Numerical study for the spatial-fractional complex Ginzburg-Landau equation in finite difference setting

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Background

Fractional problems:

- Time-fractional problems;
- Spatial-fractional problems

Spatial-fractional Ginzburg-Landau equation

- Ginzburg-Landau equation[1]
- Fractional quantum mechanics and Lévy path integrals [2],[3]
- Fractional Ginzburg-Landau equation [4]

1 V.L. Ginzburg, L.D. Landau, On the theory of superconductivity, Zh. Eksp. Teor. Fiz. 20 (1950) 1064–1082.

2 N. Laskin, Fractional quantum mechanics, Physical Review E 62 (2000) 3135–3145.

3 N. Laskin, Fractional quantum mechanics and Lévy path integrals, Physics Letters A 268 (2000) 298–305.

4 V. Tarasov, G. Zaslavsky, Fractional Ginzburg-Landau equation for fractal media, Physica A 354 (2005) 249–261.

Existed work for the spatial-fractional Ginzburg-Landau equation:

1. Pu, X.K., Guo, B.L.: Well-posedness and dynamics for the fractional Ginzburg-Landau equation. Appl. Anal. 92, 31–33 (2013)
2. X. Zhao, Z.Z. Sun, Z. Hao, A fourth-order compact ADI scheme for two-dimensional nonlinear space fractional Schrödinger equation, SIAM J. Sci. Comput. 36(6) (2014) 2865–2886.
3. Li, M., Huang, C.M., Wang, N.: Galerkin element method for the nonlinear fractional Ginzburg-Landau equation. Appl. Numer. Math. 118, 131–149 (2017)
4. Z. Hao, Z. Sun, A linearized high-order difference scheme for the fractional Ginzburg-Landau equation. Numer Methods Partial Differential Eq. 33(1) (2017) 105–124
5. Z. Xu, W. Cai, C. Jiang, Y. Wang, On the L^∞ convergence of a conservative Fourier pseudo-spectral method for the space fractional nonlinear Schrödinger equation, Numer Methods Partial Differential Eq. 37(2) 2021, 1591–1611.
6. N. Wang, C. Huang, An efficient split-step quasi-compact finite difference method for the nonlinear fractional Ginzburg-Landau equations, Comput. Math. Appl., (2018) 2223–2242
7. Zeng, W., Xiao, A., Li, X.: Error estimate of Fourier pseudo-spectral method for multidimensional nonlinear complex fractional Ginzburg-Landau equations. Appl. Math. Lett. 93, 40–45 (2019)
8. ······

Motivation

- A significant amount of relevant works on spatial fractional derivative have already been developed, they often lack theoretical support, or only offer error estimates in L^2-norm.
- L^2-error estimates do not provide insight into the local numerical error as the time evolves. Error estimates in the pointwise sense are relevant in theoretical analysis and practical application.
- Embedding theorem [1] for the fractional Riesz derivative in one dimension, is not applicable to the two-dimensional case.
- Computational efficiency for high-dimensional spatial-fractional problem is frequently low and stability is not very good.

It is necessary to explore stable, accurate numerical algorithm and numerical analysis tool for high-dimensional spatial-fractional problem!

[1] Kirkpatrick, K., Lenzmann, E., Staffilani, G.: On the continuum limit for discrete NLS with long-range lattice interactions. Commun. Math. Phys. **317**, 563–591 (2013).

Problem form:

Two-dimensional space-fractional Ginzburg-Landau equation[1]

$$\partial_t u - (\nu + \mathbf{i}\eta)(\partial_x^\alpha + \partial_y^\beta)u + (\kappa + \mathbf{i}\zeta)|u|^2 u - \gamma u = 0, (x, y) \in \Omega,\ 0 < t \leqslant T, \quad (1)$$

$$u(x, y, t) = 0, (x, y) \in \mathbb{R}^2 \backslash \Omega,\ 0 < t \leqslant T, \quad (2)$$

$$u(x, y, 0) = \varphi(x, y), (x, y) \in \mathbb{R}^2, \quad (3)$$

where $\Omega = (x_l, x_r) \times (y_d, y_u)$, $\partial\Omega$ is the boundary of Ω, $1 < \alpha, \beta \leqslant 2$, $\nu > 0,\ \kappa > 0$, η, ζ, γ are given real constants, $\varphi(x, y)$ is a given function vanishing in $\mathbb{R}^2 \backslash \Omega$. ∂_x^α in (1) denotes the Riesz fractional derivative operator for $1 < \alpha \leqslant 2$ and is defined as

$$\partial_x^\alpha u(x, y, t) = -\frac{1}{2\cos(\alpha\pi/2)\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} |x - \xi|^{1-\alpha} u(\xi, y, t) d\xi. \quad (4)$$

[1] N. Wang, C. Huang, An efficient split-step quasi-compact finite difference method for the nonlinear fractional Ginzburg-Landau equations, Comput. Math. Appl., (2018) 2223–2242

BDF2-ADI and CBDF2-ADI schemes

BDF2 scheme[1]:

$$\delta_t^+ u_{ij}^k - (\nu + \mathbf{i}\eta)(\delta_x^\alpha + \delta_y^\beta) u_{ij}^{k+1} + (\kappa + \mathbf{i}\zeta)|\widetilde{u}_{ij}^{k+1}|^2 \widetilde{u}_{ij}^{k+1} - \gamma \widetilde{u}_{ij}^{k+1} = 0,$$
$$(i,j) \in \omega,\ 1 \leqslant k \leqslant N-1, \tag{5}$$

$$\delta_t u_{ij}^{\frac{1}{2}} - (\nu + \mathbf{i}\eta)(\delta_x^\alpha + \delta_y^\beta) u_{ij}^1 + (\kappa + \mathbf{i}\zeta)|u_{ij}^0|^2 u_{ij}^0 - \gamma u_{ij}^0 = 0, \quad (i,j) \in \omega, \tag{6}$$

$$u_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \bar{\omega}, \tag{7}$$

$$u_{ij}^k = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant k \leqslant N. \tag{8}$$

where

$$\delta_t^+ u^k = \frac{3u^{k+1} - 4u^k + u^{k-1}}{2\tau}, \quad \widetilde{u}^{k+1} = 2u^k - u^{k-1},$$

$$u^{k+\frac{1}{2}} = \frac{1}{2}(u^{k+1} + u^k), \quad \delta_t u^{k+\frac{1}{2}} = \frac{1}{\tau}(u^{k+1} - u^k).$$

[1] C. Çelik C, M. Duman, Crank-Nicolson method for the fractional diffusion equation with the Riesz fractional derivative, Journal of Computational Physics 231(4) (2012) 1743–1750

Theorem (Uniqueness)

The linearized BDF2 scheme (5)–(8) *is uniquely solvable.*

Denote

$$e_{ij}^k = U_{ij}^k - u_{ij}^k, \quad (i,j) \in \bar{\omega},\ 0 \leqslant k \leqslant N.$$

We have the following convergence result.

Theorem (Convergence for BDF2)

Let $u(x,y,t)$ *be the exact solution of the problem* (1)–(3), $\{u_{ij}^k \mid (i,j) \in \bar{\omega},\ 0 \leqslant k \leqslant N\}$ *be the solution of BDF2 scheme* (5)–(8). *Denote* $h = \max\{h_x, h_y\}$, $\tau = \rho h^{\frac{1}{2}+\varepsilon}$ (ρ, ε *are positive constants*), $C_u = \sup\limits_{(x,y)\in\Omega, t\in[0,T]} |u(x,y,t)|$, $C_M = 9(1+C_u)^2$, $c_3 = 3C_M\sqrt{\kappa^2+\zeta^2} + |\gamma| + 1$ *and* $C_1 = \sqrt{\frac{(20c_2^2c_3 + 2c_1^2)(x_r - x_l)(y_u - y_d)}{c_3}} \exp(4c_3T)$. *If* $C_1(\rho^2 h^{2\varepsilon} + h) \leqslant 1$ *and* $2\tau c_3 \leqslant \frac{1}{2}$, *we have*

$$\|e^k\| \leqslant C_1(\tau^2 + h^2), \quad 0 \leqslant k \leqslant N. \tag{9}$$

The proof is based on the following results and the induction method.

Lemma

[a] *For any grid function* $v \in \mathring{\mathcal{V}}_h$, *there exist fractional symmetric positive quotient operators* $\delta_x^{\alpha/2}$ *and* $\delta_y^{\beta/2}$, *such that*

$$(-\delta_x^{\alpha} v, v) = (\delta_x^{\alpha/2} v, \delta_x^{\alpha/2} v), \quad (-\delta_y^{\beta} v, v) = (\delta_y^{\beta/2} v, \delta_y^{\beta/2} v).$$

[a] P. Wang, C. Huang, Split-step alternating direction implicit difference scheme for the fractional Schrödinger equation in two dimensions, Computers and Mathematics with Applications 71(5) (2016) 1114–1128.

Lemma

[b] *For any grid function* $v \in \mathring{\mathcal{V}}_h$, *there exists a fractional symmetric positive quotient operator* $\Lambda_h^{\frac{1}{2}}$ *such that*

$$-(\Lambda_h v, v) = (\Lambda_h^{\frac{1}{2}} v, \Lambda_h^{\frac{1}{2}} v).$$

[b] X. Zhao, Z.Z. Sun, Z. Hao, A fourth-order compact ADI scheme for two-dimensional nonlinear space fractional Schrödinger equation, SIAM Journal on Scientific Computing 36(6) (2014) 2865–2886.

Lemma

For any grid function $v \in \mathring{\mathcal{V}}_h$, *it holds that*

$$\mathrm{Re}(\delta_t^{+} v^k, v^{k+1}) = \frac{1}{4\tau}[(\|v^{k+1}\|^2 + \|2v^{k+1} - v^k\|^2) - (\|v^k\|^2 + \|2v^k - v^{k-1}\|^2) + \|v^{k+1} - 2v^k + v^{k-1}\|^2].$$

Compact BDF2 scheme:

$$\mathcal{A}_h^{\alpha,\beta}\delta_t^+ u_{ij}^k - (\nu + \mathbf{i}\eta)\Lambda_h u_{ij}^{k+1} + (\kappa + \mathbf{i}\zeta)\mathcal{A}_h^{\alpha,\beta}|\widetilde{u}_{ij}^{k+1}|^2\widetilde{u}_{ij}^{k+1} - \gamma\mathcal{A}_h^{\alpha,\beta}\widetilde{u}_{ij}^{k+1} = 0,$$
$$(i,j) \in \omega,\ 1 \leqslant k \leqslant N-1, \tag{10}$$

$$\mathcal{A}_h^{\alpha,\beta}\delta_t u_{ij}^{\frac{1}{2}} - (\nu + \mathrm{i}\eta)\Lambda_h u_{ij}^1 + (\kappa + \mathrm{i}\zeta)\mathcal{A}_h^{\alpha,\beta}|u_{ij}^0|^2 u_{ij}^0 - \gamma\mathcal{A}_h^{\alpha,\beta}u_{ij}^0 = 0, \quad (i,j) \in \omega, \tag{11}$$

$$u_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \bar{\omega}, \tag{12}$$

$$u_{ij}^k = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant k \leqslant N, \tag{13}$$

where[1]

$$\mathcal{A}_h^{\alpha,\beta} v_{ij} = \mathcal{A}_x^\alpha \mathcal{A}_y^\beta v_{ij}, \quad \Lambda_h v_{ij} = (\mathcal{A}_y^\beta \delta_x^\alpha + \mathcal{A}_x^\alpha \delta_y^\beta) v_{ij},\ (x_i, y_j) \in \Omega_h$$

with

$$\mathcal{A}_x^\alpha v_{ij} = \begin{cases} \dfrac{\alpha}{24} v_{i-1,j} + \left(1 - \dfrac{\alpha}{12}\right) v_{ij} + \dfrac{\alpha}{24} v_{i+1,j}, & 1 \leqslant i \leqslant M_1 - 1,\ 0 \leqslant j \leqslant M_2, \\ 0, \quad i = 0 \text{ or } M_1,\ 0 \leqslant j \leqslant M_2, \end{cases}$$

[1] X. Zhao, Z.Z. Sun, Z. Hao, A fourth-order compact ADI scheme for two-dimensional nonlinear space fractional Schrödinger equation, SIAM Journal on Scientific Computing 36(6) (2014) 2865–2886.

