High-order two-grid compact difference algorithm for the nonlinear time-fractional biharmonic problems

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In this talk, we consider the following nonlinear time-fractional biharmonic equations (tFBEs):

$$ {}_0^C D_t^\alpha u + \partial_x^4 u - c\partial_x^2 u = f(u) + g, \ 0 < x < L, \ 0 < t \le T, \tag{1} $$

enclosed with the second Dirichlet boundary conditions

$$ \begin{aligned} & u(0,t) = a_0(t), \quad u(L,t) = a_1(t), \\ & \partial_x^2 u(0,t) = b_0(t), \quad \partial_x^2 u(L,t) = b_1(t) \quad \text{for } 0 < t \le T. \end{aligned} \tag{2} $$

and initial condition

$$ u(x,0) = u_0(x), \ 0 \le x \le L. \tag{3} $$

- ${}_0^C D_t^\alpha$ – Caputo type fractional derivative with $0 < \alpha < 1$,

$$ {}_0^C D_t^\alpha u(x,t) = \int_0^t \omega_{1-\alpha}(t-s)\,\partial_s u(x,s) ds, \quad \omega_\beta = \frac{t^{\beta-1}}{\Gamma(\beta)}; $$

- $c \ge 0$, and the nonlinear term $f(u)$ is required to satisfy certain regularity.
  - $|f(u) - f(v)| \le K|u - v|$ for some $K > 0$ – for nonlinear algorithm.
  - $f(u) \in C^2(\mathbb{R})$, $|f(u) - f(v)| \le K|u - v|$ for some $K > 0$ – for two-grid algorithm.
  - Below analysis are based upon global Lipschitz continuous assumption, but can be extended to local assumption by employing a cutoff function technique.

- In JMAA, 2011, for linear fractional sub-diffusion equation, Sakamoto and Yamamoto show

$$u \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega)) \quad \text{and} \quad {}_0^C D_t^\alpha u \in C([0, T]; L^2(\Omega)),$$

  if $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$;

- In FCAA, 2016, Stynes show that u is smooth only if the initial function and the source term satisfy some restrictive compatibility conditions;

- For time-fractional diffusion equation, Stynes, O'Riordan and Gracia in SINUM 2017, show

$$|\partial_t^l u| \le C_u(1 + t^{\alpha - l}) \quad l = 0, 1, 2;$$

- For nonlinear equation ${}_0^C D_t^\alpha u = \Delta u + f(u)$, if f is Lipschitz continuous, Jin, Li and Zhou in SINUM 2018, prove that the solution satisfies

$$\|\partial_t u\|_{L^2} \le C_u t^{\alpha - 1};$$

- For the linear time-fractional biharmonic equation, Huang and Stynes in Numer. Algor. 2021, show
$$\|u\|_{H^k} \le C_u, \quad \text{and} \quad \|\partial_t^l u\|_{H^k} \le C_u(1 + t^{\alpha - l}), \quad l = 0, 1, 2,$$
under appropriate assumptions about u_0 and f;
- In J. Sci. Comput. 2020, Zhang, Yang and Xu give a similar regularity conclusion for the nonlinear case;
- Our regularity assumptions:
$$\|u\|_{H^6} \le C_u, \quad \|{}_0^C D_t^\alpha u\|_{H^4} \le C_u, \quad \|\partial_t^1 u\|_{H^4} \le C_u(1 + t^{\alpha-1}),$$
$$\|\partial_t^2 u\|_{H^4} \le C_u(1 + t^{\alpha-2}), \qquad \|\partial_t^3 u\|_{H^2} \le C_u(1 + t^{\alpha-3}).$$

- Jin et al. show the weak singularity of the solution will deteriorate the convergence rate of the numerical solution in IMA 2016 and SISC 2016;
- Strategy of graded mesh: Fredholm integral equation in MOC 1982 and Volterra integral equation in MOC 1985;
- Stynes, O'Riordan and Gracia considered the $L1$ scheme on a graded mesh for the linear fractional reaction-subdiffusion problem in SINUM 2017;
- Chen and Stynes presented second-order $L2\text{-}1_\sigma$ scheme on fitted mesh to solve the time fractional IBV problem in JSC 2019;
- Liao, McLean and Zhang gave a discrete fractional Gronwall inequality, which can be used to solve the nonlinear problem in SINUM 2019;
- Liao et al. presented a global consistency analysis framework by introducing the complementary discrete convolution kernels for the nonuniform $L1$ approximation in SINUM 2018 and Alikhanov approximation in CiCP 2021;
- Chen and Stynes show that the error bounds of the previous numerical methods may blow up as $\alpha \to 1^-$ and obtain α-robust error bounds for the nonuniform $L1$ and Alikhanov approximation in IMA 2021;
- An α-robust discrete fractional Gronwall inequality was derived by Huang and Stynes in JSC 2022;

- Finite difference method: Achouri et al. in AMC 2019, Ben-Artzi et al. in IMA 2020, Lu et al. in NMPDE 2022, etc;
- Local discontinuous Galerkin method: Dong and Shu in SINUM 2009, Wei and He in AMM 2014, Du et al. in JCP 2017, Tao et al. in MOC 2020, etc;
- Mixed element method: Liu et al. in AMC 2018 and CMA 2015, Keita et al. in CPC, 2021, He et al. in JSC 2021, etc;
- Virtual element method: Antonietti et al. in M3AS 2018, Li et al. in IMA 2021, Dedner et al. in IMA 2022, Adak et al. in JSC 2022, etc;
- Orthogonal spline collocation method: Yang et al. in CMA 2018, Zhang et al. in JSC 2020 and CMA 2022, etc;
- Compact difference scheme: Hu et al. in CPC, 2011, Fishelov et al. in JSC 2012, Ji et al. in JSC 2015, Liao et al. preprint, 2019, Haghi et al. in Eng. Comput. 2022, etc;

- First proposed by Xu, and was used to solve nonsymmetric indefinite problem in SINUM, 1992 and nonlinear problems in SINUM, 1994, 1996;
- Basic idea: on the coarse space V_H, solving a SMALL-SCALE nonlinear implicit problem to obtain a rough approximation $u_H \in V_H$, and then solve a LARGE-SCALE linear implicit problem based on u_H to find a corrected solution u_h on the fine space V_h;
- Works about FEMs: Chen et al. in JSC, 2011 and CiCP, 2016, Weng et al. in AMM, 2015, etc;
- Works about FVEMs: Bi et al. in NM, 2007 and AMC, 2010, Chen et al. in ANM, 2010 and CMA, 2018, 2022, etc;
- Works about FDMs: Dawson et al. in SINUM, 1998, Rui et al. in SINUM, 2015, etc;
- Recently, the method is also widely studied and applied in fractional problems: Liu et al. in CMA, 2015, Chen et al. CMA, 2020, Li et al. in JSC, 2017 and JCM, 2022, etc;

1.4 Difficulties for high-order two-grid difference algorithm

- The FEMs generate pointwise solutions in space, which implies one can directly get the rough solution at the fine space V_h in two-grid algorithm;
- However, the FDMs lead to discrete solutions only on grids, thus an appropriate mapping from the coarse space V_H to the fine space V_h is required to perform the two-grid algorithm and to preserve the spatial accuracy;
- For the second-order two-grid difference algorithm, a very simple and widely used mapping is the piecewise linear/bilinear interpolation.
- High-order two-grid difference algorithms are rarely studied, due to the lack of the corresponding analysis on the appropriate mapping operator, for instance, linearity and boundedness.

What shall we do?

