A global dynamics preserving method for a class of time-fractional epidemic model with reaction-diffusion

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Motivation

Global dynamics of fractional epidemic model

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Global dynamics preserving methods for integer order epidemic model

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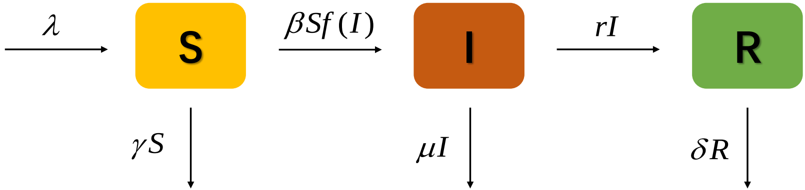
A time-fractional SIR system

A time-fractional SIR system

$S(t)$, $I(t)$, $R(t)$: number of susceptible, infected, recovered individuals.
γ, μ, δ: death rates of S, I, R.
λ: recruitment rate of the total population.
β: infection rate of S infected by I.
$f(I)$: general incidence rate.
r: recovery rate of the infected individuals.



A time-fractional SIR system

A time-fractional SIR system

Consider an time-fractional SIR system

$$\left\{ \begin{array}{l} D_t^\alpha S(t) = \lambda - \gamma S(t) - \beta S(t) f(I(t)) \quad \text{for } t \in (0, \infty) \\ D_t^\alpha I(t) = \beta S(t) f(I(t)) - (\mu + r) I(t) \quad \text{for } t \in (0, \infty) \\ D_t^\alpha R(t) = r I(t) - \delta R(t) \quad \text{for } t \in (0, \infty) \end{array} \right. \tag{1}$$

with initial conditions $S(0) = S_0$, $I(0) = I_0$ and $R(0) = R_0$.
$\lambda, \gamma, \beta, \mu, r, \delta$ are positive constants.
D_t^α is Caputo derivative with $\alpha \in (0, 1)$,

$$D_t^\alpha S(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} S'(s)\, ds.$$

Properties of the time-fractional SIR system

Positive and bounded properties

There exists an unique nonnegative and bounded solution of system (1).

Disease-free equilibrium point

$E_0 = (\frac{\lambda}{\gamma}, 0, 0)$ is the disease-free equilibrium point of system (1): if $(S_0, I_0, R_0) = (\frac{\lambda}{\gamma}, 0, 0)$ then $(S(t), I(t), R(t)) = (\frac{\lambda}{\gamma}, 0, 0)$ are solutions of system (1).

Endemic equilibrium point

If the reproduction number $R_0 = \frac{\beta\lambda f'(0)}{\gamma(\mu+r)} > 1$, system (1) has at least one endemic equilibrium $E^* = (S^*, I^*, R^*)$ with positive S^*, I^*, R^* satisfying

$$\begin{cases} \lambda - \gamma S^* - \beta S^* f(I^*) = 0, \\ \beta S^* f(I) - (\mu + r) I^* = 0, \\ r I^* - \delta R^* = 0. \end{cases}$$

Globally asymptotically stable of equilibrium point

Stable and asymptotical of equilibrium point

Let $x(t) \equiv 0$ (without losing generality) be an equilibrium point ($f(0) = 0$) of the following system:

$$D_t^\alpha x(t) = f(x(t)), \quad x(t_0) = x_0.$$

(1) The equilibrium point $x(t) \equiv 0$ is stable if for all $\epsilon > 0$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that for $t > t_0$

$$\textit{if } \|x_0\| < \delta, \textit{ then } \|x(t)\| < \epsilon.$$

(2) The equilibrium point $x(t) \equiv 0$ is (globally) asymptotical if there exists a $\delta = \delta(t_0)(= \infty) > 0$ such that

$$\textit{if } \|x_0\| < \delta, \textit{ then } \lim_{t \to \infty} x(t) = 0.$$

Global dynamic properties of the time-fractional SIR system

Globally asymptotical stability of disease-free equilibrium point E_0

If the reproduction number $R_0 = \frac{\beta\lambda f'(0)}{\gamma(\mu+r)} \leq 1$, the disease-free equilibrium point E_0 of system (1) is globally asymptotically stable.

Globally asymptotical stability of the endemic equilibrium point E^*

If the reproduction number $R_0 = \frac{\beta\lambda f'(0)}{\gamma(\mu+r)} > 1$, the endemic equilibrium point E^* of system (1) is globally asymptotically stable.