Theorem (Convergence for CBDF2)

[a] *Let $u(x,y,t)$ be the exact solution of the problem (1)–(3), $\{u_{ij}^k \mid (i,j) \in \bar{\omega},\ 0 \leqslant k \leqslant N\}$ be the solution of compact BDF2 scheme (10)–(13). Denote $h = \max\{h_x, h_y\}$, $\tau = \rho h^{\frac{1}{2}+\varepsilon}$ (ρ, ε are positive constants), $c_4 = 3\sqrt{\kappa^2+\zeta^2}C_M + |\gamma|$ and*

$$C_2 = 2\sqrt{6}C_r e^{6(11c_4+1)T}\sqrt{(45+2T)(x_r-x_l)(y_u-y_d)}.$$

If $C_2(\rho^2 h^{2\varepsilon} + h^3) \leqslant 1$ and $2(c_4+1)\tau \leqslant \frac{1}{6}$, it holds that

$$\|e^k\| \leqslant C_2(\tau^2 + h^4), \quad 0 \leqslant k \leqslant N. \tag{14}$$

[a] Q. Zhang, X. Lin, K. Pan, Y. Ren, Linearized ADI schemes for two-dimensional space-fractional nonlinear Ginzburg-Landau equation, Computers and Mathematics with Applications, 80 (2020) 1201–1220

Lemma

[a] *For any grid function* $v \in \mathring{\mathcal{V}}_h$, *it holds that*

$$\|\mathcal{A}_h^{\alpha,\beta} v\| \leqslant \|v\|$$

and

$$\frac{1}{3}\|v\|^2 \leqslant (\mathcal{A}_h^{\alpha,\beta} v, v) \leqslant \|v\|^2.$$

[a] X. Zhao, Z.Z. Sun, Z. Hao, A fourth-order compact ADI scheme for two-dimensional nonlinear space fractional Schrödinger equation, SIAM Journal on Scientific Computing 36(6) (2014) 2865–2886.

Lemma

For any grid function $v \in \mathring{\mathcal{V}}_h$, *it holds that*

$$\mathrm{Re}(\mathcal{A}_h^{\alpha,\beta}\delta_t^+ v^k, v^{k+1}) = \frac{1}{4\tau}[B^{k+1} - B^k + (\mathcal{A}_h^{\alpha,\beta}(v^{k+1} - 2v^k + v^{k-1}), v^{k+1} - 2v^k + v^{k-1})],$$

where

$$B^{k+1} = (\mathcal{A}_h^{\alpha,\beta} v^{k+1}, v^{k+1}) + (\mathcal{A}_h^{\alpha,\beta}(2v^{k+1} - v^k), 2v^{k+1} - v^k).$$

BDF2-ADI scheme:

Adding small term $\frac{4\tau^2}{9}(\nu+\mathrm{i}\eta)^2\delta_x^\alpha\delta_y^\beta\delta_t^+u_{ij}^k$ into (5), and the small term goes into the local truncation error R_{ij}^{k+1}. Thus, we have

$$\begin{aligned}&\delta_t^+u_{ij}^k-\frac{2\tau}{3}(\nu+\mathbf{i}\eta)(\delta_x^\alpha+\delta_y^\beta)\delta_t^+u_{ij}^k+\frac{4\tau^2}{9}(\nu+\mathrm{i}\eta)^2\delta_x^\alpha\delta_y^\beta\delta_t^+u_{ij}^k\\&=(\nu+\mathbf{i}\eta)(\delta_x^\alpha+\delta_y^\beta)\left(\frac{4}{3}u_{ij}^k-\frac{1}{3}u_{ij}^{k-1}\right)-(\kappa+\mathbf{i}\zeta)|\widetilde{u}_{ij}^{k+1}|^2\widetilde{u}_{ij}^{k+1}+\gamma\widetilde{u}_{ij}^{k+1},\\&\qquad(i,j)\in\omega,\ 1\leqslant k\leqslant N-1.\end{aligned}\tag{15}$$

Then we can multiply both sides of (15) by τ and introduce intermediate variable u_{ij}^*, u_{ij}^{**}, ADI method reads as follows

$$\begin{aligned}\left(\mathcal{I}-\frac{2\tau}{3}(\nu+\mathbf{i}\eta)\delta_x^\alpha\right)u_{ij}^*&=\tau(\nu+\mathbf{i}\eta)(\delta_x^\alpha+\delta_y^\beta)\left(\frac{4}{3}u_{ij}^k-\frac{1}{3}u_{ij}^{k-1}\right)\\&\quad-\tau(\kappa+\mathbf{i}\zeta)|\widetilde{u}_{ij}^{k+1}|^2\widetilde{u}_{ij}^{k+1}+\gamma\tau\widetilde{u}_{ij}^{k+1},\end{aligned}\tag{16}$$

$$\left(\mathcal{I}-\frac{2\tau}{3}(\nu+\mathbf{i}\eta)\delta_y^\beta\right)u_{ij}^{**}=u_{ij}^*,\tag{17}$$

$$u_{ij}^{k+1}=u_{ij}^{**}+\frac{4}{3}u_{ij}^k-\frac{1}{3}u_{ij}^{k-1},\quad 1\leqslant k\leqslant N-1.\tag{18}$$

CBDF2-ADI scheme:
Adding a small term $\frac{4\tau^2}{9}(\nu+\mathbf{i}\eta)^2\delta_x^\alpha\delta_y^\beta\delta_t^+ u_{ij}^k$ into (10), omitting the small term and rearranging the result, we have

$$\left(\mathcal{A}_x^\alpha - \frac{2\tau}{3}(\nu+\mathbf{i}\eta)\delta_x^\alpha\right)\left(\mathcal{A}_y^\beta - \frac{2\tau}{3}(\nu+\mathbf{i}\eta)\delta_y^\beta\right)\delta_t^+ u_{ij}^k + (\kappa+\mathbf{i}\zeta)\mathcal{A}_h^{\alpha,\beta}|\widetilde{u}_{ij}^{k+1}|^2\widetilde{u}_{ij}^{k+1}$$
$$=(\nu+\mathbf{i}\eta)\Lambda_h\left(\frac{4}{3}u_{ij}^k - \frac{1}{3}u_{ij}^{k-1}\right) + \gamma\mathcal{A}_h^{\alpha,\beta}\widetilde{u}_{ij}^{k+1}, \quad (i,j)\in\omega,\ 1\leqslant k\leqslant N-1. \tag{19}$$

Multiplying τ on both sides of (19) and introducing intermediate variables u_{ij}^*, u_{ij}^{**} yield an ADI method as follows

$$\left(\mathcal{A}_x^\alpha - \frac{2\tau}{3}(\nu+\mathbf{i}\eta)\delta_x^\alpha\right)u_{ij}^* = \tau(\nu+\mathbf{i}\eta)\Lambda_h\left(\frac{4}{3}u_{ij}^k - \frac{1}{3}u_{ij}^{k-1}\right)$$
$$- \tau(\kappa+\mathbf{i}\zeta)\mathcal{A}_h^{\alpha,\beta}|\widetilde{u}_{ij}^{k+1}|^2\widetilde{u}_{ij}^{k+1} + \gamma\tau\mathcal{A}_h^{\alpha,\beta}\widetilde{u}_{ij}^{k+1},$$
$$\left(\mathcal{A}_y^\beta - \frac{2\tau}{3}(\nu+\mathbf{i}\eta)\delta_y^\beta\right)u_{ij}^{**} = u_{ij}^*,$$
$$u_{ij}^{k+1} = u_{ij}^{**} + \frac{4}{3}u_{ij}^k - \frac{1}{3}u_{ij}^{k-1}, \quad 1\leqslant k\leqslant N-1.$$

Numerical test for BDF2-ADI and CBDF2-ADI schemes

Example

Consider the Ginzburg-Landau equation as follows

$$\partial_t u-(\nu+\mathbf{i}\eta)(\partial_x^\alpha u+\partial_y^\beta u)+(\kappa+\mathbf{i}\zeta)|u|^2u-\gamma u = f(x,y,t), \quad (x,y)\in(-1,1)\times(-1,1), \tag{20}$$

The exact solution is

$$u(x,y,t)=(x+1)^4(x-1)^4(y+1)^4(y-1)^4e^{\mathrm{i}t}, \tag{21}$$

where $\nu=\eta=\kappa=1$, $\zeta=2$, $\gamma=3$. The initial-boundary conditions and $f(x,y,t)$ are determined by (21).

Table: Example 9: L^2-norm errors and their convergence orders of difference scheme (5)–(8) and (10)–(13) for $1 < \alpha, \beta < 2$

(α, β)	(h, τ)	$E_1(h, \tau)$	Ord^1	(h, τ)	$E_2(h, \tau)$	Ord^2
	(1/10,1/10)	1.7668e-2	–	(1/32,1/5)	2.7873e-2	–
(1.2,1.8)	(1/20,1/20)	4.5364e-3	1.9616	(1/64,1/20)	2.7124e-3	3.3612
	(1/40,1/40)	1.1433e-3	1.9884	(1/128,1/80)	1.7829e-4	3.9273
	(1/10,1/10)	1.7502e-2	–	(1/32,1/5)	3.0628e-2	–
(1.5,1.5)	(1/20,1/20)	4.6298e-3	1.9185	(1/64,1/20)	3.0519e-3	3.3271
	(1/40,1/40)	1.1812e-3	1.9707	(1/128,1/80)	1.9931e-4	3.9366
	(1/10,1/10)	1.7668e-2	–	(1/32,1/5)	2.7751e-2	–
(1.8,1.2)	(1/20,1/20)	4.5364e-3	1.9616	(1/64,1/20)	2.6875e-3	3.3682
	(1/40,1/40)	1.1433e-3	1.9884	(1/128,1/80)	1.6857e-4	3.9949

Example

Consider the Ginzburg-Landau equation as follows

$$\partial_t u - (\nu + \mathbf{i}\eta)(\partial_x^\alpha u + \partial_y^\beta u) + (\kappa + \mathbf{i}\zeta)|u|^2 u - \gamma u = 0, \quad (x, y) \in \Omega, \ t \in (0, T], \tag{22}$$

$$u(x, y, t) = 0, \quad (x, y) \in \partial\Omega, \ t \in (0, T], \tag{23}$$

$$u(x, y, 0) = \mathrm{sech}(x)\mathrm{sech}(y) \exp(\mathbf{i}(x + y)), \quad (x, y) \in \bar{\Omega} = \Omega \cup \partial\Omega, \tag{24}$$

where $\Omega \in (-10, 10) \times (-10, 10)$, $\nu = \eta = \kappa = \zeta = \gamma = 1$ and $T = 1$.