- By using a model order reduction technique, we first propose a nonlinear compact difference algorithm for the nonlinear tFBEs; We prove the unconditionally and α-robust optimal-order error estimates in the discrete $L^\infty(L^2)$ and $L^\infty(H^2)$ norms via discrete energy method;
- Discuss the linearity and boundedness of the cubic spline interpolation operator used in the high-order two-grid finite difference method;
- Propose an efficient two-grid compact difference algorithms for the nonlinear tFBEs by using the cubic spline interpolation operator, and optimal-order error estimates can be retained.

Outline

- Graded mesh in time direction

$$t_n := T\left(\frac{n}{N}\right)^{\gamma}, \quad \text{for } n = 0, 1, 2, ..., N,$$

where $\gamma \geq 1$ presents the mesh grading parameter. Define

$$\tau := \max_{1\leq k\leq N} \tau_k := t_k - t_{k-1}, \quad t_{n-1+\sigma} := t_{n-1} + \sigma\tau_n.$$

For function $v(t)$ defined on $[0, T]$, denote

$$w^n := w(t_n), \quad w^{n,\sigma} := \sigma w^n + (1-\sigma)w^{n-1};$$

- Two sets of grids in space direction

$$\textit{Coarse grids}: \ x_i := iH \quad \text{with} \quad H = L/N_H, \quad \text{for } 0 \leq i \leq N_H,$$

$$\textit{Fine grids}: \ \tilde{x}_i := ih \quad \text{with} \quad h = H/M, \quad \text{for } 0 \leq i \leq N_h,$$

for $N_h := MN_H$ and $M \geq 2$;

- Discrete spaces of grid functions

$$\mathcal{V}_\kappa = \{w | w = (w_0, w_1, ..., w_{N_\kappa})\}, \quad \mathcal{V}_\kappa^0 = \{w \in \mathcal{V}_\kappa | w_0 = w_{N_\kappa} = 0\},$$

for $\kappa = H, h$.

- $L2$-1_σ formula [Alikhanov, JCP, 2015] at $t = t_{n-1+\sigma}$

$$ {}_0^C D_t^\alpha w(t_{n-1+\sigma}) \approx \sum_{k=1}^{n} A_{n-k}^{(n,\sigma)} \nabla_\tau w^k := \mathbb{D}_\tau^\alpha w^n, $$

  where $\nabla_\tau w^k := w^k - w^{k-1}$;

- The discrete convolution kernels $A_{n-k}^{(n,\sigma)}$ are monotone and positive, i.e.,

$$ A_{k-2}^{(n,\sigma)} \geq A_{k-1}^{(n,\sigma)} > 0, \quad 2 \leq k \leq n \leq N; $$

- Discrete complementary convolution (DCC) kernels: [Liao, et al., SINUM 2018, 2019 and CiCP 2021.]

$$ \sum_{j=k}^{n} P_{n-j}^{(n,\sigma)} A_{j-k}^{(j,\sigma)} = 1, \quad 1 \leq k \leq n \leq N, $$

  which satisfies $P_{n-k}^{(n,\sigma)} \geq 0$ and

$$ \sum_{j=1}^{n} P_{n-j}^{(n,\sigma)} \omega_{1+m\alpha-\alpha}(t_j) \leq \frac{11}{4} \omega_{1+m\alpha}(t_n), \quad 1 \leq n \leq N \text{ and } m = 0, 1; $$

Lemma 2.1. α-robust error bound [Chen-Stynes, IMA 2021]

If $w(t)$ satisfies $|\partial_t^l w| \le C\left(1 + t^{\alpha - l}\right)$ for $1 \le l \le 3$ and $\sigma = 1 - \alpha/2$, then the following estimate holds for $N \ge 3$, $\zeta_N = 1/(\ln N)$ and some α-robust positive constant C_w

$$\sum_{j=1}^{n} P_{n-j}^{(n,\sigma)} |\Upsilon^j[w]| \le C_w \frac{11 e^{\gamma} \Gamma(1+\zeta_N - \alpha)}{4\Gamma(1+\zeta_N)} T^{\alpha} \left(\frac{t_n}{T}\right)^{\zeta_N} N^{-\min\{\gamma\alpha, 3-\alpha\}}, \quad 1 \le n \le N,$$

where $\Upsilon^n[w] := {}_0^C D_t^{\alpha} w(t_{n-1+\sigma}) - \mathbb{D}_{\tau}^{\alpha} w^n$.

Lemma 2.2. α-robust fractional Grönwall inequality [Huang-Stynes, JSC 2022]

Suppose $\{\xi^n, \eta^n\}_{n=1}^N$ and $(\lambda_l)_{l=0}^{N-1}$ are nonnegative sequences and there exists a constant Λ such that $\sum_{l=0}^{N-1} \lambda_l \le \Lambda$. If the nonnegative grid function $(w^k)_{k=0}^N$ satisfies

$$\sum_{k=1}^{n} A_{n-k}^{(n,\sigma)} \nabla_{\tau} (w^k)^2 \le \sum_{k=1}^{n} \lambda_{n-k} (w^{k,\sigma})^2 + \xi^n w^{n,\sigma} + (\eta^n)^2, \quad \text{for } 1 \le n \le N, \tag{4}$$

the following relation holds for $\tau \le ((11/2)\Gamma(2-\alpha)\Lambda)^{-1/\alpha}$ and $1 \le n \le N$

$$w^n \le 2E_{\alpha}\left(\frac{11}{2}\Lambda t_n^{\alpha}\right)\left[w^0 + \max_{1\le k\le n} \sum_{j=1}^{k} P_{k-j}^{(k,\sigma)} (\xi^j + \eta^j) + \max_{1\le k\le n} \{\eta^k\}\right]. \tag{5}$$

- For any grid function $w \in V_\kappa \ (\kappa = H, h)$, denote

$$[d_\kappa w]_{i-\frac{1}{2}} := \tfrac{1}{\kappa}\left(w_i - w_{i-1}\right), \quad \left[d_\kappa^2 w\right]_i := \tfrac{1}{\kappa}\left([d_\kappa w]_{i+\frac{1}{2}} - [d_\kappa w]_{i-\frac{1}{2}}\right),$$

and a compact difference operator

$$[\mathcal{A}_\kappa w]_i := \begin{cases} w_i + \dfrac{\kappa^2}{12}\left[d_\kappa^2 w\right]_i, & 1 \le i \le N_\kappa - 1, \\ w_i, \quad i = 0, N_\kappa; \end{cases}$$

- The discrete inner products and discrete norms $(\kappa = H, h)$

$$< w, q >_\kappa = \kappa \sum_{i=1}^{N_\kappa - 1} w_i q_i, \quad \|w\|_\kappa = \sqrt{< w, w >_\kappa}, \quad \|w\|_{\kappa,\infty} = \max_{0 \le i \le N_\kappa} |w_i|;$$

$$(w, q)_\kappa = \frac{1}{2}\kappa w_0 q_0 + \kappa \sum_{i=1}^{N_\kappa - 1} w_i q_i + \frac{1}{2}\kappa w_{N_\kappa} q_{N_\kappa}, \quad |||w|||_\kappa = \sqrt{(w, w)_\kappa};$$

$$(w, q)_{\kappa,1} = \kappa \sum_{i=1}^{N_\kappa} [d_\kappa w]_{i-\frac{1}{2}} [d_\kappa q]_{i-\frac{1}{2}}, \quad (w, q)_{\kappa,2} = \kappa \sum_{i=1}^{N_\kappa - 1} \left[d_\kappa^2 w\right]_i \left[d_\kappa^2 q\right]_i,$$

$$\|w\|_{\kappa,i} = \sqrt{(w, w)_{\kappa,i}}, \quad \text{for } i = 1, 2, \quad \|u\|_{\mathcal{A},\kappa} = \sqrt{\langle \mathcal{A}_\kappa w, w \rangle_\kappa};$$

- The discrete inner products of binary vector: $\langle (w_1, q_1), (w_2, q_2) \rangle_\kappa = \langle w_1, w_2 \rangle_\kappa + \langle q_1, q_2 \rangle_\kappa$.

