\|y_n\|/\|y_{n-1}\|)}{\ln(t_n/t_{n-1})}, \quad t_n > 1. \tag{24}$$

Table: Observed p_α with $\tau = 1, h = 0.1$ and $a = -3, b = 1$ for initial function $\phi_1(t)$

t_n	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
100	0.0896	0.2918	0.5014	0.7069	0.9080
200	0.0902	0.2931	0.5007	0.7040	0.9041
300	0.0905	0.2938	0.5005	0.7029	0.9028
400	0.0907	0.2943	0.5003	0.7023	0.9022
500	0.0909	0.2946	0.5003	0.7019	0.9017

References:

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Thanks for your attention!