Table: Example 10: L^2-norm errors and their convergence orders of difference scheme (5)–(8) and (10)–(13) for $1 < \alpha, \beta \leqslant 2$

(α, β)	(h, τ)	$E_1(h, \tau)$	Ord^3	(h, τ)	$E_2(h, \tau)$	Ord^4
	(1/16,1/16)	–	–	(1/48,1/4)	–	–
(1.1,1.9)	(1/32,1/32)	7.0278e-2	–	(1/96,1/16)	2.0225e-3	–
	(1/64,1/64)	2.0522e-2	1.7759	(1/192,1/64)	1.2339e-4	4.0349
	(1/128,1/128)	5.4298e-3	1.9182	(1/384,1/256)	8.0891e-6	3.9311
	(1/16,1/16)	–	–	(1/48,1/4)	–	–
(1.3,1.7)	(1/32,1/32)	6.4544e-2	–	(1/96,1/16)	1.8166e-3	–
	(1/64,1/64)	1.8364e-2	1.8134	(1/192,1/64)	1.1095e-4	4.0332
	(1/128,1/128)	4.8653e-3	1.9163	(1/384,1/256)	7.0591e-6	3.9743
	(1/16,1/16)	–	–	(1/48,1/4)	–	–
(1.9,1.1)	(1/32,1/32)	7.0278e-2	–	(1/96,1/16)	2.0225e-3	–
	(1/64,1/64)	2.0522e-2	1.7759	(1/192,1/64)	1.2339e-4	4.0349
	(1/128,1/128)	5.4298e-3	1.9182	(1/384,1/256)	8.0891e-6	3.9311
	(1/16,1/16)	–	–	(1/48,1/4)	–	–
(2.0,2.0)	(1/32,1/32)	5.1266e-2	–	(1/96,1/16)	2.9644e-3	–
	(1/64,1/64)	1.5180e-2	1.7559	(1/192,1/64)	1.7928e-4	4.0474
	(1/128,1/128)	3.9922e-3	1.9269	(1/384,1/256)	1.1031e-5	4.0226

Three-level linearized difference scheme

Three-level linearized finite difference scheme reads

$$\delta_t u_{ij}^{\frac{1}{2}} - (\nu + \mathbf{i}\eta)(\delta_x^\alpha u_{ij}^{\frac{1}{2}} + \delta_y^\beta u_{ij}^{\frac{1}{2}}) + (\kappa + \mathbf{i}\zeta)|\hat{u}_{ij}^0|^2 u_{ij}^{\frac{1}{2}} - \gamma u_{ij}^{\frac{1}{2}} = 0, \quad (i,j) \in \omega, \tag{25}$$

$$\Delta_t u_{ij}^k - (\nu + \mathbf{i}\eta)(\delta_x^\alpha u_{ij}^{\bar{k}} + \delta_y^\beta u_{ij}^{\bar{k}}) + (\kappa + \mathbf{i}\zeta)|u_{ij}^k|^2 u_{ij}^{\bar{k}} - \gamma u_{ij}^{\bar{k}} = 0, \quad (i,j) \in \omega, \quad 1 \leqslant k \leqslant N-1, \tag{26}$$

$$u_{ij}^0 = \varphi(x_i, y_j), \qquad (i,j) \in \bar{\omega}, \tag{27}$$

$$u_{ij}^k = 0, \qquad (i,j) \in \partial\omega, \quad 0 \leqslant k \leqslant N, \tag{28}$$

where $\Delta_t v^k = \frac{1}{2}(v^{k+1} - v^{k-1})$, $v^{\bar{k}} = \frac{1}{2}(v^{k-1} + v^{k+1})$, $v^{k+\frac{1}{2}} = \frac{1}{2}(v^{k+1} + v^k)$, $\delta_t v^{k+\frac{1}{2}} = \frac{1}{\tau}(v^{k+1} - v^k)$.

Numerical results of the different difference schemes in 1D[1],[2]:

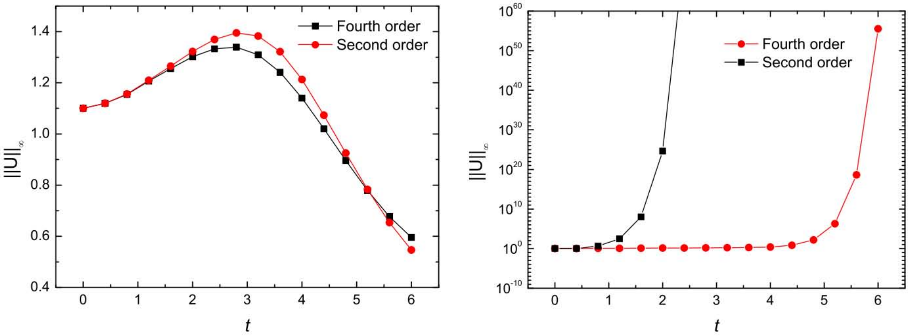

Figure: The evolution of the numerical solution for $\alpha = 1.1$ with $\tau = 0.4$, $h = 0.4$

[1] D. He, K. Pan, An unconditionally stable linearized difference scheme for the fractional Ginzburg-Landau equation, Numerical Algorithms 79(3) (2018) 899–925.

[2] Z. Hao, Z. Sun, A linearized high-order difference scheme for the fractional Ginzburg-Landau equation. Numer. Meth. Part. D. E. 33 (2017) 105–124.

Theorem

If $\gamma < 0$ or $\tau < \frac{1}{\gamma}$ for $\gamma > 0$, the three-level linearized finite difference scheme (25)–(28) *is uniquely solvable.*

Theorem

Let $\{u_{ij}^k \mid (i,j) \in \bar{\omega}, 0 \leqslant k \leqslant N\}$ be the solution of finite difference scheme (25)–(28). *If $\gamma\tau \leqslant \frac{1}{4}$, we have*

$$\|u^k\| \leqslant \sqrt{\frac{5}{3}} e^{\frac{2}{3}\gamma T} \|u^0\|, \qquad 0 \leqslant k \leqslant N. \tag{29}$$

Denote

$$e_{ij}^k = U_{ij}^k - u_{ij}^k, \quad (i,j) \in \bar{\omega},\ 0 \leqslant k \leqslant N.$$

We have the following convergence result.

Theorem (Zhang, Zhang, Sun, 2021)

[a] *Let $u(x,y,t) \in \mathscr{C}^{(2+\alpha,2+\beta,3)}(x,y,t)$ be the solution of the problem (1)–(3), $\{u_{ij}^k | (i,j) \in \bar{\omega},\ 0 \leqslant k \leqslant n\}$ be the solution of the scheme (25)–(28) and denote*

$$C_1 := \sqrt{\frac{(8c_2^2 c_4 + c_1^2)(x_r - x_l)(y_u - y_d)}{2c_4} e^{4c_4 T}}.$$

We have

$$\|e^k\| \leqslant C_1(\tau^2 + h_1^2 + h_2^2), \quad 0 \leqslant k \leqslant N. \tag{30}$$

where C_1 is a positive constant independent of τ and h.

[a] Q. Zhang, L. Zhang, H.W. Sun, A three-level finite difference method with preconditioning technique for two-dimensional nonlinear fractional complex Ginzburg-Landau equations. J. Comput. Appl. Math. 389, (2021) https://doi.org/10.1016/j.cam.2020.113355

l^∞ error estimate:

Set $L_h^2 := \left\{ u \,|\, u = \{u_{ij}\},\ \|u\|^2 < +\infty \right\}$. For $u \in L_h^2$, the semidiscrete Fourier transform of u in two dimension is defined as

$$\hat{u}(k_1, k_2) := \frac{h_x h_y}{\left(\sqrt{2\pi}\right)^2} \sum_{i \in \mathcal{Z}} \sum_{j \in \mathcal{Z}} u_{ij} \exp(-\mathbf{i} k_1 x_i - \mathbf{i} k_2 y_j).$$

The continuous inversion formula is given as

$$u_{ij} = \frac{1}{2\pi} \int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}} \int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} \hat{u}(k_1, k_2) \exp(\mathbf{i} k_1 x_i + \mathbf{i} k_2 y_j) \mathrm{d}k_1 \mathrm{d}k_2.$$

We can prove the following Parsevals identities as

$$(u, v) = \int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}} \int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} \hat{u}(k_1, k_2) \hat{v}^*(k_1, k_2) \mathrm{d}k_1 \mathrm{d}k_2,$$

$$\|u\|^2 = \int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}} \int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} |\hat{u}(k_1, k_2)|^2 \mathrm{d}k_1 \mathrm{d}k_2.$$

Denote

$$\delta_{im} = \begin{cases} 1, & \text{if} \quad m = i, \\ 0, & \text{otherwise}. \end{cases}$$

Noticing

$$\int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} \exp\big(\mathbf{i}k_1(x_m - x_i)\big)\mathrm{d}k_1 = \frac{2\pi}{h_x}\delta_{im}, \quad \int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}} \exp\big(\mathbf{i}k_2(y_n - y_j)\big)\mathrm{d}k_2 = \frac{2\pi}{h_y}\delta_{jn},$$

we have

$$\begin{aligned}
&\int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}} \int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} \hat{u}(k_1, k_2)\hat{v}^*(k_1, k_2)\mathrm{d}k_1\mathrm{d}k_2 \\
=&\int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}} \int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} \Big[\frac{h_x h_y}{2\pi} \sum_{i\in\mathcal{Z}} \sum_{j\in\mathcal{Z}} u_{ij} \exp\big(-\mathbf{i}k_1 x_i - \mathbf{i}k_2 y_j\big)\Big] \\
&\cdot \Big[\frac{h_x h_y}{2\pi} \sum_{m\in\mathcal{Z}} \sum_{n\in\mathcal{Z}} v_{mn}^* \exp\big(\mathbf{i}k_1 x_m + \mathbf{i}k_2 y_n\big)\Big]\mathrm{d}k_1\mathrm{d}k_2 \\
=&\Big(\frac{h_x h_y}{2\pi}\Big)^2 \sum_{i\in\mathcal{Z}} \sum_{j\in\mathcal{Z}} \sum_{m\in\mathcal{Z}} \sum_{n\in\mathcal{Z}} u_{ij} v_{mn}^* \int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} \exp\big(\mathbf{i}k_1(x_m - x_i)\big)\mathrm{d}k_1 \\
&\cdot \int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}} \exp\big(\mathbf{i}k_2(y_n - y_j)\big)\mathrm{d}k_2 \\
=&\Big(\frac{h_x h_y}{2\pi}\Big)^2 \sum_{i\in\mathcal{Z}} \sum_{j\in\mathcal{Z}} \sum_{m\in\mathcal{Z}} \sum_{n\in\mathcal{Z}} u_{ij} v_{mn}^* \cdot \frac{2\pi}{h_x}\delta_{im} \cdot \frac{2\pi}{h_y}\delta_{jn} \\
=&h_x h_y \sum_{i\in\mathcal{Z}} \sum_{j\in\mathcal{Z}} u_{ij} v_{ij}^*.
\end{aligned}$$

Consequently, we have Parsevals identity as

$$(u, v) = \int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}} \int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} \hat{u}(k_1, k_2)\hat{v}^*(k_1, k_2)\mathrm{d}k_1\mathrm{d}k_2, \quad \|u\|^2 = \int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}} \int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} |\hat{u}(k_1, k_2)|^2\mathrm{d}k_1\mathrm{d}k_2.$$

For given constants $\alpha,\ \beta > 1$, we define the fractional Sobolev norm $\|\cdot\|_{H_h^{\alpha,\beta}}$, semi-norms $|\cdot|_{H_h^{\alpha,\beta}}$ and $|\cdot|_{H_h^{\frac{\alpha}{2},\frac{\beta}{2}}}$ as

$$\|u\|^2_{H_h^{\alpha,\beta}} = \int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}} \int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} (1+|k_1|^\alpha+|k_2|^\beta+|k_1|^{2\alpha}+2|k_1|^\alpha|k_2|^\beta+|k_2|^{2\beta})|\hat{u}(k_1,k_2)|^2 \mathrm{d}k_1\mathrm{d}k_2,$$

$$|u|^2_{H_h^{\frac{\alpha}{2},\frac{\beta}{2}}} = \int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}} \int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} (|k_1|^\alpha + |k_2|^\beta)|\hat{u}(k_1,k_2)|^2 \mathrm{d}k_1\mathrm{d}k_2,$$

$$|u|^2_{H_h^{\alpha,\beta}} = \int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}} \int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} (|k_1|^{2\alpha} + 2|k_1|^\alpha|k_2|^\beta + |k_2|^{2\beta})|\hat{u}(k_1,k_2)|^2 \mathrm{d}k_1\mathrm{d}k_2.$$

Clearly,

$$\|u\|^2_{H_h^{\alpha,\beta}} = \|u\|^2 + |u|^2_{H_h^{\frac{\alpha}{2},\frac{\beta}{2}}} + |u|^2_{H_h^{\alpha,\beta}}. \tag{31}$$

We define $H_h^{\alpha,\beta} := \left\{ u \mid u = \{u_{ij}\},\ \|u\|_{H_h^{\alpha,\beta}} < +\infty \right\}.$

Denote the discrete l^∞-norm as

$$\|u\|_{l^\infty} = \max_{i,j\in\mathcal{Z}} |u_{ij}|.$$

A new discrete Sobolev inequality for the two-dimensional case is listed as

Lemma

(Discrete Sobolev inequality) For every α, $\beta > 1$ and $u \in H_h^{\alpha,\beta}$, it holds that

$$\|u\|_{l^\infty} \leqslant C_{\alpha,\beta}\|u\|_{H_h^{\alpha,\beta}}, \tag{32}$$

where $C_{\alpha,\beta} > 0$ is a constant depending on α, β but is independent of h_x and h_y.

Proof.