Lemma 2.3. [Sun, Science Press, 2021]

For any $w \in \mathcal{V}_\kappa^0$, we have

$$\frac{1}{3}\|w\|_\kappa^2 \le \|\mathcal{A}_\kappa w\|_\kappa^2 \le \|w\|_\kappa^2 \quad \text{and} \quad \frac{2}{3}\|u\|_\kappa^2 \le \|u\|_{\mathcal{A},\kappa}^2 \le \|u\|_\kappa^2 .$$

Lemma 2.4. [Sun, Science Press, 2021]

For any $w \in \mathcal{V}_\kappa^0$, we have

$$\|w\|_{\kappa,\infty} \le \frac{\sqrt{L}}{2}\|w\|_{\kappa,1} \quad \text{and} \quad \|w\|_{\kappa,1} \le \frac{L}{\sqrt{6}}\|w\|_{\kappa,2} .$$

Lemma 2.5.

For any $w \in \mathcal{V}_\kappa$, we have

$$\|\mathcal{A}_\kappa w\|_\kappa \le |||w|||_\kappa .$$

- Introduce an auxiliary variable $v = \partial_x^2 u$ (model order reduction)

$$\Rightarrow {}_0^C D_t^\alpha u + \partial_x^2 v - c\partial_x^2 u = f(u) + g, \quad v = \partial_x^2 u. \tag{6}$$

- Applying $L2\text{-}1_\sigma$ formula on graded mesh and compact difference technique to (6), we see

$$\mathbb{D}_\tau^\alpha \left[\mathcal{A}_h U\right]_i^n + \left[d_h^2 V\right]_i^{n,\sigma} - c\left[d_h^2 U\right]_i^{n,\sigma} = \mathcal{A}_h f(U_i^{n,\sigma}) + \mathcal{A}_h g_i^{n-1+\sigma} + (R_1)_i^n,$$
$$\left[\mathcal{A}_h V\right]_i^n = \left[d_h^2 U\right]_i^n + (R_2)_i^n, \quad 1 \le i \le N_h - 1.$$

where the local truncation errors $(R_1)_i^n := \sum_{s=1}^4 (R_{1,s})_i^n$, and

$$(R_{1,1})_i^n = \mathcal{A}_h {}_0^C D_t^\alpha u(\tilde{x}_i, t_{n-1+\sigma}) - \mathbb{D}_\tau^\alpha \left[\mathcal{A}_h U\right]_i^n = \mathcal{A}_h \Upsilon^n[u]_i,$$
$$(R_{1,2})_i^n = \mathcal{A}_h \partial_x^2 v(\tilde{x}_i, t_{n-1+\sigma}) - \left[d_h^2 V\right]_i^{n,\sigma},$$
$$(R_{1,3})_i^n = c\mathcal{A}_h \partial_x^2 u(\tilde{x}_i, t_{n-1+\sigma}) - c\left[d_h^2 U\right]_i^{n,\sigma},$$
$$(R_{1,4})_i^n = \mathcal{A}_h f(u(\tilde{x}_i, t_{n-1+\sigma})) - \mathcal{A}_h f(U_i^{n,\sigma}),$$
$$(R_2)_i^n = \mathcal{A}_h \partial_x^2 u(\tilde{x}_i, t_n) - \left[d_h^2 U\right]_i^n.$$

- Define operator $[\mathcal{L}_\kappa w]_i = \mathcal{A}_\kappa \partial_x^2 w(x_i) - d_\kappa^2 w(x_i)$. By the well-known Bramble-Hilbert Lemma, we can prove

$$\|\mathcal{L}_\kappa w\|_\kappa \leq C\kappa^s \|w\|_{H^s(0,L)}, \quad w \in H^s(0,L), \quad 1 \leq s \leq 6; \tag{7}$$

Lemma 2.7.

Under our regularity assumptions, the following estimates hold for $1 \leq n \leq N$

$$(a)\ \|R_2^{n,\sigma}\|_h \leq Ch^4; \quad (b)\ \sum_{s=2}^{4} \|R_{1,s}^n\|_h \leq C\left(N^{-\min\{\gamma\alpha,2\}} + h^4\right);$$

$$(c)\ \sum_{k=1}^{n} P_{n-k}^{(n,\sigma)} \left(\|R_1^k\|_h + \|R_2^k\|_h\right) \leq C\left(N^{-\min\{\gamma\alpha,2\}} + h^4\right);$$

$$(d)\ \sum_{k=1}^{n} P_{n-k}^{(n,\sigma)} \|\Upsilon^k[u]\|_{h,2} \leq C\left(N^{-\min\{\gamma\alpha,3-\alpha\}} + h^4\right)$$

where C is some α-robust positive constant.

Sketch of Proof.

- One can easily derive conclusions (a), (b), (c) using the error bounds yielded by different difference operators and the properties of complementary discrete convolution kernels $P_{n-k}^{n,\sigma}$;

- To derive (d), we define a time-dependent grid function $\varphi(t)$ with component $[\varphi(t)]_i := [\mathcal{A}_h V]_i - \left[d_h^2 U\right]_i$, $1 \le i \le N_h - 1$. Let $\varphi^n := \varphi(t_n)$. By (7), it is easy to verify $\|\partial_t^l \varphi(t)\|_h \le Ch^4(1 + t^{\alpha-l})$ for $l = 0, 1$;
  - Note that $\Upsilon^k[u]_i^n = {}_0^C D_t^\alpha u(\tilde{x}_i, t_{n-1+\sigma}) - \mathbb{D}_\tau^\alpha U_i^n$. We write
$$d_h^2 \Upsilon^k[u] = -{}_0^C D_t^\alpha \varphi(t_{n-1+\sigma}) + \mathbb{D}_\tau^\alpha \varphi^n + \Upsilon^n[\mathcal{A}_h V].$$
$$\Rightarrow \|\Upsilon^k[u]\|_{h,2} \le \|{}_0^C D_t^\alpha \varphi(t_{n-1+\sigma})\|_h + \|\mathbb{D}_\tau^\alpha \varphi^n\|_h + \|\Upsilon^n[\mathcal{A}_h V]\|_h := I_1^n + I_2^n + I_3^n;$$
  - It is obvious that
$$\sum_{k=1}^n P_{n-k}^{(n,\sigma)} I_1^k \le Ch^4 \quad \text{and} \quad \sum_{k=1}^n P_{n-k}^{(n,\sigma)} I_3^k \le CN^{-\min\{\gamma\alpha, 3-\alpha\}};$$
  - Pay special attention to I_2^n
$$\sum_{k=1}^n P_{n-k}^{(n,\sigma)} I_2^k = \sum_{k=1}^n P_{n-k}^{(n,\sigma)} \left\| \sum_{j=1}^k A_{k-j}^{(k,\sigma)} \nabla_\tau \varphi^j \right\|_h \le \sum_{k=1}^n P_{n-k}^{(n,\sigma)} \sum_{j=1}^k A_{k-j}^{(k,\sigma)} \left\|\varphi^j - \varphi^{j-1}\right\|_h$$
$$= \sum_{j=1}^n \left\| \int_{t_{j-1}}^{t_j} \partial_t^1 \varphi(s) ds \right\|_h \le \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left\|\partial_t^1 \varphi(s)\right\|_h ds \le C_u \left(t_n + t_n^\alpha/\alpha\right) h^4.$$

