Numerical Analysis of Some Singular PDEs with Logarithmic Nonlinearity

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Outline

Some PDEs with Logarithmic Nonlinearity[1]

- Heat equation with a logarithmic nonlinearity (Log-Heat)

$$u_t - \Delta u = \lambda u \ln|u| := \lambda f(u), \quad \lambda \neq 0, \ \lambda \in \mathbb{R}$$

- Logarithmic Schrödinger equation (LogSE)

$$\mathrm{i}u_t + \Delta u = \lambda u \ln|u|$$

or LogSE with a potential V (e.g., $V = |x|^2$): $\Delta + V \to \Delta$.

[1] **Applications:** nonlinear wave, quantum mechanics & optics; nuclear physical; superfluids & Bose-Einstein condensation, ... (cf. Białynicki-Birula & J. Mycielski, 1975)

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- Logarithmic Klein-Gordon equation (LogKGE)

$$u_{tt} - \Delta u + u = \lambda u \ln|u|$$

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- Cahn-Hilliard equation with logarithmic Flory-Huggins potential[2]

$$u_t = \Delta(-\Delta u + f(u)) \text{ with } f(u) = F'(u) = \frac{\theta_0}{2} \ln \frac{1+u}{1-u} - \theta_1 u,$$

and $F(u) = \frac{\theta_0}{2}\big((1+u)\ln(1+u) + (1-u)\ln(1-u)\big) - \frac{\theta_1}{2}u^2.$

[2] If $\|u_0\|_\infty < 1$, then $\|u\|_\infty < 1$, see Elliot-Garcke'94, Debussche-Detttori'95.

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- Wasserstein gradient flows
  - Poisson-Nernst-Planck (PNP) system: $(c_1, c_2 > 0)$

$$E[c_1, c_2, \phi] = \int_\Omega \Big(c_1(\ln c_1 - 1) + c_2(\ln c_2 - 1) + \frac{1}{2}|\nabla\phi|^2\Big)dx$$

  - Keller-Segel system with free energy: $(0 < u < 1)$

$$E[u, \phi] = \int_\Omega \Big(u \ln u + (1-u)\ln(1-u) - u\phi + \frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}\phi^2\Big)dx$$

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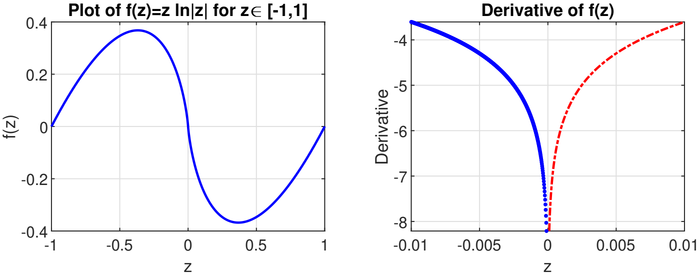
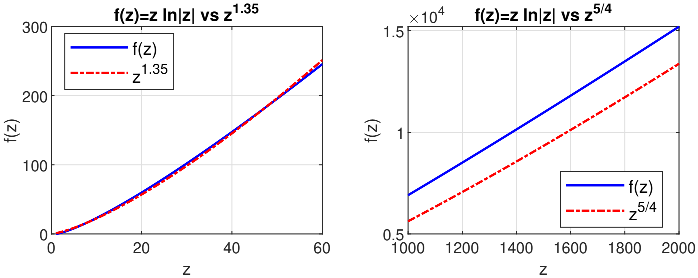
Some Observations



- Let $f(z) = z \ln|z|$. Then $f(0) = 0$ and $f(-z) = -f(z)$.
- Non-differentiable at $z = 0$, as

$$f'(z) = 1 + \ln|z|, \quad z \neq 0$$

Growth of $f(z)$ for $z > 1$



- The function $f(z) = z \ln |z|$ grows like

$$f(z) = |z|^p z, \quad 0 < p = p(z) < 1,$$

and $f(z) = o(z^{1+\epsilon})$ for $z \gg 1$.