An L1 scheme for time-fractional SIR system

An L1 scheme for time-fractional SIR system

Consider the time-fractional SIR system discretized by L1 scheme

$$\begin{cases} \delta_n^\alpha S_n = \lambda - \gamma S_n - \beta S_n f(I_{n-1}), \\ \delta_n^\alpha I_n = \beta S_n f(I_{n-1}) - (\mu + r) I_n, \\ \delta_n^\alpha R_n = r I_n - \delta R_n \end{cases} \tag{2}$$

for $n = 1, 2, \ldots$ with initial conditions S_0, I_0 and R_0.

δ_n^α is the L1 scheme: set $b_i^{(1-\alpha)} = \frac{(\Delta t)^{-\alpha}}{\Gamma(1-\alpha)} [(i+1)^{1-\alpha} - i^{1-\alpha}]$

$$\delta_n^\alpha S_n := \frac{1}{\Gamma(1-\alpha)} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (t_n - s)^{-\alpha} \frac{S_i - S_{i-1}}{\Delta t}\, ds$$

$$= \sum_{i=1}^{n} b_{i-1}^{(1-\alpha)} (S_{n-i+1} - S_{n-i}) = S_n - \sum_{i=1}^{n-1} (b_{i-1}^{(1-\alpha)} - b_i^{(1-\alpha)}) S_{n-i} - b_{n-1}^{(1-\alpha)} S_0.$$

Positivity preserving property

Equivalent form of L1 discrete system (2)

$$\begin{cases} (1+\gamma+\beta f(I_{n-1}))S_n = \lambda + \sum_{i=1}^{n-1}(b_{i-1}^{(1-\alpha)} - b_i^{(1-\alpha)})S_{n-i} + b_{n-1}^{(1-\alpha)}S_0, \\ (1+\mu+r)I_n = \beta S_n f(I_{n-1}) + \sum_{i=1}^{n-1}(b_{i-1}^{(1-\alpha)} - b_i^{(1-\alpha)})I_{n-i} + b_{n-1}^{(1-\alpha)}I_0, \\ (1+\delta)R_n = rI_n + \sum_{i=1}^{n-1}(b_{i-1}^{(1-\alpha)} - b_i^{(1-\alpha)})R_{n-i} + b_{n-1}^{(1-\alpha)}R_0. \end{cases}$$

Positivity preserving property

The discrete system (2) preserve the positivity property of the continuous system.

Boundness preserving property

Equivalent form of L1 discrete system (2)

$$\begin{cases} (1+\gamma+\beta f(I_{n-1}))S_n = \lambda + \sum_{i=1}^{n-1}(b_{i-1}^{(1-\alpha)} - b_i^{(1-\alpha)})S_{n-i} + b_{n-1}^{(1-\alpha)}S_0, \\ (1+\mu+r)I_n = \beta S_n f(I_{n-1}) + \sum_{i=1}^{n-1}(b_{i-1}^{(1-\alpha)} - b_i^{(1-\alpha)})I_{n-i} + b_{n-1}^{(1-\alpha)}I_0, \\ (1+\delta)R_n = rI_n + \sum_{i=1}^{n-1}(b_{i-1}^{(1-\alpha)} - b_i^{(1-\alpha)})R_{n-i} + b_{n-1}^{(1-\alpha)}R_0. \end{cases}$$

Key point of proof

Set $S_n + I_n + R_n = G_n$, then

$$(1+\min\{\gamma,\mu,\delta\})\, G_n \le \lambda + \sum_{i=1}^{n-1}\left(b_{i-1}^{(1-\alpha)} - b_i^{(1-\alpha)}\right) G_{n-i} + b_{n-1}^{(1-\alpha)} G_0$$

Boundness preserving property

Key point of proof

By using mathematical induction,

$$G_n \leq (G_0 + \lambda) \sum_{i=1}^{n} \frac{i}{(1 + \min\{\gamma, \mu, \delta\})^i}.$$

By using $\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}$ $(-1 < x < 1)$,

$$\lim_{n \to \infty} G_n \leq (\lambda + G_0) \sum_{i=1}^{\infty} \frac{i}{(1 + \min\{\gamma, \mu, \delta\})^i} = \frac{(\lambda + G_0)(1 + \min\{\gamma, \mu, \delta\})}{(\min\{\gamma, \mu, \delta\})^2}.$$

Boundness preserving properties

The discrete system (2) preserve the boundness property of the continuous system.