A nonlinear HOC difference algorithm

$$\mathbb{D}_\tau^\alpha [\mathcal{A}_h u]_i^n + \left[d_h^2 v\right]_i^{n,\sigma} - c\left[d_h^2 u\right]_i^{n,\sigma} = \mathcal{A}_h f(u_i^{n,\sigma}) + \mathcal{A}_h g_i^{n-1+\sigma},$$
$$1 \le i \le N_h - 1,\ 1 \le n \le N,$$
$$[\mathcal{A}_h v]_i^n = \left[d_h^2 u\right]_i^n, \quad 1 \le i \le N_h - 1,\ 1 \le n \le N,$$
$$u_i^0 = u_0(\tilde{x}_i), \quad 1 \le i \le N_h - 1,$$
$$u_0^n = a_0(t_n), \quad u_{N_h}^n = a_1(t_n), \quad v_0^n = b_0(t_n), \quad v_{N_h}^n = b_1(t_n), \quad 1 \le n \le N.$$

- Eliminating the intermediate variable v_i^n from the above algorithm, one obtain a nonlinear system only about u^n (decoupled):

$$\mathbb{D}_\tau^\alpha \left[\mathcal{A}_h^2 u\right]_i^n + \left[d_h^4 u\right]_i^{n,\sigma} - c\left[\mathcal{A}_h d_h^2 u\right]_i^{n,\sigma} = \mathcal{A}_h^2 f(u_i^{n,\sigma}) + \mathcal{A}_h^2 g_i^{n-1+\sigma},$$
$$1 \le i \le N_h - 1,\ 1 \le n \le N, \tag{8}$$

where $[d_h^2 u]_0^n := b_0(t_n), \ \ [d_h^2 u]_{N_h}^n := b_1(t_n)$.

- The nonlinear compact difference algorithm is reduced to a real symmetric five-point nonlinear algebraic system.

Theorem 2.1. Existence Result

The nonlinear compact difference method is solvable if the maximum temporal stepsize satisfies

$$\tau \le \sqrt[\alpha]{\frac{4}{33\Gamma(2-\alpha)K\sigma}},$$

where $K > 0$ is the Lipschitz continuous constant.

Sketch of Proof.

- Denote $w_i = \sigma u_i^n + (1-\sigma)u_i^{n-1}$, $q_i = \sigma v_i^n + (1-\sigma)v_i^{n-1}$ and

$$G_i^n := \frac{1-\sigma}{\sigma} A_0^{(n,\sigma)} \left[\mathcal{A}_h^2 u\right]_i^{n-1} + \sum_{k=1}^{n-1} \left(A_{n-k-1}^{(n,\sigma)} - A_{n-k}^{(n,\sigma)}\right) \left[\mathcal{A}_h^2 u\right]_i^k + A_{n-1}^{(n,\sigma)} u_i^0.$$

- Define a binary mapping $T(w,q) : \mathcal{V}_h^0 \times \mathcal{V}_h^0 \to \mathcal{V}_h^0 \times \mathcal{V}_h^0$ by $T(w,q) := (T^1(w,q), T^2(w,q))$:

$$\left[T^1(w,q)\right]_i := \frac{A_0^{(n,\sigma)}}{\sigma} \left[\mathcal{A}_h w\right]_i - G_i^n + \left[d_h^2 q\right]_i - c\left[d_h^2 w\right]_i - \mathcal{A}_h f(w_i) - \mathcal{A}_h g_i^{n-1+\sigma}$$

$$\left[T^2(w,q)\right]_i := \left[\mathcal{A}_h q\right]_i - \left[d_h^2 w\right]_i.$$

- We prove that for $\tau \leq \sqrt[\alpha]{4/\left(33\Gamma(2-\alpha)K\sigma\right)}$,

$$\langle T(w,q),(w,q)\rangle_h \geq \left(\frac{A_0^{(n,\sigma)}}{2\sigma}\|w\|_{\mathcal{A},h} - Q\right)\|w\|_{\mathcal{A},h} + \|q\|_{\mathcal{A},h}^2 \geq 0,$$

for $\|w\|_{\mathcal{A},h} = \frac{2\sigma}{A_0^{(n,\sigma)}}Q$ and any q, where

$$Q := \frac{\sqrt{6}}{2}\left(K\|u^0\|_h + |||f(u^0) + g^{n-1+\sigma}|||_h\right) + \frac{1-\sigma}{\sigma}A_0^{(n,\sigma)}\left\|u^{n-1}\right\|_{\mathcal{A},h} + \sum_{k=1}^{n-1}\left(A_{n-k-1}^{(n,\sigma)} - A_{n-k}^{(n,\sigma)}\right)\left\|u^k\right\|_{\mathcal{A},h} + A_{n-1}^{(n,\sigma)}\left\|u^0\right\|_{\mathcal{A},h};$$

- Then, Browder's Lemma shows there exists some $(w^*,q^*) \in \mathcal{V}_h^0 \times \mathcal{V}_h^0$ such that $T(w^*,q^*) = 0$ and

$$\|w^*\|_h \leq \frac{\sqrt{6}}{2}\|w\|_{\mathcal{A},h} \leq \frac{2\sqrt{3}\sigma}{A_0^{(n,\sigma)}}Q.$$

- The nonlinear compact difference scheme is solvable and we have

$$u_i^n = \frac{1}{\sigma}w_i - \frac{1-\sigma}{\sigma}u_i^{n-1}, \quad v_i^n = \frac{1}{\sigma}q_i - \frac{1-\sigma}{\sigma}v_i^{n-1}.$$

Theorem 2.2. Uniqueness Result

Assume the conditions in Theorem 2.1 hold. Furthermore, if the maximum temporal stepsize satisfies

$$\tau \le \frac{1}{\sqrt[\alpha]{11/2\Gamma(2-\alpha)K}} \quad \text{and} \quad h \le \sqrt{12/c},$$

the solution of the nonlinear compact difference method is unique.

Sketch of Proof.

- Let $\{X_u^n, X_v^n\}$, $\{Y_u^n, Y_v^n\} \in \mathcal{V}_h^0 \times \mathcal{V}_h^0$ be two group solutions and denote the difference by $Z_u^n = X_u^n - Y_u^n$, $Z_v^n = X_v^n - Y_v^n$,

$$\mathbb{D}_\tau^\alpha \left[\mathcal{A}_h Z_u\right]_i^n + \left[d_h^2 Z_v\right]_i^{n,\sigma} - c\left[d_h^2 Z_u\right]_i^{n,\sigma} = \mathcal{A}_h f(X_u^{n,\sigma}) - \mathcal{A}_h f(Y_u^{n,\sigma}),$$
$$\left[\mathcal{A}_h Z_v\right]_i^{n,\sigma} = \left[d_h^2 Z_u\right]_i^{n,\sigma}.$$

- Taking inner product with $\mathcal{A}_h Z_u^{n,\sigma}$ and $\mathcal{A}_h Z_v^{n,\sigma}$ respectively, summing them and utilizing Cauch-Schwarz inequality and Lipschitz continuous of $f(\cdot)$

$$\frac{1}{2}\sum_{k=1}^{n} A_{n-k}^{(n,\sigma)} \nabla_\tau \|\mathcal{A}_h Z_u^k\|_h^2 \le K\|\mathcal{A}_h Z_u^{n,\sigma}\|_h^2;$$

- Application of Lemma 2.2, i.e., the discrete fractional Gronwall inequality yields

$$\|\mathcal{A}_h Z_u^n\|_h = 0 \Rightarrow Z_u^n = 0 \Rightarrow Z_v^n = 0, \quad \text{for} \quad \tau \le 1/\sqrt[\alpha]{11/2\Gamma(2-\alpha)K}.$$

L^2-norm estimate of the nonlinear algorithm

- Denote

$$e_u^n = U^n - u^n \quad \text{and} \quad e_v^n = V^n - v^n,\ 1 \le n \le N;$$

Theorem 2.3. Error estimate under discrete L^2 norm

Let $\sigma = 1 - \alpha/2$. Under the conditions

$$\tau \le \frac{1}{\sqrt[\alpha]{11\sqrt{3}\Gamma(2-\alpha)K}} \quad \text{and} \quad h \le \sqrt{3/c},$$

the following estimate holds for some α-robust positive constant C

$$\|e_u^n\|_h \le C(N^{-\min\{r\alpha,2\}} + h^4), \quad 1 \le n \le N.$$

Sketch of Proof.