- Composition: $g(x) = f(u(x))$:

$$g'(x) = f'(u)u'(x) = (1 + \ln|u(x)|)u'(x), \quad u \neq 0 \text{ or } u'(x) \neq 0.$$

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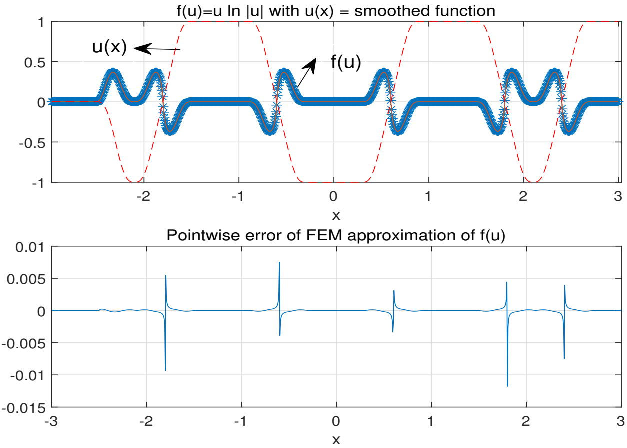
- Maximum point-wise errors of FEM approximation (globally $\approx O(h)$) occur at the locations where u changes sign. Very localised!



Locally Holder Continuous

Lemma: *Let* $f(z) = z \ln |z|$ *for* $z \in \mathbb{R}$.

- *If* $0 \le |z_1| \le |z_2| \le \epsilon$ *for* $\epsilon > 0$, *then for any* $\alpha \in (0,1)$,

$$|f(z_1) - f(z_2)| \le (2\epsilon)^{1-\alpha}(|\ln \epsilon| + 1)\,|z_1 - z_2|^{\alpha},$$

i.e., α*-Holder continuous on any finite interval containing* 0.

Outline

1. Examples and Motivations
2. **Existing Results: Theory & Numerics**
3. Direct Linearised CN Scheme
4. Positive and Ground-State Solutions
5. Numerical Results

Log-PDE: H^1-Regularity or More?

- Consider, for example, the Log-Heat equation:

$$\begin{cases} u_t - \Delta u = u \ln |u| & \text{in } \Omega, \ t > 0, \\ u(x,t) = 0 \ \text{ at } \partial\Omega, \ t \geq 0; & u(x,0) = u_0(x) \ \text{ on } \bar{\Omega}, \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary.

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  where $\Omega \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary.

- **Global H^1-solution** (cf. Chen-Luo-Liu'15): $u \in L^\infty(0,\infty; H_0^1(\Omega))$ and $u_t \in L^2(0,\infty; L^2(\Omega))$, if $u_0 \in H_0^1(\Omega)$, $J[u_0] < (2\pi)^{d/2} e^d/4$, and $I[u_0] \ge 0$, where **("Energy" has no definite sign ☹)**

$$I[u] := \|\nabla u\|^2 - \int_\Omega |u|^2 \ln|u|^2 \,\mathrm{d}x, \quad J[u] = \frac{1}{2} I[u] + \frac{1}{4}\|u\|^2.$$

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- **Blowup** (cf. Han'19): If $u_0 \in H_0^1(\Omega)$ and $I[u_0] < 0$, $\|u(\cdot,t)\| \to \infty$ as $t \to \infty$.

- **Super-exponential decay/growth** (cf. Alfaro-Carles'17): There exists $u_0 \in H_0^1$ with some $\eta > 0$, such that the unique solution decays super-exponentially

$$\|u(\cdot,t)\|_\infty \leq Ce^{-\eta e^t}.$$

Moreover, there exist $u_0 \in H_0^1$ and $\exists \xi > 0$, s.t.

$$\|u(\cdot,t)\|_\infty \geq Ce^{\xi e^t}.$$

- **Remark:** Some recent results on LogSE, e.g., H^1-regularity (cf. Carles-Gallagher'18)[3], but no much on numerical methods.