Globally asymptotically stable of equilibrium point

Stable and asymptotical of equilibrium point

Let $x_n = 0$ be an equilibrium point ($f(x_n) = 0$) of the following system:

$$\delta_n^\alpha x_n = f(x_n).$$

(1) The equilibrium point $x_n = 0$ is stable if for all $\epsilon > 0$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that for $n \geq 0$

$$if \ |x_0| < \delta, \ then \ |x_n| < \epsilon.$$

(2) The equilibrium point $x_n = 0$ is (globally) asymptotical if there exists a $\delta = \delta(t_0)(= \infty) > 0$ such that

$$if \ |x_0| < \delta, \ then \ \lim_{n \to \infty} x_n = 0.$$

Discrete fractional Lyapunov method

Discrete fractional Lyapunov method

Consider the following L1 discrete fractional systems:

$$\delta_n^\alpha x_n = g(x_n),$$

with initial condition x_0 and $g(0) = 0$ (equilibrium point $x_n \equiv 0$). If there exists a positive define functions $V(x_n)$ and $W(x_n)$ such that

$$\delta_n^\alpha V(x_n) \leq -W(x_n).$$

Then the equilibrium point $x_n \equiv 0$ is globally asymptotically stable.

Positive define function

Let $U(x)$ be continuously differentiable function and $U(0) = 0$. If all $x \neq 0$ satisfies $U(x) > 0$, $U(x)$ is said to be positive definite.

Globally asymptotically stable of disease-free equilibrium point

Discrete Lyapunov function for disease-free equilibrium point

$$V_n = \sum_{i=1}^{n} b_{n-i}^{(1-\alpha)} \left[\Phi\left(\frac{S_i}{S_0}\right) S_0 + I_i \right] + \beta S_0 f'(0) I_n,$$

where

$$\Phi(w) = w - 1 - lnw.$$

Globally asymptotically stable of disease-free equilibrium point

Key point of proof

$$\delta_n^\alpha V^n \leq -\sum_{i=1}^{n} b_{n-i}^{(1-\alpha)} \left[\frac{\gamma(S_i - S_0)^2}{S_i} + (\mu + r)(1 - R_0)I_i\right].$$

Globally asymptotically stable of disease-free equilibrium point

If the reproduction number $R_0 = \frac{\beta\lambda f'(0)}{\gamma(\mu+r)} \leq 1$, the disease-free equilibrium point E_0 of system (2) is globally asymptotically stable.

Globally asymptotically stable of endemic equilibrium point

Discrete Lyapunov function for endemic equilibrium point

$$V_n = \sum_{i=1}^{n} b_{n-i}^{(1-\alpha)} \left[\Phi\left(\frac{S_i}{S^*}\right) S^* + \Phi\left(\frac{I_i}{I^*}\right) I^* \right] + \beta S^* f(I^*) \Phi\left(\frac{I_n}{I^*}\right),$$

where

$$\Phi(w) = w - 1 - lnw.$$

Globally asymptotically stable of endemic equilibrium point

Key point of proof

$$\delta_n^\alpha V^n \leq -\sum_{i=1}^{n} b_{n-i}^{(1-\alpha)} \Psi_i,$$

where

$$\Psi_i = \beta S^* f(I^*) \left[\Phi\left(\frac{S^*}{S_i}\right) + \Phi\left(\frac{I_{i-1} f(I^*)}{I^* f(I_{i-1})}\right) + \Phi\left(\frac{S_i I^* f(I_{i-1})}{S^* I_i f(I^*)}\right)\right] + \frac{\gamma(S_i - S^*)^2}{S_i}.$$

Globally asymptotically stable of endemic equilibrium point

If the reproduction number $R_0 = \frac{\beta\lambda f'(0)}{\gamma(\mu+r)} > 1$, the endemic equilibrium point E^* of system (2) is globally asymptotically stable.