- Error equation

$$\mathbb{D}_\tau^\alpha \left[\mathcal{A}_h e_u\right]_i^n + \left[d_h^2 e_v\right]_i^{n,\sigma} - c\left[d_h^2 e_u\right]_i^{n,\sigma} = \mathcal{A}_h\left(f(U_i^{n,\sigma}) - f(u_i^{n,\sigma})\right) + (R_1)_i^n, \tag{9}$$

$$\left[\mathcal{A}_h e_v\right]_i^{n,\sigma} = \left[d_h^2 e_u\right]_i^{n,\sigma} + (R_2)_i^{n,\sigma}, \quad 1 \le i \le N_h - 1,\ 1 \le n \le N; \tag{10}$$

- The equations (9) (taking inner product with $\mathcal{A}_h e_u^{n,\sigma}$) and (10) (taking inner product with $\mathcal{A}_h e_v^{n,\sigma}$) are used to estimate $\|e_u^n\|_h$:

$$\langle \mathbb{D}_\tau^\alpha \mathcal{A}_h e_u^n, \mathcal{A}_h e_u^{n,\sigma}\rangle_h + \langle \mathcal{A}_h e_v^{n,\sigma}, \mathcal{A}_h e_v^{n,\sigma}\rangle_h - c\left\langle d_h^2 e_u^{n,\sigma}, \mathcal{A}_h e_u^{n,\sigma}\right\rangle_h$$
$$= \langle \mathcal{A}_h\left(f(U^{n,\sigma}) - f(u^{n,\sigma})\right), \mathcal{A}_h e_u^{n,\sigma}\rangle_h + \langle R_1^n, \mathcal{A}_h e_u^{n,\sigma}\rangle_h + \left\langle R_2^{n,\sigma}, \mathcal{A}_h e_v^{n,\sigma}\right\rangle_h.$$

- Standard estimates based upon Cauchy-Schwartz inequality and previous lemmas yield

$$\sum_{k=1}^{n} A_{n-k}^{(n,\sigma)} \nabla_\tau \left(\|\mathcal{A}_h e_u^k\|_h^2\right) + 2c\,\|e_u^{n,\sigma}\|_{1,h}^2$$
$$\leq 2\sqrt{3}K\,\|\mathcal{A}_h e_u^{n,\sigma}\|_h^2 + 2\,\|R_1^n\|_h\,\|\mathcal{A}_h e_u^{n,\sigma}\|_h + 2\left\|R_2^{n,\sigma}\right\|_h^2;$$

- Note that the above inequality has the form of (4) in Lemma 2.2 of the α-robust fractional Grönwall inequality. Thus

$$\|e_u^n\|_h \leq \|\mathcal{A}_h e_u^n\|_h \leq C\left(N^{-\min\{\gamma\alpha,2\}} + h^4\right).$$

Theorem 2.4. Error estimate under discrete H^2 norm

Let $\sigma = 1 - \alpha/2$. Under the conditions

$$\tau \leq \min\left\{ \frac{1}{\sqrt[\alpha]{11\sqrt{3}\Gamma(2-\alpha)K}}, \frac{1}{\sqrt[\alpha]{11\Gamma(2-\alpha)c^2}} \right\} \quad \text{and} \quad h \leq \sqrt{3/c},$$

the following error estimates hold for some α-robust positive constant C and $1 \leq n \leq N$

$$\|e_v^n\|_h \leq C(N^{-\min\{r\alpha,2\}} + h^4) \quad \text{and} \quad \|e_u^n\|_{h,2} \leq C(N^{-\min\{r\alpha,2\}} + h^4).$$

Sketch of Proof.

- To estimate e_v^n, a new error equation needs to be established. Act the operator $\mathbb{D}_\tau^\alpha$ on both side of $[\mathcal{A}_h v]_i^n = \left[d_h^2 u\right]_i^n$, and noting $v = \partial_x^2 u$ at $(\cdot, t_{n-1+\sigma})$, we derive a new error equation

$$\mathbb{D}_\tau^\alpha \left[\mathcal{A}_h e_v\right]_i^n = \mathbb{D}_\tau^\alpha \left[d_h^2 e_u\right]_i^n + (R_3)_i^k, \qquad (11)$$

where

$$(R_3)_i^n := \mathbb{D}_\tau^\alpha \left([\mathcal{A}_h V]_i^n - \left[d_h^2 U\right]_i^n \right) = I_2^n \;\Rightarrow \sum_{k=1}^{n} P_{n-k}^{(n,\sigma)} \|R_3^n\|_h \leq Ch^4;$$

- Taking inner products with $\mathcal{A}_h e_v^{n,\sigma}$ and $d_h^2 e_v^{n,\sigma}$ for the error equation (11) and the original error equations (9) to estimate $\|e_v^n\|_h$:

$$(\mathbb{D}_\tau^\alpha \mathcal{A}_h e_v^n, \mathcal{A}_h e_v^{n,\sigma}\rangle_h + \langle d_h^2 e_v^{n,\sigma}, d_h^2 e_v^{n,\sigma}\rangle_h - c\langle d_h^2 e_u^{n,\sigma}, d_h^2 e_v^{n,\sigma}\rangle_h$$
$$= \langle \mathcal{A}_h(f(U^{n,\sigma}) - f(u^{n,\sigma})), d_h^2 e_v^{n,\sigma}\rangle_h + \langle R_1^n, d_h^2 e_v^{n,\sigma}\rangle_h + \langle R_3^n, \mathcal{A}_h e_v^{n,\sigma}\rangle_h .$$

- Considering the weak regularity of the solution at $t = 0$, we pay special attention to the $R_{1,1}^n$ term

$$\langle R_1^n, d_h^2 e_v^{n,\sigma}\rangle_h \le \|\Upsilon_2^n[u]\|_{h,2} \|\mathcal{A}_h e_v^{n,\sigma}\|_h + \left\|\sum_{s=2}^{4} R_{1,s}^n\right\|_h^2 + \frac{1}{4}\|e_v^{n,\sigma}\|_{h,2}^2 ;$$

$\Rightarrow$

$$\sum_{k=1}^{n} A_{n-k}^{(n,\sigma)} \nabla_\tau \left(\|\mathcal{A}_h e_v^k\|_h^2\right) \le 2c^2 \|\mathcal{A}_h e_v^{n,\sigma}\|_h^2 + 2\left(\|R_3^n\|_h + \|\Upsilon_2^n[u]\|_{h,2}\right)\|\mathcal{A}_h e_v^{n,\sigma}\|_h$$
$$+ 6K^2 \|\mathcal{A}_h e_u^{n,\sigma}\|_h^2 + 2\left\|\sum_{s=2}^{4} R_{1,s}^n\right\|_h^2 + 2c^2 \left\|R_2^{n,\sigma}\right\|_h^2 .$$

The conclusion of Theorem 2.3 and the α-robust Gronwall Lemma 2.2 implies

$$\|\mathcal{A}_h e_v^n\|_h \le C\left(N^{-\min\{\gamma\alpha,2\}} + h^4\right);$$

- Finally, error equation (10) is used to estimate $\|e_u^n\|_{h,2}$:

$$\|e_u^n\|_{h,2} \le \|\mathcal{A}_h e_v^n\|_h + \left\|R_2^{n,\sigma}\right\|_h \le C\left(N^{-\min\{\gamma\alpha,2\}} + h^4\right).$$

Denote $\{u_H^n, v_H^n\}_{n=1}^N$ and $\{u_h^n, v_h^n\}_{n=1}^N$ as difference approximations to $\{U^n, V^n\}_{n=1}^N$ on the coarse grids and fine grids respectively.