[3] R. Carles and I. Gallagher, Universal dynamics for the defocusing logarithmic Schrödinger equation, Duke Math. J., 2018.

Regularized Numerical Methods

- **Bao-Carles-Su-Tang** (SINUM'19): Regularized LogSE:

$$\begin{cases} iu_t^\varepsilon(x,t) + \Delta u^\varepsilon(x,t) = \lambda u^\varepsilon(x,t)\ln(\varepsilon+|u^\varepsilon(x,t)|) & \text{in } \Omega, \quad t>0, \\ u^\varepsilon(x,t)=0 \quad \text{at } \partial\Omega, \quad t\ge 0; \quad u^\varepsilon(x,0)=u_0(x) & \text{on } \bar{\Omega}. \end{cases}$$

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- **Regularization error**: If $u_0 \in H^2(\Omega)$, then

$$\|u^\varepsilon - u\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1 \varepsilon, \quad \|u^\varepsilon - u\|_{L^\infty(0,T;H^1(\Omega))} \leq C_2 \sqrt{\varepsilon},$$

where $C_1, C_2 > 0$ are independent of ε.

- Crank-Nicolson-Leap-Frog in time with FD in space:

THEOREM 3.1 (main result). *Assume that the solution u^ε is smooth enough over $\Omega_T := \Omega \times [0,T]$, i.e.,*

$$(A) \qquad u^\varepsilon \in C\left([0,T]; H^5(\Omega)\right) \cap C^2\left([0,T]; H^4(\Omega)\right) \cap C^3\left([0,T]; H^2(\Omega)\right),$$

and there exist $\varepsilon_0 > 0$ and $C_0 > 0$ independent of ε such that

$$\|u^\varepsilon\|_{L^\infty(0,T;H^5(\Omega))} + \|\partial_t^2 u^\varepsilon\|_{L^\infty(0,T;H^4(\Omega))} + \|\partial_t^3 u^\varepsilon\|_{L^\infty(0,T;H^2(\Omega))} \le C_0,$$

uniformly in $0 \le \varepsilon \le \varepsilon_0$. Then there exist $h_0 > 0$ and $\tau_0 > 0$ sufficiently small with $h_0^2 \sim \varepsilon e^{-CT|\ln(\varepsilon)|^2}$ and $\tau_0^2 \sim \varepsilon e^{-CT|\ln(\varepsilon)|^2}$ such that, when $0 < h \le h_0$ and $0 < \tau \le \tau_0$ satisfying the stability condition (3.7), we have the following error estimates

$$(3.9) \qquad \begin{aligned} &\|e^{\varepsilon,k}\| \le C_3(\varepsilon,T)(h^2+\tau^2), \qquad 0 \le k \le \frac{T}{\tau}, \\ &\|e^{\varepsilon,k}\|_{H^1} \le C_4(\varepsilon,T)(h^2+\tau^2), \quad \|u^{\varepsilon,k}\|_\infty \le \Lambda + 1, \end{aligned}$$

where $\Lambda = \|u^\varepsilon\|_{L^\infty(\Omega_T)}$, $C_3(\varepsilon,T) \sim e^{CT|\ln(\varepsilon)|^2}$, $C_4(\varepsilon,T) \sim \frac{1}{\varepsilon} e^{CT|\ln(\varepsilon)|^2}$, and C depends on C_0.

$$\|u^n - u_h^{\varepsilon,n}\| \le e^{C_\varepsilon T (\ln \varepsilon)^2}(\varepsilon + \tau^2 + h^2).$$

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- Bao-Carles-Su-Tang (Numer. Math.'19): Regularized Lie-Trotter time splitting method: L^2-error bound: $O(\tau^{\frac{1}{2}} \ln \varepsilon^{-1})$.