We have

$$\begin{aligned}
|u_{ij}| =& \Big|\frac{1}{2\pi}\int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}}\int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} \hat{u}(k_1,k_2)e^{\mathrm{i}k_1x_i+\mathrm{i}k_2x_j}\mathrm{d}k_1\mathrm{d}k_2\Big| \\
\leqslant& \frac{1}{2\pi}\int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}}\int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} |\hat{u}(k_1,k_2)|\mathrm{d}k_1\mathrm{d}k_2 \\
=& \frac{1}{2\pi}\int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}}\int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} (1+|k_1|^{\alpha})^{-\frac{1}{2}}(1+|k_2|^{\beta})^{-\frac{1}{2}}(1+|k_1|^{\alpha})^{\frac{1}{2}}(1+|k_2|^{\beta})^{\frac{1}{2}}|\hat{u}(k_1,k_2)|\mathrm{d}k_1\mathrm{d}k_2 \\
\leqslant& \frac{1}{2\pi}\Big(\int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}}\int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} \frac{1}{(1+|k_1|^{\alpha})(1+|k_2|^{\beta})}\mathrm{d}k_1\mathrm{d}k_2\Big)^{\frac{1}{2}} \\
&\cdot\Big(\int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}}\int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} (1+|k_1|^{\alpha})(1+|k_2|^{\beta})|\hat{u}(k_1,k_2)|^2\mathrm{d}k_1\mathrm{d}k_2\Big)^{\frac{1}{2}} \\
\leqslant& C_{\alpha,\beta}\|u\|_{H_h^{\alpha,\beta}},
\end{aligned}$$

where

$$C_{\alpha,\beta} = \frac{1}{2\pi}\Big(\int_{-\infty}^{+\infty}\frac{1}{1+|k_1|^{\alpha}}\mathrm{d}k_1\Big)^{\frac{1}{2}}\cdot\Big(\int_{-\infty}^{+\infty}\frac{1}{1+|k_2|^{\beta}}\mathrm{d}k_2\Big)^{\frac{1}{2}}$$

is bounded provided $\alpha,\beta > 1$. □

Remark

- When $\alpha = \beta = 2$, (32) reduces to the case of integer order, see Lemma 2.5 in [H.-L. Liao, Z.-Z. Sun, S.-S. Han, *SIAM J. Numer. Anal.*, 47(6) (2010), pp. 4381–4401] which can be used to analyze the pointwise errors in two- and three-dimensional Schrödinger equations or Ginzburg-Landau equations.
- For the scope of application of fractional Sobolev imbedding inequality in different dimensions of problems, we list in Table.

Norms \ Dimensions	$d = 1$	$d = 2$	$d = 3$
$\|\cdot\|_{H_h^\alpha}$, $1/2 < \alpha \leqslant 1$	Valid	Invalid	Invalid
$\|\cdot\|_{H_h^{\alpha,\beta}}$, $1 < \alpha, \beta \leqslant 3/2$	Valid	Valid	Invalid
$\|\cdot\|_{H_h^{\alpha,\beta}}$, $\alpha, \beta > 3/2$ (Case of integers $\alpha = \beta = 2$)	Valid	Valid	Valid

Lemma

(Equivalence of the fractional seminorm) For every α, $\beta > 1$ and $u \in H_h^{\alpha,\beta}$, the second-order fractional centeral difference operator for the space fractional derivatives is defined as

$$\delta_x^\alpha u_{ij} := -\frac{1}{h_x^\alpha} \sum_{l=-\infty}^{\infty} g_l^{(\alpha)} u_{i-l,j}, \quad \delta_y^\beta u_{ij} := -\frac{1}{h_y^\beta} \sum_{m=-\infty}^{\infty} g_m^{(\beta)} u_{i,j-m}.$$

Then it holds

$$\widetilde{C}_{\alpha,\beta} |u|^2_{H_h^{\frac{\alpha}{2},\frac{\beta}{2}}} \leqslant (-(\delta_x^\alpha + \delta_y^\beta)u, u) \leqslant |u|^2_{H_h^{\frac{\alpha}{2},\frac{\beta}{2}}},$$

and

$$(\widetilde{C}_{\alpha,\beta})^2 |u|^2_{H_h^{\alpha,\beta}} \leqslant ((\delta_x^\alpha + \delta_y^\beta)u, (\delta_x^\alpha + \delta_y^\beta)u) \leqslant |u|^2_{H_h^{\alpha,\beta}},$$

where $\widetilde{C}_{\alpha,\beta} = (2/\pi)^{\max\{\alpha,\beta\}}$.

Convergence in l^∞

Theorem (Convergence)

[a] *Let $u(x, y, t)$ be the solution of the problem (1)–(3), $\{u_{ij}^n \mid (i, j) \in \bar{\omega},\ 0 \leqslant n \leqslant N\}$ be the solution of the scheme (25)–(28). There are constants $\tau_0 = \min\{1/(8c_1),\ 1/\sqrt[4]{9c_3},\ 1/(8c_4),\ 1/(32c_9)\}$ and $h_0 = 1/\sqrt[4]{9c_3}$ when $\tau \leqslant \tau_0$, $h_x \leqslant h_0$, $h_y \leqslant h_0$ and $\tau^2 + h_x^2 + h_y^2 \leqslant 1/c_{12}$, we have*

$$\|e^n\|_{l^\infty} \leqslant c_{12}(\tau^2 + h_x^2 + h_y^2), \quad 0 \leqslant n \leqslant N. \tag{33}$$

[a] Q. Zhang, J. S. Hesthaven, Z. Sun, Y. Ren, Pointwise error estimate in difference setting for the two-dimensional nonlinear fractional complex Ginzburg-Landau equation, Advances in Computational Mathematics, (2021)

Boundedness in l^∞

Corollary (Boundedness)

Let $u(x, y, t)$ be the solution of the problem (1)–(3) and $\{u_{ij}^n \,|\, (i, j) \in \bar{\omega},\ 0 \leqslant n \leqslant N\}$ be the solution of the scheme (25)–(28). Denote $c_{13} = C_u + 1$. Then for $\tau \leqslant \tau_0$ and $\max\{h_x, h_y\} \leqslant h_0$, when $\tau^2 + h_x^2 + h_y^2 \leqslant 1/c_{12}$ we have

$$\|u^n\|_{l^\infty} \leqslant c_{13}, \quad 0 \leqslant n \leqslant N,$$

where c_{12}, τ_0 and h_0 are some positive constants.

From convergence result in l^∞ norm, we have

$$\|u^n\|_{l^\infty} = \|U^n - e^n\|_{l^\infty} \leqslant \|U^n\|_{l^\infty} + \|e^n\|_{l^\infty} \leqslant C_u + c_{12}(\tau^2 + h_x^2 + h_y^2) \leqslant c_{13}, \quad 0 \leqslant n \leqslant N.$$

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$$\|u^n\|_{l^\infty} = \|U^n - e^n\|_{l^\infty} \leqslant \|U^n\|_{l^\infty} + \|e^n\|_{l^\infty} \leqslant C_u + c_{12}(\tau^2 + h_x^2 + h_y^2) \leqslant c_{13}, \quad 0 \leqslant n \leqslant N.$$

Numerical implementation

Let u^k be the $\hat{m}$-dimensional vectors defined by

$$u^k = \left[u^k_{1,1}, \ldots, u^k_{\hat{m}_1,1}, u^k_{1,2}, \ldots, u^k_{\hat{m}_1,\hat{m}_2}\right]^{\mathsf{T}}.$$

Then the discretized formulation (25)–(28) can be expressed in the following matrix form

$$(D_1^1 + A)u^1 = (D_2^1 - A)u^0, \tag{34}$$

$$(D_1^k + A)u^{k+1} = (D_2^k - A)u^{k-1}, \quad 1 \leqslant k \leqslant N-1, \tag{35}$$

where

$$D_1^1 = \mathrm{diag}\left(\frac{2 + (\kappa + \mathbf{i}\zeta)|u^0_{ij}|^2\tau - \gamma\tau}{2(\nu + \mathbf{i}\eta)}\right),$$

$$D_2^1 = \mathrm{diag}\left(\frac{2 - (\kappa + \mathbf{i}\zeta)|u^0_{ij}|^2\tau + \gamma\tau}{2(\nu + \mathbf{i}\eta)}\right),$$

$$D_1^k = \mathrm{diag}\left(\frac{1 + (\kappa + \mathbf{i}\zeta)|u^k_{ij}|^2\tau - \gamma\tau}{2(\nu + \mathbf{i}\eta)}\right),$$

$$D_2^k = \mathrm{diag}\left(\frac{1 - (\kappa + \mathbf{i}\zeta)|u^k_{ij}|^2\tau + \gamma\tau}{2(\nu + \mathbf{i}\eta)}\right)$$

are diagonal matrices.

$$A = I_{\hat{m}_2} \otimes \left(\frac{\tau}{2h_1^{\alpha}} G_{\alpha,\hat{m}_1} \right) + \left(\frac{\tau}{2h_2^{\beta}} G_{\beta,\hat{m}_2} \right) \otimes I_{\hat{m}_1}, \tag{36}$$

"$\otimes$" denotes the Kronecker product, and I_m is the identity matrix of order m.

$$G_{\mu,m} = \begin{bmatrix} g_0^{(\mu)} & g_{-1}^{(\mu)} & g_{-2}^{(\mu)} & \cdots & g_{-m+2}^{(\mu)} & g_{-m+1}^{(\mu)} \\ g_1^{(\mu)} & g_0^{(\mu)} & g_{-1}^{(\mu)} & \cdots & \cdots & g_{-m+2}^{(\mu)} \\ g_2^{(\mu)} & g_1^{(\mu)} & g_0^{(\mu)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & g_{-2}^{(\mu)} \\ g_{m-2}^{(\mu)} & \ddots & \ddots & \ddots & g_0^{(\mu)} & g_{-1}^{(\mu)} \\ g_{m-1}^{(\mu)} & g_{m-2}^{(\mu)} & \cdots & g_2^{(\mu)} & g_1^{(\mu)} & g_0^{(\mu)} \end{bmatrix} \tag{37}$$

is a symmetric positive definite Toeplitz matrix with $\mu = \alpha$ or β.

GMRES+Strang's block circulant preconditioner!!!

The Strang's block circulant preconditioner for A is defined by

$$S = I_{\hat{m}_2} \otimes S\left(\frac{\tau}{2h_1^{\alpha}} G_{\alpha,\hat{m}_1}\right) + S\left(\frac{\tau}{2h_2^{\beta}} G_{\beta,\hat{m}_2}\right) \otimes I_{\hat{m}_1}. \tag{38}$$

Note that the discretized coefficient matrix of linear system (48) or (49) is diagonal-plus-Toeplitz matrix. However, standard block circulant preconditioners may not work for diagonal-plus-Toeplitz systems. In the following, we propose an effective preconditioner for such Toeplitz-like matrices. For the matrix $D_1^1 + A$ in (48), let

$$\tilde{d} = \frac{1}{\hat{m}} \sum_{i=1}^{\hat{m}_1} \sum_{j=1}^{\hat{m}_2} \frac{2 + (\kappa + \mathbf{i}\zeta)|u_{ij}^0|^2\tau - \gamma\tau}{2(\nu + \mathbf{i}\eta)},$$

then using the S in (50), the preconditioner for linear systems (48) is given as follows

$$P = \tilde{d} I_{\hat{m}} + S.$$

It is easy to see that P is a block circulant matrix with circulant blocks matrix.