TG-HOC difference algorithm

- **Step 1.** On the coarse grid, solve a small-scale nonlinear compact finite difference scheme to find a rough solution u_H^n by

$$\mathbb{D}_\tau^\alpha [\mathcal{A}_H u_H]_i^n + \left[d_H^2 v_H\right]_i^{n,\sigma} - c\left[d_H^2 u_H\right]_i^{n,\sigma} = \mathcal{A}_H f([u_H]_i^{n,\sigma}) + \mathcal{A}_H g_i^{n-1+\sigma},$$

$$[\mathcal{A}_H v_H]_i^n = \left[d_H^2 u_H\right]_i^n, \quad 1 \le i \le N_H - 1,\ 1 \le n \le N,$$

- **Step 2.** On the fine grid, based on the obtained rough solution u_H^n, solve a large-scale linearized compact difference scheme to produce a corrected solutio u_h^n by

$$\mathbb{D}_\tau^\alpha [\mathcal{A}_h u_h]_i^n + \left[d_h^2 v_h\right]_i^{n,\sigma} - c\left[d_h^2 u_h\right]_i^{n,\sigma} = [\mathcal{A}_h F]_i^{n,\sigma} + \mathcal{A}_h g_i^{n-1+\sigma},$$

$$[\mathcal{A}_h v_h]_i^n = \left[d_h^2 u_h\right]_i^n, \quad 1 \le i \le N_h - 1,\ 1 \le n \le N,$$

  where $F_i^{n,\sigma}$ represents a Newton linearization of $f\left([u_h]_i^{n,\sigma}\right)$ about $[\Pi_H u_H]_i^{n,\sigma}$, defined as

$$F_i^{n,\sigma} := f\left([\Pi_H u_H]_i^{n,\sigma}\right) + \partial_u^1 f([\Pi_H u_H]_i^{n,\sigma})\left([u_h]_i^{n,\sigma} - [\Pi_H u_H]_i^{n,\sigma}\right).$$

- For any grid function $w \in \mathcal{V}_H$, Π_H is defined as the cubic spline interpolation operator satisfying the second-order derivative boundary condition, i.e.,

$$[\Pi_H w](x) = M_{i-1}\frac{(x_i - x)^3}{6H} + M_i\frac{(x - x_{i-1})^3}{6H} + \left(w_{i-1} - \frac{M_{i-1}H^2}{6}\right)\frac{x_i - x}{H} + \left(w_i - \frac{M_i H^2}{6}\right)\frac{x - x_{i-1}}{H}, \quad x \in (x_{i-1}, x_i), \quad i = 1, 2, ..., N_H$$

with given conditions $M_0 = \partial_x^2 w_0$ and $M_{N_H} = \partial_x^2 w_{N_H}$, and $\mathbf{M} := [M_1, M_2, \ldots, M_{N_H-1}]^T$ is the solution of the following linear system

$$\mathbf{AM} = \mathbf{d}, \quad \mathbf{A} := \frac{1}{2}\text{tridiag}(1,4,1), \quad \mathbf{d} := [d_1, d_2, \ldots, d_{N_H-1}]^T,$$

with

$$d_1 = 3[d_H^2 w]_1 - \frac{1}{2}M_0, \quad d_{N_H-1} = 3[d_H^2 w]_{N_H-1} - \frac{1}{2}M_{N_H}$$
$$d_i = 3[d_H^2 w]_i, \quad i = 2, 3, \cdots N_H - 2$$

.

Lemma 3.1. Interpolation error of Π_H [Quarteroni-Sacco-Saleri, 2007]

Let $\Pi_H w$ be the cubic spline interpolation of function $w(x) \in C^4[0, L]$ which satisfying the second-order derivative boundary condition. Then, the following estimate holds for some positive constant C_0

$$\|w - \Pi_H w\|_{L^\infty} \leq C_0 H^4 \|\partial_x^4 w\|_{L^\infty}.$$

Lemma 3.2. Linearity property of Π_H

For any gird functions $w, \tilde{w} \in \mathcal{V}_H$, the linearity property $[\Pi_H (w + \tilde{w})](x) = [\Pi_H w](x) + [\Pi_H \tilde{w}](x)$ holds on $x \in (x_{i-1}, x_i)$, $i = 1, 2, ..., N_H$.

Lemma 3.3. Boundedness property of Π_H

For any grid function $w \in \mathcal{V}_H$, the following estimates hold

$$|||\, \Pi_H w|||_h^2 \leq 48\, |||\, w\, |||_H^2 + \frac{2}{9}\left(M_0^2 + M_{N_H}^2\right) H^5, \tag{12}$$

$$\|\Pi_H w\|_{h,\infty} \leq \max\left\{2\,\|w\|_{H,\infty} + \frac{H^2}{3}\,|M_0|\,, 2\|w\|_{H,\infty} + \frac{H^2}{3}\left|M_{N_H}\right|, 6\,\|w\|_{H,\infty}\right\}. \tag{13}$$

In particular, if $M_0 = M_H = 0$, the operator $\Pi_H : \mathcal{V}_H \to \mathcal{V}_h$ is bounded in the sense that

$$|||\Pi_H w|||_h \leq 4\sqrt{3}\, |||\, w\, |||_H \quad \text{and} \quad \|\Pi_H w\|_{h,\infty} \leq 6\, \|w\|_{H,\infty}.$$

Sketch of Proof.

- By definitions of $\|\cdot\|_h$ and Π_H

$$\|\Pi_H w\|_h^2 \le \left(4H + \frac{1}{2}h\right) w_0^2 + \left(4H - \frac{1}{2}h\right) w_{N_H}^2 + \frac{H^5}{9}\left(M_0^2 + M_{N_H}^2\right)$$
$$+ 8H \sum_{i=1}^{N_H-1} w_i^2 + \frac{2H^5}{9} \sum_{i=1}^{N_H-1} M_i^2;$$

- As $\mathbf{A}$ is symmetric and diagonally dominant, by Gerschgorin theorem, we know the eigenvalues of $(\mathbf{A}^{-1})^2$ belong to $[1/9, 1]$;

- By the Rayleigh-Ritz theorem, the last term of the above inequality can be estimated

$$\sum_{i=1}^{N_H-1} M_i^2 = \mathbf{d}^T(\mathbf{A}^{-1})^2\mathbf{d} \le \mathbf{d}^T\mathbf{d} = \sum_{i=1}^{N_H-1} d_i^2$$

$$\Longrightarrow \|\Pi_H w\|_h^2 \le 48\|w\|_H^2 + \frac{2H^5}{9}\left(M_0^2 + M_{N_H}^2\right);$$