Outline

Our Approach & Error Analysis

- **Time discretization**: Find $U^{n+1} \in H_0^1(\Omega)$ for $0 \le n \le N_t - 1$, s.t.

$$\frac{U^{n+1} - U^n}{\tau} - \Delta\Big(\frac{U^{n+1} + U^n}{2}\Big) = \lambda U^n \ln |U^n| \quad \text{in} \;\; \Omega,$$

with $U^0 = u_0$. FD, FEM or spectral method can be used in space.

- **Remarks**
  - It is first-order discretized at $t = t_n$. If $u_0 \equiv 0$, then $U^{n+1} \equiv 0$.

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  - The non-differentiability of the log-term is not in favour of extrapolation or Newton iteration (for an implicit scheme).
  - Higher-order schemes can be used for e.g., positive, regular solutions.

Essential Tools for Error Analysis

- **Theorem** (Locally Holder Continuity, W.-Yan'22)

Let $f(z) = z \ln |z|$. *If* $u, v \in L^\infty(\Omega)$, *then for any* $\epsilon > 0$ *and any* $\alpha \in (0,1)$,

$$\|f(u) - f(v)\| \le (2\epsilon)^{1-\alpha}(|\ln \epsilon| + 1)\,\|u - v\|^\alpha + \Big(\max_{\epsilon \le z \le \Lambda_\infty}\{|\ln z| + 1\}\Big)\,\|u - v\|$$

where $\|\cdot\|$ *is the* L^2*-norm and*

$$\Lambda_\infty := \max\{\|u\|_\infty, \|v\|_\infty\}.$$

- **Lemma** (Nonlinear Gronwall's inequality, W.-Yan'22)

  Let c_1, c_2, c_3 *be positive constants, and let* $\{y(n)\}$ *satisfy*

$$y(n) \le c_1 + c_2 \sum_{m=0}^{n-1} y^{\alpha}(m) + c_3 \sum_{m=0}^{n-1} y(m), \quad n \ge 1, \ \alpha \in (0,1].$$

  Then for any $\alpha \in (0,1]$,

$$y(n) \le c_1 \left(1 + (c_1^{\alpha-1} c_2 + c_3) \frac{(1 + \alpha c_1^{\alpha-1} c_2 + c_3)^n - 1}{\alpha c_1^{\alpha-1} c_2 + c_3} \right), \quad n \ge 1.$$

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- **Remarks**

  (i) If $\alpha = 1$, it reduces to the linear Gronwall's inequality.

  (ii) A similar continuous integral version also holds.

- **Lemma** (Log-Sobolev inequality, Gross'76)

  If $u \in H_0^1(\Omega)$, then for any $a > 0$,

$$2\int_\Omega |u(x)|^2 \ln\left(\frac{|u(x)|}{\|u\|}\right) \mathrm{d}x + d(1+\ln a)\|u\|^2 \le \frac{a^2}{\pi}\|\nabla u\|^2,$$

  or equivalently,

$$\int_\Omega u^2 \ln u^2 \mathrm{d}x + \big(d(1+\ln a) - \ln\|u\|^2\big)\|u\|^2 \le \frac{a^2}{\pi}\|\nabla u\|^2.$$

Main Result: FEM in Space

Theorem [Log-Heat] (W.-Yan'22) : Under the condition of global existence as in Chen et al.'15, we further assume that the solution of the Log-Heat equation has the regularity:

$$u \in C^2([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^2(\Omega)), \tag{1}$$

and $(\ln \tau)^2 + (\ln |\ln h|)^2 \leq c$, then for $\alpha \in (1/2, 1)$,

$$\|u^n - u_h^n\| \leq C_\tau(\tau + h^{2\alpha}), \quad C_\tau \sim e^{cT(\ln \tau)^2}.$$

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Remarks:

- Used the argument for parabolic problems by introducing an auxiliary semi-discretised elliptic problem of U^n as in B. Li-W. Sun'13.
- The extra log-factor appears inevitable, but seems insignificant.