Numerical tests in l^2

Define the discrete L^2-norm for the numerical error as follows

$$\|\mathrm{E}(h,\tau)\| = \sqrt{h^2 \sum_{i=0}^{M_1} \sum_{j=0}^{M_2} (U_{ij}^N - u_{ij}^N)^2}. \tag{39}$$

When the exact solution is known, we define the convergence orders, respectively, as follows

$$\mathrm{Ord}_h^1 = \log_2 \left(\frac{\|\mathrm{E}(h,\tau)\|}{\|\mathrm{E}(h/2,\tau)\|} \right), \quad \mathrm{Ord}_\tau^1 = \log_2 \left(\frac{\|\mathrm{E}(h,\tau)\|}{\|\mathrm{E}(h,\tau/2)\|} \right).$$

Otherwise, if the exact solution is unknown, we adopt the following posteriori error to test the convergence orders in temporal and spatial dimensions.

$$\mathrm{Ord}_h^2 = \log_2 \frac{\|u(h,\tau) - u(h/2,\tau)\|}{\|u(h/2,\tau) - u(h/4,\tau)\|}, \quad \mathrm{Ord}_\tau^2 = \log_2 \frac{\|u(h,\tau) - u(h,\tau/2)\|}{\|u(h,\tau/2) - u(h,\tau/4)\|}.$$

Example (1)

Consider the problem as

$$\partial_t u - (\nu + \mathbf{i}\eta)(\partial_x^\alpha u + \partial_y^\beta u) + (\kappa + \mathbf{i}\zeta)|u|^2 u - \gamma u = f(x, y, t),$$
$$(x, y) \in (-1, 1) \times (-1, 1), \ t \in (0, 1],$$

where $\nu = \eta = \kappa = 1$, $\zeta = 2$, $\gamma = 3$. The initial-boundary value conditions of the above truncated problem are determined by the exact solution
$u(x, y, t) = (x+1)^4(x-1)^4(y+1)^4(y-1)^4 \exp(-\mathbf{i}t)$.

Example (2)

Consider the nonlinear fractional Ginzburg-Landau equation as

$$\begin{cases} \partial_t u - (\nu + \mathbf{i}\eta)(\partial_x^\alpha u + \partial_y^\beta u) + (\kappa + \mathbf{i}\zeta)|u|^2 u - \gamma u = 0, & (x, y) \in \Omega, \ t \in (0, T], \\ u(x, y, t) = 0, \quad (x, y) \in \partial\Omega, \ t \in (0, T], \\ u(x, y, 0) = \operatorname{sech}(x)\operatorname{sech}(y) \exp(\mathbf{i}(x + y)), & (x, y) \in \bar{\Omega} = \Omega \cup \partial\Omega, \end{cases}$$

where $\Omega = (-10, 10) \times (-10, 10)$, $\nu = \eta = \kappa = \zeta = \gamma = 1$. The exact solution is unknown.

Table: **Example 1 with the exact solution:** L^2-norm error behavior versus spatial grid size reduction and their convergence orders when $1 < \alpha,\ \beta < 2$ with fixed temporal step size $\tau = 1/128$ and compared CPU time as well as average number of iteration times

		Direct Solver			Preconditioned Solver			
(α, β)	h	$\mathrm{E}_1(h,\tau)$	Ord_h^1	**CPU**	$\mathrm{E}_2(h,\tau)$	Ord_h^1	Iter	**CPU**
	1/4	2.9835e − 2	∗	0.172	2.9835e − 2	∗	4.9	0.15
	1/8	6.5814e − 3	2.1805	0.640	6.5814e − 3	2.1805	5.0	0.281
(1.2, 1.8)	1/16	1.6033e − 3	2.0373	10.766	1.6033e − 3	2.0373	7.0	0.873
	1/32	4.0057e − 4	2.0009	358.02	4.0057e − 4	2.0009	8.0	2.028
	1/64	1.0299e − 4	1.9596	16773.64	1.0298e − 4	1.9597	9.0	11.094
	1/4	2.9381e − 2	∗	0.125	2.9381e − 2	∗	4.4	0.125
	1/8	6.5864e − 3	2.1573	0.655	6.5864e − 3	2.1573	5.0	0.203
(1.5, 1.5)	1/16	1.6139e − 3	2.0289	11.11	1.6139e − 3	2.0289	6.0	0.764
	1/32	4.0342e − 4	2.0002	357.102	4.0334e − 3	2.0003	6.0	1.544
	1/64	1.0303e − 4	1.9693	16549.49	1.0302e − 4	1.9692	7.0	8.475
	1/4	2.9835e − 2	∗	0.062	2.9835e − 2	∗	4.9	0.14
	1/8	6.5814e − 3	2.1805	0.686	6.5814e − 3	2.1805	5.0	0.203
(1.8, 1.2)	1/16	1.6033e − 3	2.0373	11.25	1.6033e − 3	2.0373	7.0	0.873
	1/32	4.0057e − 4	2.0009	357.655	4.0057e − 4	2.0009	8.0	1.998
	1/64	1.0299e − 4	1.9596	16543.64	1.0298e − 4	1.9597	9.0	11.048

Table: **Example 1 with the exact solution:** L^2-norm error behavior versus temporal grid size reduction and time convergence orders when $1 < \alpha,\ \beta < 2$ with fixed spatial step size $h = 1/64$ and compared CPU time as well as average number of iteration times

		Direct Solver			Preconditioned Solver			
(α, β)	τ	$E_1(h,\tau)$	Ord_τ^1	**CPU**	$E_2(h,\tau)$	Ord_τ^1	Iter	**CPU**
(1.2, 1.8)	1/2	1.0769e − 1	∗	258.13	1.0769e − 1	∗	19.0	0.593
	1/4	2.5522e − 2	2.0771	523.638	2.5522e − 2	2.0771	18.8	0.874
	1/8	5.4955e − 3	2.2154	1046.717	5.4955e − 3	2.2154	18.6	1.747
	1/16	1.2627e − 3	2.1217	2106.978	1.2627e − 3	2.1217	16.9	2.996
	1/32	3.2688e − 4	1.9497	4180.057	3.2688e − 4	1.9497	14.9	5.025
(1.5, 1.5)	1/2	1.0767e − 1	∗	257.479	1.0767e − 1	∗	12.0	0.265
	1/4	2.5777e − 2	2.0625	516.465	2.5777e − 2	2.0625	12.0	0.483
	1/8	5.7237e − 3	2.1711	1031.176	5.7237e − 3	2.1711	11.9	0.952
	1/16	1.3184e − 3	2.1182	2075.022	1.3184e − 3	2.1182	10.9	1.779
	1/32	3.3475e − 4	1.9776	4137.345	3.3475e − 4	1.9776	9.9	3.182
(1.8, 1.2)	1/2	1.0769e − 1	∗	258.361	1.0769e − 1	∗	18.5	0.452
	1/4	2.5522e − 2	2.0771	524.868	2.5522e − 2	2.0771	18.8	0.874
	1/8	5.4955e − 3	2.2154	1050.958	5.4955e − 3	2.2154	18.5	1.686
	1/16	1.2627e − 3	2.1217	2097.169	1.2627e − 3	2.1217	16.9	3.012
	1/32	3.2688e − 4	1.9497	4201.392	3.2688e − 4	1.9497	14.9	7.071

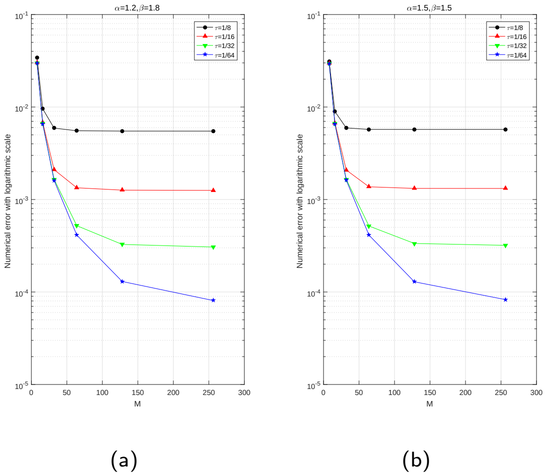

Figure: The numerical error obtained by proposed numerical scheme with preconditioner for fixed τ but varying h

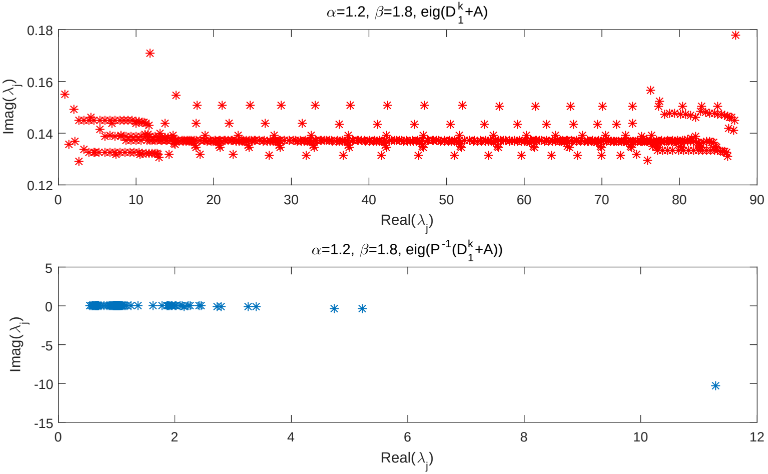

Figure: Spectrum of both original and preconditioned matrices at the final time level, respectively, when $\mathrm{M} = 24$, $\alpha = 1.2$, $\beta = 1.8$. *Upper* original matrix; *Lower* block circulant preconditioned matrix

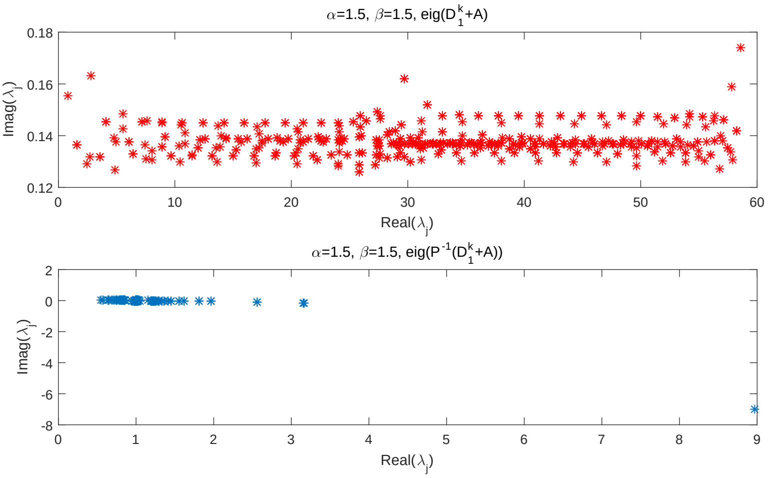

Figure: Spectrum of both original and preconditioned matrices at the final time level, respectively, when $\mathrm{M} = 24$, $\alpha = 1.5$, $\beta = 1.5$. *Upper* original matrix; *Lower* block circulant preconditioned matrix

Table: **Example 2 without exact solution: spatial convergence order**

(α, β)	h	Direct Solver $E_1(h,\tau)$	Direct Solver Ord_h^2	Direct Solver CPU	Preconditioned Solver $E_2(h,\tau)$	Preconditioned Solver Ord_h^2	Preconditioned Solver Iter	Preconditioned Solver CPU
(1.2, 1.8)	1	$*$	$*$	0.297	$*$	$*$	3.0	0.032
	1/2	7.3319e − 2	$*$	4.01	7.3319e − 2	$*$	3.0	0.046
	1/4	1.9148e − 2	1.9370	143.897	1.9148e − 2	1.9370	3.0	0.281
	1/8	4.8318e − 3	1.9866	19273.864	4.8318e − 3	1.9866	3.9	1.234
	1/16	—	—	—	1.2104e − 3	1.9970	4.7	4.04
	1/32	—	—	—	3.0280e − 4	1.9991	6.5	32.69
(1.5, 1.5)	1	$*$	$*$	0.296	$*$	$*$	3.0	0.031
	1/2	4.8318e − 3	$*$	4.056	6.8802e − 2	$*$	3.0	0.047
	1/4	1.7956e − 2	1.9380	143.294	1.7956e − 2	1.9380	3.0	0.297
	1/8	4.5374e − 3	1.9845	18557.16	4.5374e − 3	1.9845	3.3	1.045
	1/16	—	—	—	1.1372e − 3	1.9964	4.0	3.496
	1/32	—	—	—	2.8447e − 4	1.9991	4.9	24.216
(1.8, 1.2)	1	$*$	$*$	0.296	$*$	$*$	3.0	0.031
	1/2	7.3319e − 2	$*$	3.964	7.3319e − 2	$*$	3.0	0.063
	1/4	1.9148e − 2	1.9370	143.081	1.9148e − 2	1.9370	3.0	0.265
	1/8	4.8318e − 3	1.9866	19450.98	4.8318e − 3	1.9866	3.9	1.232
	1/16	—	—	—	1.2104e − 3	1.9970	4.7	4.105
	1/32	—	—	—	3.0280e − 4	1.9991	6.5	32.705
(2.0, 2.0)	1	$*$	$*$	0.062	$*$	$*$	3.0	0.039
	1/2	1.0938e − 1	$*$	0.110	1.0938e − 1	$*$	3.0	0.063
	1/4	2.8443e − 2	1.9432	0.484	2.8443e − 2	1.9432	3.9	0.296
	1/8	7.1502e − 3	1.9920	2.512	7.1502e − 3	1.9920	4.1	1.280
	1/16	1.7893e − 3	1.9986	13.029	1.7893e − 3	1.9986	5.8	5.025
	1/32	4.4744e − 4	1.9996	70.988	4.4744e − 4	1.9996	7.8	38.933