A time-fractional dynamic system with diffusion

Time-fractional SIR system with diffusion

Consider an time-fractional SIR system with diffusion

$$\begin{cases} D_t^\alpha S = d_1 \Delta S + \lambda - \gamma S - \beta S f(I) \quad \text{for } (x,t) \in \Omega \times (0,\infty) \\ D_t^\alpha I = d_2 \Delta I + \beta S f(I) - (\mu + r) I \quad \text{for } (x,t) \in \Omega \times (0,\infty) \\ D_t^\alpha R = d_3 \Delta R + rI - \delta R \quad \text{for } (x,t) \in \Omega \times (0,\infty) \\ \frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial R}{\partial n} = 0 \quad \text{for } (x,t) \in \partial\Omega \times (0,\infty). \end{cases} \tag{3}$$

with initial conditions $S(x,0) = S_0(x)$, $I(x,0) = I_0(x)$ and $R(x,0) = R_0(x)$.
In which, $\lambda, \gamma, \beta, \mu, r, \delta$ are positive constants.
D_t^α is Caputo derivative with $\alpha \in (0,1)$,

$$D_t^\alpha S(x,t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial}{\partial s} S(x,s)\, ds.$$

Time fractional discrete initial boundary value problem

L1 finite difference scheme

Consider the time-fractional SIR system with diffusion discretized by L1 finite difference scheme

$$\begin{cases} \delta_n^\alpha S_n^m = \frac{S_n^{m+1}-2S_n^m+S_n^{m-1}}{(\Delta x)^2} + \lambda - \gamma S_n^m - \beta S_n^m f(I_{n-1}^m), \\ \delta_n^\alpha I_n^m = \frac{I_n^{m+1}-2I_n^m+I_n^{m-1}}{(\Delta x)^2} + \beta S_n^m f(I_{n-1}^m) - (\mu + r) I_n^m, \\ \delta_n^\alpha R_n^m = \frac{R_n^{m+1}-2R_n^m+R_n^{m-1}}{(\Delta x)^2} + r I_n^m - \delta R_n^m \end{cases} \quad (4)$$

for $n = 1, 2, ...$ and $m = 1, ..., M-1$ with initial conditions $S_0^m = S_0(x_m)$, $I_0^m = I_0(x_m)$ and $R_0^m = R_0(x_m)$.

Positivity preserving property

The discrete system (4) preserve the positivity property of the continuous system (3). M-matrix

Boundness preserving property

The discrete system (4) preserve the boundness property of the continuous system (3). Mathematical induction

Globally asymptotical stability preserving property

The discrete system (4) preserve the Globally asymptotically stable property of the continuous system (3). Sum of initial value problem case

A time-fractional dynamic system

Time-fractional SIR system

Consider an time-fractional SIR system

$$\begin{cases} D_t^{0.8} S = \Delta S + \lambda - \gamma S - \beta SI \\ D_t^{0.8} I = \Delta I + \beta SI - (\mu + r) I \\ D_t^{0.8} R = \Delta R + rI - \delta R \end{cases}$$

with initial conditions $S(x, 0) = 0.5$, $I(x, 0) = e^{-x}$ and $R(x, 0) = e^{-x}$. Let $\lambda = 0.2$, $\beta = 0.2144$, $\gamma = \delta = 0.2$, $\mu = 0.2$, $r = 0.25$. In this case, the reproduction number $R_0 = \frac{\beta\lambda}{\gamma(\mu+r)} = 0.7238 < 1$ and the disease-free equilibrium point $E_0 = (\frac{\lambda}{\mu}, 0, 0) = (1, 0, 0)$.

The disease-free equilibrium $E_0 = (1, 0, 0)$ is globally asymptotically stables

Set $\Delta t = 0.1$ and $\Delta x = 0.1$. Initial conditions of susceptible individuals: $S(x, 0) = 0.5$. The disease-free equilibrium of susceptible individuals: 1.

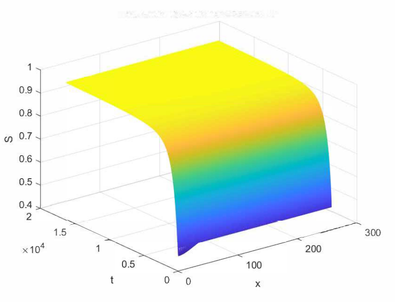


Figure: Susceptible individuals

The disease-free equilibrium $E_0 = (1, 0, 0)$ is globally asymptotically stables

Set $\Delta t = 0.1$ and $\Delta x = 0.1$. Initial conditions of susceptible individuals: $I(x, 0) = e^{-x}$. The disease-free equilibrium of susceptible individuals: 0.