Theorem (LogSE) : Assume the regularity condition (1) holds, and

$$C_1 h^{2\alpha} \le \tau \le C_2 h^{d/2}, \quad (\ln \tau)^2 + (\ln |\ln h|)^2 \le C_3. \tag{2}$$

Then we have that for $\alpha \in (1/2, 1)$,

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Remark:

- The argument of Li-Sun'13 and J. Wang'14 cannot be applied due to the non-differentiability of the logarithmic nonlinear term.
- This led to the conditions in (2) largely from the use of inverse inequality.
- Compared with regularized approach, the conditions are on the original PDEs (at least positive solutions having such regularity).

Outline

Motivations

- The LogSE for N particles (Bialynicki et al' 1979):

$$\mathrm{i}\hbar\,\partial_t\psi(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N,t) = \left[-\hbar^2\sum_{k=1}^{N}\frac{1}{2m_k}\Delta_k - b\ln\left(|\psi|^2a^{3N}\right)\right]\psi(\boldsymbol{r}_1\ldots,\boldsymbol{r}_N,t),$$

- Dimensionless LogSE in $d = 3N$ dimensions

$$\mathrm{i}\partial_t\Psi(\boldsymbol{r},t) = \left[-\Delta - \ln|\Psi|^2\right]\Psi(\boldsymbol{r},t), \quad \boldsymbol{r}\in\mathbb{R}^{3N}, \tag{3}$$

where $\Psi = a^{3N/2}\psi$ and

$$\boldsymbol{r} = \hbar^{-1}\sqrt{2b}\left(\boldsymbol{r}_1\sqrt{m_1}\ldots,\boldsymbol{r}_N\sqrt{m_N}\right).$$

- Time-harmonic problem: if $\Psi(\boldsymbol{r},t) = e^{\frac{\omega}{2\lambda}-\mathrm{i}\omega t}u(\boldsymbol{r})$ is a solution of (3), then

$$-\Delta u(\boldsymbol{r}) + \lambda u(\boldsymbol{r})\ln|u(\boldsymbol{r})|^2 = 0, \quad \boldsymbol{r}\in\mathbb{R}^d. \tag{4}$$

- The problem of finding a positive ground state solution $u > 0$,

$$\Delta u + u \ln |u| = 0 \ \text{ in } \ \mathbb{R}^d, \quad u \to 0, \ \ r = |x| \to +\infty$$

  is of fundamental importance.

- It has a unique solution $u(r) = \exp(d/2 - r^2/4)>0$ if $d \in [1, 9]$ (cf. Troy'16): Ground-state solution:

$$E(v) = \|\nabla v\|^2 - \int_{\mathbb{R}^d} |v|^2 \ln |v|^2 \mathrm{d}x \geq E[u] = d(1 + \ln \pi/2).$$

- Related to the steady state problem of nonlinear Klein-Gordon and parabolic equations as $p \to 0^+$:

$$u_{tt} = \Delta u + u|u|^p - u, \quad u_t = \Delta u + u|u|^p - u$$

  Remark: The existence of positive solutions for the above was studied e.g., by Berestycki-Lions'83, Coffman'96, etc..

- **Our Goal:** Find analytically or compute numerically positive solutions in more general setting, e.g., with a potential V.

- For example, we consider the steady-state problem:

$$-\Delta u + V u = \lambda u \ln |u|, \quad u > 0 \tag{5}$$

- We also consider time-dependent problems, e.g.,

$$u_t - \Delta u + V u = \lambda u \ln |u|, \quad u > 0 \tag{6}$$

- Ideally to design **positivity-preserving schemes**, but non-trivial for some schemes.

Our Approach: Exponential Substitution

- Introduce the exponential substitution: $u = e^v > 0$.[4]
- It favours the log-term:

$$u \ln |u| = uv, \quad u_t = uv_t, \quad \Delta u = u(\Delta v + |\nabla v|^2)$$

- The steady-state equation (5) reads

$$\Delta v + |\nabla v|^2 + \lambda v = V \tag{7}$$

- The time-dependent problem (6) becomes

$$v_t - (\Delta v + |\nabla v|^2) - \lambda v = -V \tag{8}$$

[4] Huang-Shen (SISC'21) introduced the exponential substitution for constructing positivity preserving schemes for Poisson-Nernst-Planck (PNP) equations.