- To prove conclusion (13), let $\|\Pi_H w\|_{h,\infty} = |[\Pi_H w]_j|$ for some j satisfying $(i-1)M \le j \le iM$. By the definition of Π_H, it holds

$$\begin{aligned}\|\Pi_H w\|_{h,\infty} &\le |w_{i-1}| + |w_i| + \frac{H^2}{6}|M_{i-1}| + \frac{H^2}{6}|M_i| \\ &\le 2\|w\|_{H,\infty}^2 + \frac{H^2}{6}|M_{i-1}| + \frac{H^2}{6}|M_i|;\end{aligned}$$

- Let $\|\mathbf{M}\|_{H,\infty} = |M_p|$ for some index p such that $|M_p| \ge |M_i|$ for $i = 1, 2, \ldots, N_H - 1$.
  - If $|M_p| \le |M_0|$ or $|M_{N_H}|$

$$\Rightarrow \|\Pi_H w\|_{h,\infty} \le 2\|w\|_{H,\infty} + \frac{H^2}{3}\max\{|M_0|, |M_{N_H}|\};$$

  - If $|M_p| \ge \max\{|M_0|, |M_{N_H}|\}$, as $M_{p-1} + 4M_p + M_{p+1} = 6\left[d_H^2 w\right]_p$

$$\Rightarrow |M_p| \le 3\left|\left[d_H^2 w\right]_p\right| = 3\left|\frac{w_{p-1} - 2w_p + w_{p+1}}{H^2}\right| \le \frac{12}{H^2}\|w\|_{H,\infty},$$

$$\Longrightarrow \|\Pi_H w\|_{h,\infty} \le 2\|w\|_{H,\infty} + \frac{H^2}{3}|M_p| \le 6\|w\|_{H,\infty}.$$

- Denote

$$\left[e_{u,H}\right]_i^n = U_i^n - [u_H]_i^n, \quad \left[e_{v,H}\right]_i^n = V_i^n - [v_H]_i^n, \quad 0 \le i \le N_H,\ 1 \le n \le N,$$
$$\left[e_{u,h}\right]_i^n = U_i^n - [u_h]_i^n, \quad \left[e_{v,h}\right]_i^n = V_i^n - [v_h]_i^n, \quad 0 \le i \le N_h,\ 1 \le n \le N.$$

Theorem 3.1. Error estimate for the nonlinear scheme on coarse grid

Let $\sigma = 1 - \alpha/2$. Under the conditions

$$\tau \le \min\left\{ \frac{1}{\sqrt[\alpha]{11\sqrt{3}\Gamma(2-\alpha)K}}, \frac{1}{\sqrt[\alpha]{11\Gamma(2-\alpha)c^2}} \right\},$$

and $H \le \sqrt{3/c}$, the following estimates hold for some α-robust positive constant C

$$\|e_{u,H}^n\|_H \le C(N^{-\min\{r\alpha,2\}} + H^4), \quad \|e_{v,H}^n\|_H \le C(N^{-\min\{r\alpha,2\}} + H^4)$$
$$\|e_{u,H}^n\|_{H,2} \le C(N^{-\min\{r\alpha,2\}} + H^4).$$

for $1 \le n \le N$.

Corollary 3.1. Conclusions about interpolation solution $\Pi_H u_H^n$

Assume the conditions in Theorem 3.1 hold, then the numerical solution u_H^n yielded by nonlinear scheme on the coarse grid satisfies

$$\|U^n - \Pi_H u_H^n\|_h \le C\left(N^{-\min\{r\alpha,2\}} + H^4\right), \quad \text{for } 1 \le n \le N, \tag{14}$$

and

$$\|U^n - \Pi_H u_H^n\|_{h,\infty} \le C\left(N^{-\min\{r\alpha,2\}} + H^4\right), \quad \text{for } 1 \le n \le N, \tag{15}$$

so that the interpolation solution $\Pi_H u_H^n$ is bounded that

$$\|\Pi_H u_H^n\|_{h,\infty} \le K^*, \quad \text{for } 1 \le n \le N, \tag{16}$$

where C and K^* are all α-robust positive constants dependent on u and T.

Sketch of Proof.

- For conclusion (14), utilize the linearity and boundedness of Π_H

$$\begin{aligned}\left\|U^n - \Pi_H u_H^n\right\|_h &\le \left\|U^n - \Pi_H U^n\right\|_h + \left\|\Pi_H U^n - \Pi_H u_H^n\right\|_h \\ &\le CH^4 + \|\Pi_H e_{u,H}^n\|_h \le C\left(N^{-\min\{r\alpha,2\}} + H^4\right);\end{aligned}$$

- Conclusion (15) can be similarly proved, which implies conclusion (16).

Theorem 3.2. Error estimate under the discrete L^2 norm

Assume the conditions in Theorem 3.1 hold, then the following estimates hold for some α-robust positive constant C

$$\left\| e_{u,h}^n \right\|_h \leq C(N^{-\min\{r\alpha,2\}} + h^4 + H^8), \quad \text{for } 1 \leq n \leq N.$$

Sketch of Proof.

- Compared with the proof in Theorem 2.3, the difference lies in the treatment of the nonlinear difference term between $f(U_i^{n,\sigma})$ and $F_i^{n,\sigma}$.
- A Taylor expansion about $[\Pi_H u_H]_i^{n,\sigma}$ gives

$$f(U_i^{n,\sigma}) - F_i^{n,\sigma} = \partial_u^1 f([\Pi_H u_H]_i^{n,\sigma}) \left(U_i^{n,\sigma} - [u_h]_i^{n,\sigma}\right) + \frac{1}{2}\partial_u^2 f(\theta_i^{n,\sigma}) \left(U_i^{n,\sigma} - [\Pi_H u_H]_i^{n,\sigma}\right)^2,$$

where $\theta_i^{n,\sigma}$ is a constant between $U_i^{n,\sigma}$ and $[\Pi_H u_H]_i^{n,\sigma}$, and $\|\theta^{n,\sigma}\|_{h,\infty} \leq \max\left\{\|U^{n,\sigma}\|_{h,\infty}, K^*\right\}$ if τ satisfies the condition in Theorem 3.1.

- The estimate of the nonlinear term

$$\begin{aligned}
&\left\langle \mathcal{A}_h(f(U^{n,\sigma}) - F^{n,\sigma}), \mathcal{A}_h e_{u,h}^{n,\sigma} \right\rangle_h \\
&= \left\langle \mathcal{A}_h \partial_u^1 f(\Pi_H u_H^{n,\sigma}) \left(U^{n,\sigma} - u_h^{n,\sigma}\right), \mathcal{A}_h e_{u,h}^{n,\sigma} \right\rangle_h \\
&\quad + \frac{1}{2} \left\langle \mathcal{A}_h \partial_u^2 f(\theta^{n,\sigma}) \left(U^{n,\sigma} - \Pi_H u_H^{n,\sigma}\right)^2, \mathcal{A}_h e_{u,h}^{n,\sigma} \right\rangle_h \\
&\le \sqrt{3} K \|\mathcal{A}_h e_{u,h}^{n,\sigma}\|_h^2 + C \left\| U^{n,\sigma} - \Pi_H u_H^{n,\sigma} \right\|_{h,\infty} \left\| \mathcal{A}_h \left(U^{n,\sigma} - \Pi_H u_H^{n,\sigma} \right) \right\|_h \|\mathcal{A}_h e_{u,h}^{n,\sigma}\|_h \\
&\le \sqrt{3} K \|\mathcal{A}_h e_{u,h}^{n,\sigma}\|_h^2 + C \left(N^{-\min\{\gamma\alpha, 2\}} + H^4 \right)^2 \|\mathcal{A}_h e_{u,h}^{n,\sigma}\|_h;
\end{aligned}$$

- Note: the linearity, boundedness of Π_H and optimal H^2 error estimate for nonlinear algorithm jointly ensure the unconditional and optimal L^2 error estimate for the two-grid algorithm.