Table: **Example 2 without exact solution: temporal convergence order:**

(α, β)	τ	Direct Solver			Preconditioned Solver			
		$E_1(h, \tau)$	Ord_τ^2	CPU	$E_2(h, \tau)$	Ord_τ^2	Iter	CPU
(1.2, 1.8)	1/20	*	*	259.715	*	*	6.0	0.109
	1/40	8.8857e − 2	*	526.638	8.8857e − 2	*	5.0	0.219
	1/80	2.1990e − 2	2.0147	1052.693	2.1990e − 2	2.0147	4.1	0.312
	1/160	5.4323e − 3	2.0172	2097.946	5.4323e − 3	2.0172	3.9	0.609
	1/320	1.3517e − 3	2.0067	4208.828	1.3517e − 3	2.0068	3.0	0.952
	1/640	3.3731e − 4	2.0027	7937.098	3.3735e − 4	2.0025	3.0	1.919
(1.5, 1.5)	1/20	*	*	258.151	*	*	5.5	0.156
	1/40	7.4470e − 2	*	516.093	7.4470e − 2	*	4.8	0.188
	1/80	1.8532e − 2	2.0066	1035.547	1.8532e − 2	2.0066	3.9	0.297
	1/160	4.6019e − 3	2.0097	2082.214	4.6019e − 3	2.0097	3.0	0.483
	1/320	1.1467e − 3	2.0048	4147.171	1.1467e − 3	2.0048	3.0	0.952
	1/640	2.8621e − 4	2.0023	7967.377	2.8621e − 4	2.0023	3.0	1.92
(1.8, 1.2)	1/20	*	*	256.806	*	*	6.0	0.14
	1/40	8.8857e − 2	*	522.534	8.8857e − 2	*	5.0	0.203
	1/80	2.1990e − 2	2.0147	1048.303	2.1990e − 2	2.0147	4.1	0.312
	1/160	5.4323e − 3	2.0172	2083.151	5.4323e − 3	2.0172	3.9	0.624
	1/320	1.3517e − 3	2.0067	4145.526	1.3517e − 3	2.0068	3.0	0.967
	1/640	3.3731e − 4	2.0027	7989.022	3.3735e − 4	2.0025	3.0	1.965
(2.0, 2.0)	1/20	*	*	0.202	*	*	7.0	0.156
	1/40	1.7328e − 1	*	0.359	1.7328e − 1	*	5.8	0.219
	1/80	4.2391e − 2	2.0312	0.749	4.2391e − 2	2.0312	4.9	0.390
	1/160	1.0303e − 2	2.0407	1.419	1.0303e − 2	2.0407	4.0	0.639
	1/320	2.5530e − 3	2.0128	2.872	2.5526e − 3	2.0130	3.3	1.061
	1/640	6.3647e − 4	2.0040	5.820	6.3680e − 4	2.0031	3.0	1.904

Numerical tests in l^∞

- When the exact solution is known, we will test the order of convergence in space and time, respectively, by

$$\mathrm{Ord}^h_{l^\infty} = \log_2 \left(\frac{\|E(h,\tau)\|_{l^\infty}}{\|E(h/2,\tau)\|_{l^\infty}} \right),$$
$$\mathrm{Ord}^\tau_{l^\infty} = \log_2 \left(\frac{\|E(h,\tau)\|_{l^\infty}}{\|E(h,\tau/2)\|_{l^\infty}} \right). \tag{40}$$

- When the exact solution is unknown, we can test the order of convergence in space and time, respectively, by the following posterior error estimation

$$\mathrm{Ord}^h_{l^\infty} = \log_2 \frac{\|u(h,\tau)-u(h/2,\tau)\|_{l^\infty}}{\|u(h/2,\tau)-u(h/4,\tau)\|_{l^\infty}},$$
$$\mathrm{Ord}^\tau_{l^\infty} = \log_2 \frac{\|u(h,\tau)-u(h,\tau/2)\|_{l^\infty}}{\|u(h,\tau/2)-u(h,\tau/4)\|_{l^\infty}}. \tag{41}$$

Numerical results

Table: **Example 1**: the maximum norm errors versus temporal grid size reduction and convergence orders of the difference scheme (25)–(28) in time with fixed $h = 1/256$

(α, β)	τ	$\|E(h,\tau)\|_{l^\infty}$	$\mathrm{Ord}^{\tau}_{l^\infty}$
(1.2, 1.8)	1/2	1.9363e − 1	*
	1/4	3.4756e − 2	2.4780
	1/8	8.2517e − 3	2.0745
	1/16	1.8508e − 3	2.1565
	1/32	4.5568e − 4	2.0220
(1.5, 1.5)	1/2	1.9339e − 1	*
	1/4	3.4412e − 2	2.4905
	1/8	8.6226e − 3	1.9967
	1/16	1.9118e − 3	2.1732
	1/32	4.6982e − 4	2.0247
(1.9, 1.1)	1/2	1.9385e − 1	*
	1/4	3.4969e − 2	2.4708
	1/8	8.0120e − 3	2.1258
	1/16	1.7961e − 3	2.1573
	1/32	4.4442e − 4	2.0148
(2.0, 2.0)	1/2	2.0293e − 1	*
	1/4	3.4126e − 2	2.5720
	1/8	7.9055e − 3	2.1100
	1/16	1.6649e − 3	2.2474
	1/32	4.2330e − 4	1.9757

Numerical results

Table: **Example 1**: the maximum norm errors versus spatial grid size reduction and convergence orders of the difference scheme (25)–(28) in space with fixed $\tau = 1/512$

(α, β)	h	$\|E(h,\tau)\|_{l^\infty}$	$\mathrm{Ord}^h_{l^\infty}$
(1.2, 1.8)	1/4	2.9701e − 2	*
	1/8	7.7677e − 3	1.9350
	1/16	1.9633e − 3	1.9842
	1/32	4.9292e − 4	1.9939
	1/64	1.2444e − 4	1.9859
(1.5, 1.5)	1/4	2.8914e − 2	*
	1/8	7.4816e − 3	1.9504
	1/16	1.8818e − 3	1.9912
	1/32	4.7148e − 4	1.9968
	1/64	1.1902e − 4	1.9860
(1.9, 1.1)	1/4	3.0190e − 2	*
	1/8	7.9503e − 3	1.9250
	1/16	2.0157e − 3	1.9797
	1/32	5.0667e − 4	1.9922
	1/64	1.2793e − 4	1.9856
(2.0, 2.0)	1/4	4.7915e − 2	*
	1/8	1.2276e − 2	1.9647
	1/16	3.0937e − 3	1.9884
	1/32	7.7619e − 4	1.9948
	1/64	1.9535e − 4	1.9903

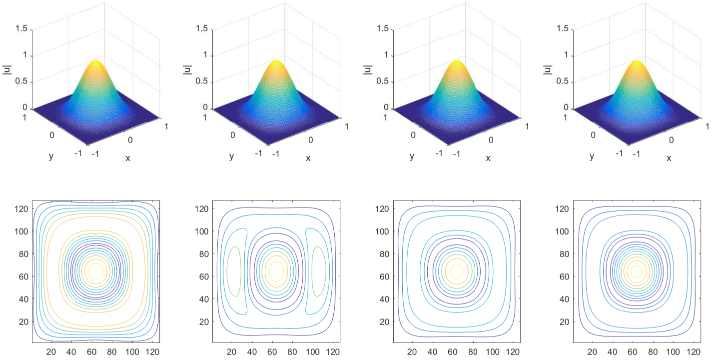

Figure: The numerical solutions and corresponding contour profiles. The parameters are taken with $(\alpha, \beta) = (1.1, 1.1)$, $(1.2, 1.8)$, $(1.5, 1.5)$, $(1.9, 1.9)$ from left to right; The final time is $T = 1$ and the grid sizes $M = 128$, $N = 200$

Table: **Example 2**: the maximum norm errors versus temporal grid size reduction and convergence orders of the difference scheme (25)–(28) in time with fixed $h = 5/64$, $T = 1$

(α, β)	τ	$\|E(h,\tau)\|_{l^\infty}$	$\mathrm{Ord}^{\tau}_{l^\infty}$
	1/4	1.1760e − 2	*
	1/8	2.7348e − 3	2.1044
(1.1, 1.9)	1/16	6.4542e − 4	2.0831
	1/32	1.5920e − 4	2.0194
	1/64	3.9065e − 5	2.0269
	1/4	9.7617e − 3	*
	1/8	2.3333e − 3	2.0647
(1.5, 1.5)	1/16	5.6950e − 4	2.0346
	1/32	1.4103e − 4	2.0137
	1/64	3.5390e − 5	1.9946
	1/4	1.0267e − 2	*
	1/8	2.4244e − 3	2.0823
(1.7, 1.3)	1/16	5.8758e − 4	2.0448
	1/32	1.4563e − 4	2.0125
	1/64	3.6091e − 5	2.0126
	1/4	1.4561e − 2	*
	1/8	2.2635e − 3	2.6855
(2.0, 2.0)	1/16	5.7728e − 4	1.9712
	1/32	1.4615e − 4	1.9819
	1/64	3.6704e − 5	1.9934

Table: **Example 2**: the maximum norm errors versus spatial grid size reduction and convergence orders of the difference scheme (25)–(28) in space with fixed $\tau = 1/16,\ T = 1$

(α, β)	h	$\|E(h,\tau)\|_{l^\infty}$	$\mathrm{Ord}^h_{l^\infty}$
(1.1, 1.9)	1/4	1.0814e − 2	*
	1/8	2.7407e − 3	1.9803
	1/16	6.8252e − 4	2.0056
	1/32	1.7081e − 4	1.9985
	1/64	4.2711e − 5	1.9997
(1.5, 1.5)	1/4	1.0396e − 2	*
	1/8	2.6967e − 3	1.9468
	1/16	6.7326e − 4	2.0020
	1/32	1.6831e − 4	2.0001
	1/64	4.2093e − 5	1.9994
(1.7, 1.3)	1/4	1.0369e − 2	*
	1/8	2.6864e − 3	1.9485
	1/16	6.7586e − 4	1.9909
	1/32	1.6888e − 4	2.0007
	1/64	4.2235e − 5	1.9995
(2.0, 2.0)	1/4	3.7243e − 2	*
	1/8	7.0375e − 3	2.4038
	1/16	1.7180e − 3	2.0343
	1/32	4.2597e − 4	2.0119
	1/64	1.0626e − 4	2.0031

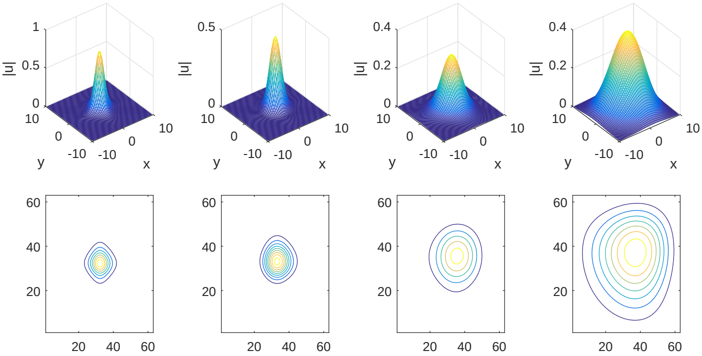

Figure: The numerical solutions and corresponding contour profiles. The parameters are taken with $T = 0.1$, 0.4, 1.6, 3.2 from left to right and the grid sizes $\tau = 0.01$, $h = 5/16$, $\alpha = \beta = 1.5$