Selected Analytic Solutions

- If $V = V(r)$, we seek the axis-symmetric solution of

$$v'' + \frac{d-1}{r}v' + (v')^2 + \lambda v = V(r), \quad r > 0$$

- For example, if $V = \kappa r^{-1}$ (Coulomb's potential), then

$$u(r) = \exp\Big(-\frac{\lambda}{4}r^2 + \frac{\kappa}{\varepsilon^2(d-1)}r - \frac{\kappa^2}{\lambda(d-1)^2} + \frac{d}{2}\Big), \quad d \geq 2, \lambda > 0.$$

- We can also find Gaussian solutions for polynomial $V = \kappa|x|^m$ when $m = 0, 1, 2$, or more general

$$.V(r) = \kappa_1 r^2 + \kappa_2 r + \kappa_3 + \kappa_4 r^{-1}, \;\; V(r) = \lambda a_2 \log r + a_2(a_2 + d - 2)r^{-2}.$$

- We can show some solutions are ground-state, but some are excited state solutions for respective $V(r)$.

- Time-dependent problem (8) with $V = \kappa|x|^m$ has the analytic positive solution $u(r,t) = e^{v(r,t)}$ with

$$v(r,t) = A(t)e^{\lambda t} + B(t)P(r), \quad P(r) = r^2 + \alpha, \tag{9}$$

where A, B can be solved out from some solvable ODEs.

- The same techniques can be applied to study the LogSE with $V = \kappa|x|^m$ that can enrich the studies by Carles-Ferriere'21.

- Computing positive solutions for initial-valued BVPs in more general setting (Ongoing)!

Outline

Numerical Results: Accuracy Test

- **Scheme**: Find $u_h^{n+1} \in V_h^0$ for $0 \le n \le N_t - 1$, s.t.

$$\Big(\frac{u_h^{n+1} - u_h^n}{\tau}, v_h\Big) + \Big(\nabla\Big(\frac{u_h^{n+1} + u_h^n}{2}\Big), \nabla v_h\Big) = \big(\lambda u_h^n \ln|u_h^n|, v_h\big), \quad \forall v_h \in V_h^0,$$

  with $u_h^0 = I_h u_0$.

- Test the positive (smooth) exact solution given in (9), and expect to get the optimal order: $O(\tau + h^2)$!

Table 1: Piecewise Linear FEM ($h = 2^{-M}, \tau = h^2, T = 1$)

M	$\|e\|_2$	Order	$\|e\|_\infty$	Order
4	4.43e-3	-	5.91e-3	-
8	1.02e-3	2.12	1.41e-3	2.07
16	2.47e-4	2.05	3.47e-4	2.02
32	6.08e-5	2.02	8.65e-5	2.00
64	1.51e-5	2.01	2.16e-5	2.00
128	3.77e-6	2.00	5.40e-6	2.00

Table 2: Piecewise Quadratic FEM ($h = 2^{-M}$, $\tau = h^3, T = 1$)

M	$\|e\|_2$	Order	$\|e\|_\infty$	Order
2	8.05e-3	-	9.93e-3	-
4	8.65e-4	3.22	1.21e-3	3.03
8	1.00e-4	3.11	1.52e-4	2.99
16	1.21e-5	3.06	1.91e-5	3.00
32	1.48e-6	3.03	2.37e-6	3.00
64	1.83e-7	3.01	2.96e-7	3.00

Spectral Methods in Space

Table 3: Convergence order in space ($\tau = 1 \times 10^{-5}, T = 1$)