Theorem 3.3. Error estimates under the discrete H^2 norm

Assume the conditions in Theorem 3.1 hold, then the following estimates hold for some α-robust positive constant C

$$\|e_v^n\|_h \le C(N^{-\min\{r\alpha, 2\}} + h^4 + H^8), \quad \|e_u^n\|_{h,2} \le C(N^{-\min\{r\alpha, 2\}} + h^4 + H^8), \quad \text{for } 1 \le n \le N.$$

Outline

- The nonlinear time fractional biharmonic equation

$$ {}_{0}^{C}D_t^{\alpha} u + \partial_x^4 u - \partial_x^2 = u - u^3 + g(x), \quad 0 < x < \pi, \ 0 < t \leq 1; $$

- Exact solution

$$ u(x, t) = \omega_{1+\alpha}(t) \sin x; $$

- Define

$$ E_0(N, h) = \max_{1 \leq n \leq N} \|U^n - u^n\|_h \quad \text{and} \quad E_2(N, h) = \max_{1 \leq n \leq N} \|U^n - u^n\|_{h,2}, $$

where u^n represents the numerical solution yielded by the nonlinear algorithm or by the two-grid algorithm;

- Choose mesh grading parameter $\gamma = 2/\alpha$ and $h = CH^2$.

表: Numerical spatial convergence for $\alpha = 0.5$ and $\gamma = 4$

Algorithm	N_H	N_h	$E_0(N,h)$	Order	$E_2(N,h)$	Order
Nonlinear		9	2.2185×10^{-4}	—	2.1961×10^{-4}	—
		16	2.5847×10^{-5}	3.7365	2.5764×10^{-5}	3.7244
		25	4.3574×10^{-6}	3.9891	4.3517×10^{-6}	3.9849
		36	1.0199×10^{-6}	3.9826	1.0192×10^{-6}	3.9807
		49	2.9694×10^{-7}	4.0022	2.9684×10^{-7}	4.0013
		64	1.0209×10^{-7}	3.9979	1.0207×10^{-7}	3.9974
Two-Grid	3	9	2.2185×10^{-4}	—	2.1961×10^{-4}	—
	4	16	2.5847×10^{-5}	3.7365	2.5764×10^{-5}	3.7244
	5	25	4.3574×10^{-6}	3.9891	4.3517×10^{-6}	3.9849
	6	36	1.0199×10^{-6}	3.9826	1.0192×10^{-6}	3.9807
	7	49	2.9694×10^{-7}	4.0022	2.9684×10^{-7}	4.0013
	8	64	1.0209×10^{-7}	3.9979	1.0207×10^{-7}	3.9974

表: Numerical temporal convergence for $\alpha = 0.4$, $\gamma = 5$, $h = \frac{\pi}{400}$ and $H = \frac{\pi}{20}$

Algorithm	N	$E_0(N,h)$	Order	$E_2(N,h)$	Order
Nonlinear	40	1.7817×10^{-3}	—	1.7817×10^{-3}	—
	80	5.1187×10^{-4}	1.7994	5.1187×10^{-4}	1.7994
	160	1.4788×10^{-4}	1.7913	1.4788×10^{-4}	1.7913
	320	3.9646×10^{-5}	1.8992	3.9646×10^{-5}	1.8992
Two-Grid	40	1.7817×10^{-3}	—	1.7817×10^{-3}	—
	80	5.1187×10^{-4}	1.7994	5.1187×10^{-4}	1.7994
	160	1.4788×10^{-4}	1.7913	1.4788×10^{-4}	1.7913
	320	3.9646×10^{-5}	1.8992	3.9646×10^{-5}	1.8992

表: Numerical temporal convergence for $\alpha = 0.8$, $\gamma = 5/2$, $h = \frac{\pi}{400}$ and $H = \frac{\pi}{20}$

Algorithm	N	$E_0(N, h)$	Order	$E_2(N, h)$	Order
Nonlinear	40	2.0380×10^{-4}	—	2.0380×10^{-4}	—
	80	5.6574×10^{-5}	1.8489	5.6574×10^{-5}	1.8489
	160	1.4995×10^{-5}	1.9157	1.4995×10^{-5}	1.9157
	320	3.8556×10^{-6}	1.9594	3.8556×10^{-6}	1.9594
Two-Grid	40	2.0380×10^{-4}	—	2.0380×10^{-4}	—
	80	5.6574×10^{-5}	1.8489	5.6574×10^{-5}	1.8489
	160	1.4995×10^{-5}	1.9157	1.4995×10^{-5}	1.9157
	320	3.8556×10^{-6}	1.9594	3.8556×10^{-6}	1.9594

表: CPU times for nonlinear and two-grid compact difference algorithms

Algorithm	N_H	N_h	$E_0(N, h)$	$E_2(N, h)$	CPU times
Nonlinear		361	1.3106×10^{-5}	1.3106×10^{-5}	21.03 s
		400	1.0487×10^{-5}	1.0487×10^{-5}	1 m 53 s
		441	8.7653×10^{-6}	8.7652×10^{-6}	6 m 49 s
		484	7.2904×10^{-6}	7.2904×10^{-6}	29 m 29 s
		529	6.0492×10^{-6}	6.0492×10^{-6}	2 h 25 m 10 s
		576	5.0998×10^{-6}	5.0998×10^{-6}	14 h 10 m 33 s
Two-Grid	19	361	1.3106×10^{-5}	1.3106×10^{-5}	3.22 s
	20	400	1.0487×10^{-5}	1.0487×10^{-5}	5.66 s
	21	441	8.7653×10^{-6}	8.7652×10^{-6}	7.13 s
	22	484	7.2903×10^{-6}	7.2903×10^{-6}	9.17 s
	23	529	6.0492×10^{-6}	6.0492×10^{-6}	11.53 s
	24	576	5.0998×10^{-6}	5.0998×10^{-6}	14.69 s

表: The α-robustness for $\alpha \to 1$

Algorithm		$\alpha = 0.95$	$\alpha = 0.99$	$\alpha = 0.999$	$\alpha = 0.9999$
Nonlinear	$E_0(N, h)$	2.2495×10^{-7}	2.2725×10^{-7}	2.2919×10^{-7}	2.2942×10^{-7}
	$E_2(N, h)$	2.2539×10^{-7}	2.2768×10^{-7}	2.2963×10^{-7}	2.2986×10^{-7}
Two-Grid	$E_0(N, h)$	2.3586×10^{-7}	2.3725×10^{-7}	2.3898×10^{-7}	2.3918×10^{-7}
	$E_2(N, h)$	2.3619×10^{-7}	2.3758×10^{-7}	2.3931×10^{-7}	2.3952×10^{-7}

Conclusions:

- A nonlinear compact difference algorithm for the nonlinear tFBEs is proposed and α-robust error estimates under discrete $L^\infty(L^2)$ and $L^\infty(H^2)$ norms are proved;
- An efficient and accurate two-grid compact difference algorithm for the nonlinear tFBEs is presented by introducing a cubic spline interpolation operator;
- The linearity and boundedness properties of the cubic spline interpolation operator are discussed; and then Unconditional and optimal α-robust error estimates in the sense of discrete $L^\infty(L^2)$ and $L^\infty(H^2)$ norms with the accuracy $\mathcal{O}\left(N^{-\min\{r\alpha,2\}} + h^4 + H^8\right)$ are proved;
- Numerical experiments are given to show accuracy and efficiency of the method.

Future work:

- Variable coefficient tFBEs ?
- Other kinds of boundary conditions?
- Rough initial and source data?

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Thanks for your attention!