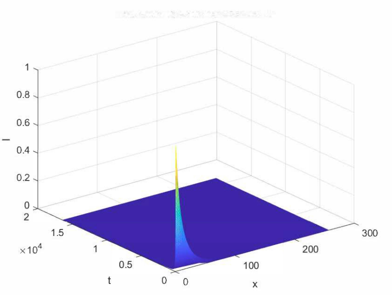


Figure: Infected individuals

The disease-free equilibrium $E_0 = (1, 0, 0)$ is globally asymptotically stables

Set $\Delta t = 0.1$ and $\Delta x = 0.1$. Initial conditions of susceptible individuals: $R(x, 0) = e^{-x}$. The disease-free equilibrium of susceptible individuals: 0.

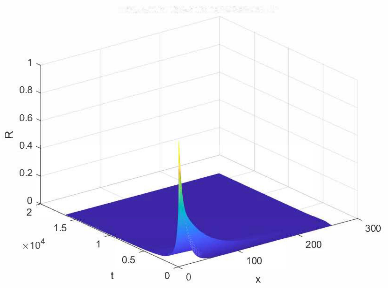


Figure: Recovered individuals

A time-fractional dynamic system

Time-fractional SIR system

Consider an time-fractional SIR system

$$\begin{cases} D_t^{0.8} S = \Delta S + \lambda - \gamma S - \beta S I \\ D_t^{0.8} I = \Delta I + \beta S I - (\mu + r) I \\ D_t^{0.8} R = \Delta R + r I - \delta R \end{cases}$$

with initial conditions $S(x,0) = 0.5$, $I(x,0) = e^{-x}$ and $R(x,0) = e^{-x}$. Let $\lambda = 0.2$, $\beta = 0.6271$, $\gamma = \delta = 0.2$, $\mu = 0.2$, $r = 0.25$. In this case, the reproduction number $R_0 = \frac{\beta\lambda}{\gamma(\mu+r)} = 1.3816 > 1$ and the disease-free equilibrium point
$E^* = (\frac{\mu+r}{\beta}, \frac{\lambda}{\mu+r} - \frac{\gamma}{\beta}, \frac{\lambda r}{(\mu+r)\delta} - \frac{r\gamma}{\beta\delta}) = (0.7238, 0.1227, 0.1534)$.

The endemic equilibrium $E^* = (0.7238, 0.1227, 0.1534)$ is globally asymptotically stables.

Let $\Delta t = 0.1$ and $\Delta x = 0.1$. Initial conditions of susceptible individuals: $S(x, 0) = 0.5$. The disease-free equilibrium of susceptible individuals: 0.7238.

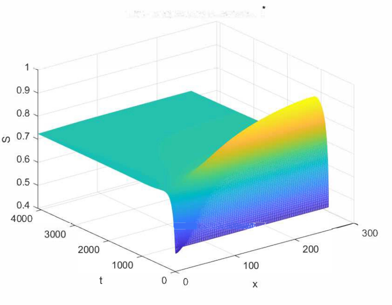


Figure: Susceptible individuals

The endemic equilibrium $E^* = (0.7238, 0.1227, 0.1534)$ is globally asymptotically stables.

Let $\Delta t = 0.1$ and $\Delta x = 0.1$. Initial conditions of susceptible individuals: $S(x, 0) = e^{-x}$. The disease-free equilibrium of susceptible individuals: 0.1227.

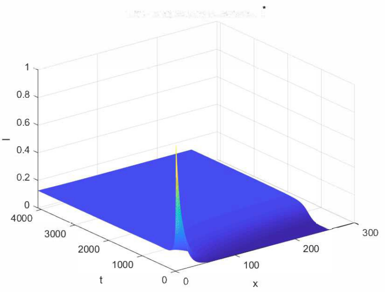


Figure: Infected individuals

The endemic equilibrium $E^* = (0.7238, 0.1227, 0.1534)$ is globally asymptotically stables.

Let $\Delta t = 0.1$ and $\Delta x = 0.1$. Initial conditions of susceptible individuals: $R(x, 0) = e^{-x}$. The disease-free equilibrium of susceptible individuals: 0.1534.

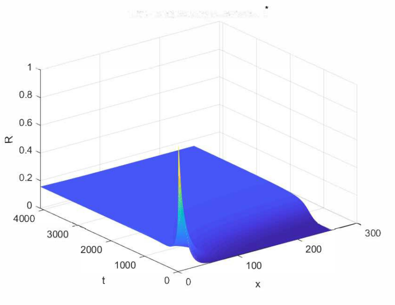


Figure: Recovered individuals

Thank you for your attention!

For more details, please see
'Global dynamics for a class of discrete fractional epidemic model with reaction-diffusion'
on https://arxiv.org/abs/2208.06548