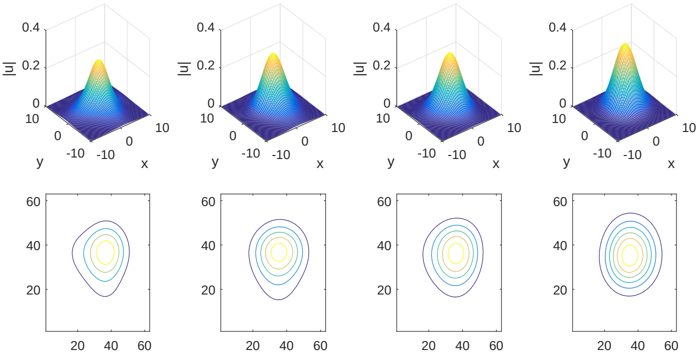

Figure: The numerical solutions at the fixed time $T = 2$ and corresponding contour profiles. The parameters are taken with $(\alpha, \beta) = (1.1, 1.1)$, $(1.2, 1.8)$, $(1.5, 1.5)$, $(1.9, 1.9)$ from left to right and the grid sizes $M = 64$, $N = 100$

Three-level linearized compact difference scheme reads

$$\mathcal{A}_h^{\alpha,\beta}\delta_t u_{ij}^{\frac{1}{2}} - (\nu + \mathbf{i}\eta)\left(\mathcal{A}_y^{\beta}\delta_x^{\alpha} u_{ij}^{\frac{1}{2}} + \mathcal{A}_x^{\alpha}\delta_y^{\beta} u_{ij}^{\frac{1}{2}}\right) + (\kappa + \mathbf{i}\zeta)\mathcal{A}_h^{\alpha,\beta}(|\hat{u}_{ij}^{0}|^2 u_{ij}^{\frac{1}{2}}) - \gamma\mathcal{A}_h^{\alpha,\beta} u_{ij}^{\frac{1}{2}} = 0, \qquad (i,j) \in \omega. \tag{42}$$

$$\mathcal{A}_h^{\alpha,\beta}\Delta_t u_{ij}^{k} - (\nu + \mathbf{i}\eta)\left(\mathcal{A}_y^{\beta}\delta_x^{\alpha} u_{ij}^{\bar{k}} + \mathcal{A}_x^{\alpha}\delta_y^{\beta} u_{ij}^{\bar{k}}\right) + (\kappa + \mathbf{i}\zeta)\mathcal{A}_h^{\alpha,\beta}(|u_{ij}^{k}|^2 u_{ij}^{\bar{k}}) - \gamma\mathcal{A}_h^{\alpha,\beta} u_{ij}^{\bar{k}} = 0, \qquad (i,j) \in \omega,\ 1 \leqslant k \leqslant N-1, \tag{43}$$

$$u_{ij}^{0} = \varphi(x_i, y_j), \qquad (i,j) \in \bar{\omega}, \tag{44}$$

$$u_{ij}^{k} = 0, \qquad (i,j) \in \partial\omega, \quad 0 \leqslant k \leqslant N. \tag{45}$$

Theorem

Denote $c_r = \frac{1}{2}\sqrt{\kappa^2 + \zeta^2} c_0^2$. *When* $\tau \leqslant \frac{1}{6c_r}$ *for* $\gamma \leqslant 0$ *or* $\tau \leqslant \frac{1}{3\gamma + 6c_r}$ *for* $\gamma > 0$, *the linearized finite difference scheme* (42)–(45) *is uniquely solvable.*

Theorem

Let $\{u_{ij}^k \,|\, (i,j) \in \bar{\omega},\ 0 \leqslant k \leqslant N\}$ *be the solution of the three-level linearized compact scheme* (42)–(45). *Denote*

$$c_l = \sqrt{\kappa^2 + \zeta^2} c_0^2 + |\gamma|, \quad c_3 = 3\sqrt{2}\exp\left(4c_l T\right).$$

If $\gamma\tau \leqslant \frac{1}{4}$, *we have*

$$\|u^k\| \leqslant c_3 \|u^0\|, \quad 0 \leqslant k \leqslant N. \tag{46}$$

Theorem

[a] *Let $u(x,y,t)$ be the solution of the problems (1)–(3), $\{u_{ij}^k \mid (i,j) \in \bar{\omega},\ 0 \leqslant k \leqslant N\}$ be the solution of (42)–(45). Denote*

$$c_4 = \sqrt{\kappa^2 + \zeta^2}(c_0^2 + c_0 c_3 + c_3^2) + |\gamma| + \frac{1}{2},$$
$$c_5 = 2\sqrt{3}\exp\big(2(2c_4 + 1)T\big)c_2\sqrt{T(x_r - x_l)(y_u - y_d)}.$$

When $\tau < \frac{1}{6c_4}$, we have

$$\|e^k\| \leqslant c_5(\tau^2 + h_1^4 + h_2^4), \quad 0 \leqslant k \leqslant N. \tag{47}$$

[a] L. Zhang, Q. Zhang*, H.-W. Sun, A fast compact difference method for two-dimensional nonlinear space-fractional complex Ginzburg-Landau equations, Journal of Computational Mathematics, (2021)

Numerical implementation

The matrix form for the scheme (42)–(45) is expressed as follows

$$\left((A_\beta \otimes A_\alpha)D_1^1 + A\right) u^1 = \left((A_\beta \otimes A_\alpha)D_2^1 - A\right) u^0, \tag{48}$$

$$\left((A_\beta \otimes A_\alpha)D_1^k + A\right) u^{k+1} = \left((A_\beta \otimes A_\alpha)D_2^k - A\right) u^{k-1}, \quad 1 \leqslant k \leqslant N-1, \tag{49}$$

where

$$D_1^1 = \operatorname{diag}\left(\frac{2+(\kappa+\mathbf{i}\zeta)|u_{ij}^0|^2\tau-\gamma\tau}{2(\nu+\mathbf{i}\eta)}\right), \quad D_2^1 = \operatorname{diag}\left(\frac{2-(\kappa+\mathbf{i}\zeta)|u_{ij}^0|^2\tau+\gamma\tau}{2(\nu+\mathbf{i}\eta)}\right),$$

$$D_1^k = \operatorname{diag}\left(\frac{1+(\kappa+\mathbf{i}\zeta)|u_{ij}^k|^2\tau-\gamma\tau}{2(\nu+\mathbf{i}\eta)}\right), \quad D_2^k = \operatorname{diag}\left(\frac{1-(\kappa+\mathbf{i}\zeta)|u_{ij}^k|^2\tau+\gamma\tau}{2(\nu+\mathbf{i}\eta)}\right)$$

are diagonal matrices, A_α, A_β are tridiagonal matrices

$$A_\alpha = \operatorname{tridiag}\left(\frac{\alpha}{24}, 1-\frac{\alpha}{12}, \frac{\alpha}{24}\right), \quad A_\beta = \operatorname{tridiag}\left(\frac{\beta}{24}, 1-\frac{\beta}{12}, \frac{\beta}{24}\right),$$

$$A = A_\beta \otimes \left(\frac{\tau}{2h_1^\alpha} G_{\alpha,\hat{m}_1}\right) + \left(\frac{\tau}{2h_2^\beta} G_{\beta,\hat{m}_2}\right) \otimes A_\alpha,$$

- A is a BTTB matrix, which can be stored in $\mathcal{O}(M_1 M_2)$ of memory and the matrix-vector multiplication $A\mathbf{v}$ can be performed in $\mathcal{O}\big(M_1 M_2(\log M_1 + \log M_2)\big)$ operations for some vector $\mathbf{v}$ by the two-dimensional FFT.
- The coefficient matrix of linear systems (48) or (49) is non-Hermitian, which is efficient to be solved by GMRES method.
- The preconditioner technique could be exploited to accelerate the convergence rate of the GMRES method.

For the matrix $\left((A_\beta \otimes A_\alpha)D_1^1 + A\right) u^1$ in (48), let

$$\tilde{d} = \frac{1}{\hat{m}} \sum_{i=1}^{\hat{m}_1} \sum_{j=1}^{\hat{m}_2} \frac{2 + (\kappa + \mathbf{i}\zeta)|u_{ij}^0|^2\tau - \gamma\tau}{2(\nu + \mathbf{i}\eta)},$$

then using S in (50), the preconditioner for the linear system (48) is given as follows

$$P = \tilde{d} S(A_\beta) \otimes S(A_\alpha) + S,$$

where

$$S = S(A_\beta) \otimes S\left(\frac{\tau}{2h_1^\alpha} G_{\alpha,\hat{m}_1}\right) + S\left(\frac{\tau}{2h_2^\beta} G_{\beta,\hat{m}_2}\right) \otimes S(A_\alpha) \tag{50}$$

is a Strang block circulant preconditioner of A.

Numerical results

(α, β)	(h, τ)	PGMRES − 4 E(h, τ)	PGMRES − 4 Ord_h^2	PGMRES − 4 Iter	PGMRES − 4 CPU	PGMRES − 2 E(h, τ)	PGMRES − 2 Ord_h^2	PGMRES − 2 Iter	PGMRES − 2 CPU
(1.2, 1.8)	(10/9, 1/20)	*	*	4.0	0.014	*	*	4.0	0.037
	(5/9, 1/80)	5.7124e − 2	*	3.0	0.016	9.3753e − 2	*	3.0	0.091
	(5/18, 1/320)	3.9889e − 3	3.8400	3.0	0.515	2.3900e − 2	1.9718	3.0	0.312
	(5/36, 1/1280)	2.4034e − 4	4.0528	3.0	2.171	5.9805e − 3	1.9987	2.0	1.359
	(5/72, 1/5120)	1.5713e − 5	3.9351	2.1	58.78	1.4953e − 3	1.9998	2.0	37.23
(1.5, 1.5)	(10/9, 1/20)	*	*	4.0	0.015	*	*	4.0	0.015
	(5/9, 1/80)	5.3407e − 2	*	3.0	0.016	8.8358e − 2	*	3.0	0.036
	5/18, 1/320)	3.8886e − 3	3.7797	3.0	0.484	2.2404e − 2	1.9796	3.0	0.312
	(5/36, 1/1280)	2.3735e − 4	4.0341	2.7	2.049	5.6165e − 3	1.9960	2.0	1.297
	(5/72, 1/5120)	1.4989e − 5	3.9850	2.0	56.32	1.4051e − 3	1.9990	2.0	37.26
(1.8, 1.2)	(10/9, 1/20)	*	*	4.0	0.014	*	*	4.0	0.032
	(5/9, 1/80)	5.7124e − 2	*	3.0	0.016	9.3753e − 2	*	3.0	0.101
	5/18, 1/320)	3.9889e − 3	3.8400	3.0	0.468	2.3900e − 2	1.9718	3.0	0.313
	(5/36, 1/1280)	2.4034e − 4	4.0528	3.0	2.156	5.9805e − 3	1.9987	2.0	1.359
	(5/72, 1/5120)	1.5713e − 5	3.9351	2.1	58.64	1.4953e − 3	1.9998	2.0	38.15

Table: **Example 2**: L^2-norm error behavior with reduced temporal step size and spatial grid mesh size and their convergence orders when $1 < \alpha, \beta < 2$ and compared CPU time(in seconds) as well as average numbers of iterations

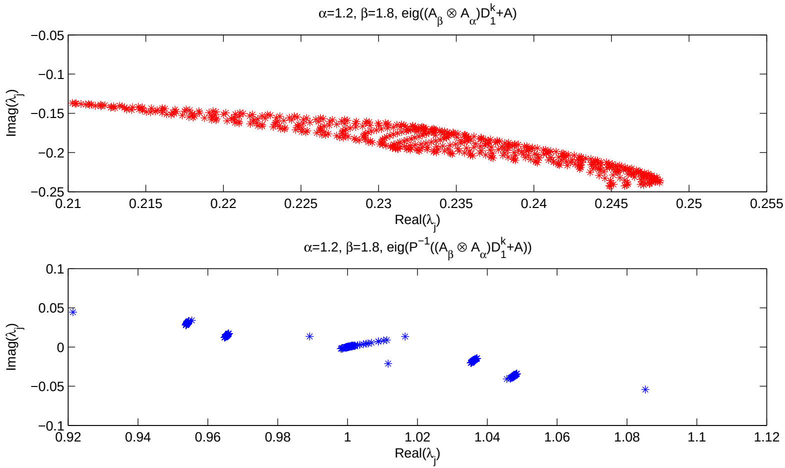

Figure: Spectrum of both original and preconditioned matrices at the final time level, respectively, when $M = 24$, $\alpha = 1.2$, $\beta = 1.8$. *Red* (upper) original matrix; *Blue* (lower) block circulant preconditioned matrix