N	$\|e\|_2$	Order	$\|e\|_\infty$	Order
20	8.16e-3	-	8.16 e-3	-
24	3.91e-3	4.04	3.91e-3	4.21
28	1.66e-3	5.57	1.66e-3	5.80
32	6.21e-4	7.34	6.21e-4	7.58
36	2.07e-4	9.32	2.07e-4	9.56
40	6.19e-5	11.48	6.19e-5	11.71
44	1.66e-5	13.82	1.66e-5	13.98
48	4.00e-6	16.33	4.00e-6	16.09

Table 4: Convergence order in time ($N = 2^{10}, T = 1$)

τ = 1.25e-3	$\|e\|_2$	Order	$\|e\|_\infty$	Order
τ	1.02e-3	-	1.02e-3	-
$\tau/2^1$	5.12e-4	1.00	5.04e-3	1.00
$\tau/2^2$	2.56e-4	1.00	2.52e-4	1.00
$\tau/2^3$	1.28e-4	1.00	1.26e-4	1.00
$\tau/2^4$	6.41e-5	1.00	6.31e-4	1.00
$\tau/2^5$	3.21e-5	1.00	3.16e-5	1.00

Non-positive Solutions: H^{1+}-Regularity

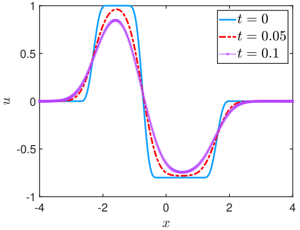


Table 5: Convergence in space (it is first-order in time)

N	$\|e\|_2$	Order	$\|e\|_\infty$	Order
2^6	2.96e-2	-	2.16e-2	-
2^7	1.45e-2	1.03	1.09e-2	0.99
2^8	6.93e-3	1.07	5.19e-3	1.06
2^9	3.25e-3	1.09	2.43e-3	1.09
2^{10}	1.39e-3	1.22	1.04e-3	1.22

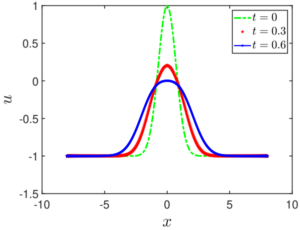


Table 6: Convergence in space (it is first-order in time)

N	$\|e\|_2$	Order	$\|e\|_\infty$	Order
2^6	8.01e-2	-	8.01e-2	-
2^7	4.35e-2	0.94	4.35e-2	0.88
2^8	1.64e-2	1.55	1.64e-2	1.41
2^9	8.11e-3	1.10	8.11e-3	1.01
2^{10}	3.48e-3	1.22	3.48e-3	1.22

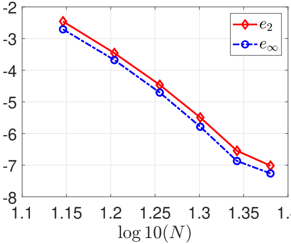
Numerical Results on LogSE

We test the numerical scheme on the LogSE:

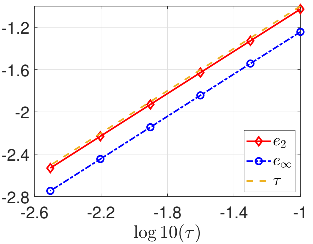
$$\mathrm{i}u_t + \Delta u = \lambda u \ln|u|, \quad t > 0; \quad u(x,0) = u_0(x),$$

with the exact Gaussian solution:

$$u(x,t) = b_0 e^{\frac{\lambda}{2}(x-2vt)^2 + \mathrm{i}(vx-(\phi_0+v^2)t)}, \quad t \geq 0.$$