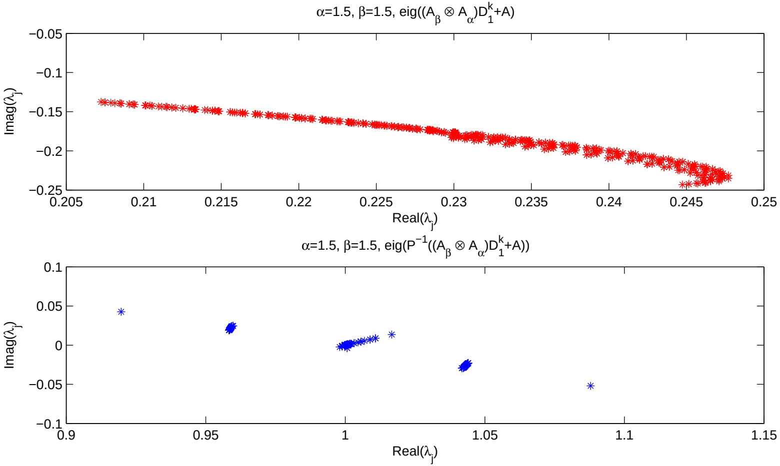

Figure: Spectrum of both original and preconditioned matrices at the final time level, respectively, when $M = 24$, $\alpha = 1.5$, $\beta = 1.5$. *Red* (upper) original matrix; *Blue* (lower) block circulant preconditioned matrix

Exponential Runge-Kutta method

Spatial semi-discretized formulation:

$$\begin{cases} \dfrac{\mathrm{d}u_{i,j}(t)}{\mathrm{d}t} = & -\dfrac{\nu + \mathbf{i}\eta}{2h_1^{\alpha}\cos(\frac{\alpha\pi}{2})}\left[\sum_{\ell=1}^{4}\lambda_\ell\left(\sum_{k=0}^{i+p_\ell} q_k^{(\alpha)} u_{i-k+p_\ell,j}(t) + \sum_{k=0}^{M_x-i+p_\ell} q_k^{(\alpha)} u_{i+k-p_\ell,j}(t)\right)\right] \\ & -\dfrac{\nu + \mathbf{i}\eta}{2h_2^{\beta}\cos(\frac{\beta\pi}{2})}\left[\sum_{\ell=1}^{4}\lambda_\ell\left(\sum_{k=0}^{j+p_\ell} q_k^{(\beta)} u_{i,j-k+p_\ell}(t) + \sum_{k=0}^{M_y-j+p_\ell} q_k^{(\beta)} u_{i,j+k-p_\ell}(t)\right)\right] \\ & -(\kappa + \mathbf{i}\zeta)|u_{ij}(t)|^2 u_{ij}(t) + \gamma u_{ij}(t), \\ u_{i,j}(0) = & \varphi(x_i, y_j), \end{cases} \tag{51}$$

where $1 \leqslant i \leqslant M_x - 1$, $1 \leqslant j \leqslant M_y - 1$.

Matrix form

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{u}(t)}{\mathrm{d}t} = -A_{x,y}\mathbf{u}(t) + \mathbf{f}\big(\mathbf{u}(t)\big), \quad t \in (0, T], \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{52}$$

with the initial value

$$\mathbf{u}_0 = \left[\varphi(x_1, y_1), \ldots, \varphi(x_{\hat{M}_x}, y_1), \varphi(x_1, y_2), \ldots, \varphi(x_{\hat{M}_x}, y_{\hat{M}_y})\right]^{\mathsf{T}}.$$

A typical fourth-order exponential integrator method proposed by Krogstad and rewrite it as follows

$$\mathbf{u}_{i+1} = \mathbf{u}_i + \tau[\varphi_1 - 3\varphi_2 + 4\varphi_3, 2\varphi_2 - 4\varphi_3, 2\varphi_2 - 4\varphi_3, -\varphi_2 + 4\varphi_3] \begin{bmatrix} \mathbf{f}_{i1} - A\mathbf{u}_i \\ \mathbf{f}_{i2} - A\mathbf{u}_i \\ \mathbf{f}_{i3} - A\mathbf{u}_i \\ \mathbf{f}_{i4} - A\mathbf{u}_i \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{u}_{i1} \\ \mathbf{u}_{i2} \\ \mathbf{u}_{i3} \\ \mathbf{u}_{i4} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_i \\ \mathbf{u}_i \\ \mathbf{u}_i \end{bmatrix} + \tau \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\varphi_{1,2} & 0 & 0 \\ 0 & \frac{1}{2}\varphi_{1,3} - \varphi_{2,3} & \varphi_{2,3} & 0 \\ 0 & \varphi_{1,4} - 2\varphi_{2,4} & 0 & 2\varphi_{2,4} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{f}_{i1} - A\mathbf{u}_i \\ \mathbf{f}_{i2} - A\mathbf{u}_i \\ \mathbf{f}_{i3} - A\mathbf{u}_i \end{bmatrix}, \quad (53)$$

$$\mathbf{f}_{ik} = \mathbf{f}(\mathbf{u}_{ik}, t_i + c_k\tau), \ [c_1, c_2, c_3, c_4] = \left[0, \frac{1}{2}, \frac{1}{2}, 1\right], \ k = 1, \ldots, 4,$$

where

$$\varphi_{j,k} = \varphi_j(-c_k\tau A), \quad \varphi_j = \varphi_j(-\tau A).$$

Table: **Example 1:** L^2-norm error behavior, convergence orders when $1 < \alpha,\ \beta < 2$, CPU time (in seconds) and average numbers of iterations

(α, β)	**ERKo**[5]					**ERKn1**				
	(h, τ)	$E(h, \tau)$	Ord_o^I	It	Ctime	(h, τ)	$E(h, \tau)$	Ord_n^I	It	Ctime
(1.2, 1.8)	(1/12, 1/4)	3.0289e − 3	∗	11.5	0.891	(1/16, 1/4)	1.2055e − 3	∗	20.4	0.765
	(1/48, 1/8)	2.0400e − 4	3.8922	19.2	20.010	(1/32, 1/8)	1.3063e − 4	3.2060	27.1	7.920
	(1/192, 1/16)	1.4296e − 5	3.8349	33.3	473.715	(1/64, 1/16)	1.0050e − 5	3.7002	33.2	67.358
	(1/768, 1/32)	9.3240e − 7	3.9385	55.9	82015.362	(1/128, 1/32)	6.7888e − 7	3.8879	42.6	573.169
(1.5, 1.5)	(1/12, 1/4)	3.1116e − 3	∗	9.7	1.172	(1/16, 1/4)	1.2024e − 3	∗	14.9	0.703
	(1/48, 1/8)	2.1138e − 4	3.8798	14.3	18.669	(1/32, 1/8)	1.2673e − 4	3.2461	17.5	7.388
	(1/192, 1/16)	1.4272e − 5	3.8886	20.0	462.116	(1/64, 1/16)	9.6358e − 6	3.7172	20.9	59.538
	(1/768, 1/32)	9.4263e − 7	3.9203	27.2	108575.225	(1/128, 1/32)	6.5294e − 7	3.8834	25.6	530.836
(1.8, 1.2)	(1/12, 1/4)	3.0289e − 3	∗	11.5	0.938	(1/16, 1/4)	1.2055e − 3	∗	20.4	0.765
	(1/48, 1/8)	2.0400e − 4	3.8922	19.8	17.386	(1/32, 1/8)	1.3063e − 4	3.2060	27.2	7.141
	(1/192, 1/16)	1.4287e − 5	3.8357	33.1	425.652	(1/64, 1/16)	1.0050e − 5	3.7002	33.1	68.751
	(1/768, 1/32)	9.3249e − 7	3.9375	55.6	75445.401	(1/128, 1/32)	6.7888e − 7	3.8879	42.5	566.824

"∗" denotes that we could not obtain the datum.

[5] Zhang, Zhang, Sun, 2020, J. Sci. Comput.

Table: **Example 1:** L^2-norm error behavior, convergence orders when $1 < \alpha, \beta < 2$, CPU time (in seconds) and average numbers of iterations

(α, β)	(h, τ)	ERKn2				ERKn1			
		$E(h, \tau)$	Ord_n^I	It	Ctime	$E(h, \tau)$	Ord_n^I	It	Ctime
(1.2, 1.8)	(1/16, 1/4)	1.2108e − 3	*	15.7	0.667	1.2055e − 3	*	20.4	0.765
	(1/32, 1/8)	1.3042e − 4	3.2147	19.8	3.258	1.3063e − 4	3.2060	27.1	7.920
	(1/64, 1/16)	9.9989e − 6	3.7052	24.2	37.397	1.0050e − 5	3.7002	33.2	67.358
	(1/128, 1/32)	6.7748e − 7	3.8835	28.9	276.916	6.7888e − 7	3.8879	42.6	573.169
(1.5, 1.5)	(1/16, 1/4)	31.2060e − 3	*	12.3	0.532	1.2024e − 3	*	14.9	0.703
	(1/32, 1/8)	1.2669e − 4	3.2509	14.0	2.289	1.2673e − 4	3.2461	17.5	7.388
	(1/64, 1/16)	9.6161e − 6	3.7197	16.1	24.815	9.6358e − 6	3.7172	20.9	59.538
	(1/128, 1/32)	6.5276e − 7	3.8808	18.0	171.031	6.5294e − 7	3.8834	25.6	530.836
(1.8, 1.2)	(1/16, 1/4)	1.2108e − 3	*	15.6	0.691	1.2055e − 3	*	20.4	0.765
	(1/32, 1/8)	1.3042e − 4	3.2147	19.6	3.256	1.3063e − 4	3.2060	27.2	7.141
	(1/64, 1/16)	9.9987e − 6	3.7053	24.0	37.243	1.0050e − 5	3.7002	33.1	68.751
	(1/128, 1/32)	6.7733e − 7	3.8838	28.7	277.469	6.7888e − 7	3.8879	42.5	566.824

“*” denotes that we could not obtain the datum.

Conclusion

We summary this work as follows

- We develop several numerical methods including BDF2-ADI, CBDF2-ADI, three-level average scheme, compact three-level average scheme and exponential Runge-Kutta methods for two-dimensional spatial fractional Ginzburg-Landau equation.
- The numerical theoretical results of these methods are proved and compared according to CPU time and numerical accuracy.
- We prove a fractional Sobolev inequality at a discrete frame, which is useful for the pointwise error estimate of two-dimensional spatial fractional problems.

Ref: 2D fractional complex Ginzburg-Landau equation:

- **Q. Zhang***, J. S. Hesthaven, Z. Sun, Y. Ren, Pointwise error estimate in difference setting for the two-dimensional nonlinear fractional complex Ginzburg-Landau equation, **Advances in Computational Mathematics**, (2021) (Pointwise error estimate)
- **Q. Zhang**, L. Zhang, H.-W. Sun*, A three-level finite difference method with preconditioning technique for two-dimensional nonlinear fractional complex Ginzburg-Landau equations, **Journal of Computational and Applied Mathematics** 389 (2021) (Three-level average scheme)
- **Q. Zhang***, X. Lin, K. Pan, Y. Ren, Linearized ADI schemes for two-dimensional space-fractional nonlinear Ginzburg-Landau equation, **Computers and Mathematics with Applications**, 80 (2020) 1201–1220 (BDF2-ADI, compact BDF2-ADI)
- L. Zhang, **Q. Zhang***, H.-W. Sun, A fast compact difference method for two-dimensional nonlinear space-fractional complex Ginzburg-Landau equations, **Journal of Computational Mathematics**, 39(5) (2021), 697–721 (Compact three-level average scheme)
- L. Zhang, **Q. Zhang**, H.-W. Sun*, Exponential Runge-Kutta method for two-dimensional nonlinear fractional complex Ginzburg-Landau equations, **Journal of Scientific Computing**, 83(59) (2020) (Exponential Runge-Kutta method)

Thank you

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Questions or comments?