(a) Spatial Conv.: $\tau = 10^{-7}$



(b) Temporal Conv.: $N = 256$

Take the initial data:

$$u_0(x) = \sum_k b_k e^{-\frac{a_k}{2}(x-x_k)^2 + \mathrm{i}v_k x}$$

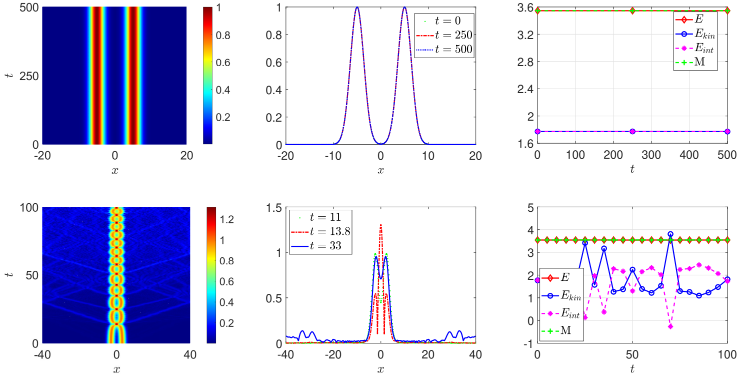


Figure 1: $\sqrt{|u(x,t)|}$, $|u(x,t)|$ and energies & mass at different time: (i) $x_1 = -x_2 = -5, v_k = 0, b_k = a_k = 1$; (ii) $x_1 = -x_2 = -3, v_k = 0, b_k = a_k = 1$.

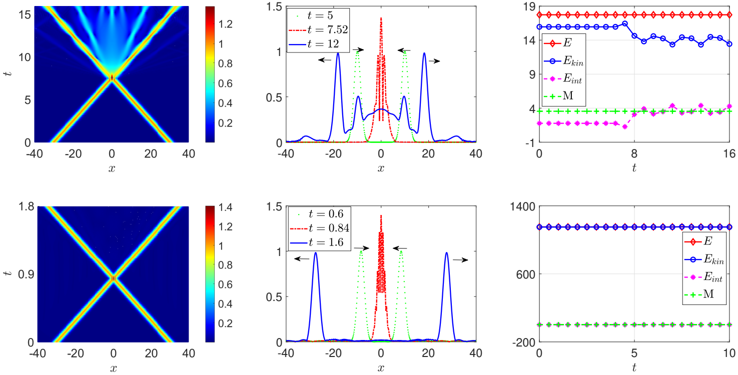


Figure 2: $\sqrt{|u(x,t)|}$, $|u(x,t)|$ and energies & mass at different time:
(iii) $v_1 = -v_2 = 2, x_1 = -x_2 = -30, b_k = a_k = 1 (k = 1, 2)$;
(iv) $v_1 = 18, v_2 = -18, x_1 = -30, x_2 = 30, b_2 = b_1 = 1, a_1 = a_2 = 1$.

Mass Preserving Scheme (ongoing)

- **IMEX Relaxed Runge-Kutta Scheme:**
  - **Step 1**: Find $\tilde{u}^{n+1}$ through BDF1:

$$\mathrm{i}\frac{\tilde{u}^{n+1} - u^n}{\tau} + \Delta\tilde{u}^{n+1} = \lambda f(u^n). \tag{10}$$

  - **Step 2**: Find u^{n+1} through correction:

$$u^{n+1} = u^n + \mathrm{i}\tau\gamma_n(\Delta\tilde{u}^{n+1} - \lambda f(\tilde{u}^{n+1})). \tag{11}$$

- Relaxation parameter:

$$\gamma_n = \begin{cases} 1, & \|\Delta\tilde{u}^{n+1} - \lambda f(\tilde{u}^{n+1})\| = 0, \\ \frac{2\,\mathrm{Im}\{(\Delta\tilde{u}^{n+1} - \lambda f(\tilde{u}^{n+1}), u^n)\}}{\tau\|\Delta\tilde{u}^{n+1} - \lambda f(\tilde{u}^{n+1})\|^2}, & \|\Delta\tilde{u}^{n+1} - \lambda f(\tilde{u}^{n+1})\| \neq 0. \end{cases} \tag{12}$$

We have

$$\|u^{n+1}\|^2 = \|u^n\|^2. \tag{13}$$

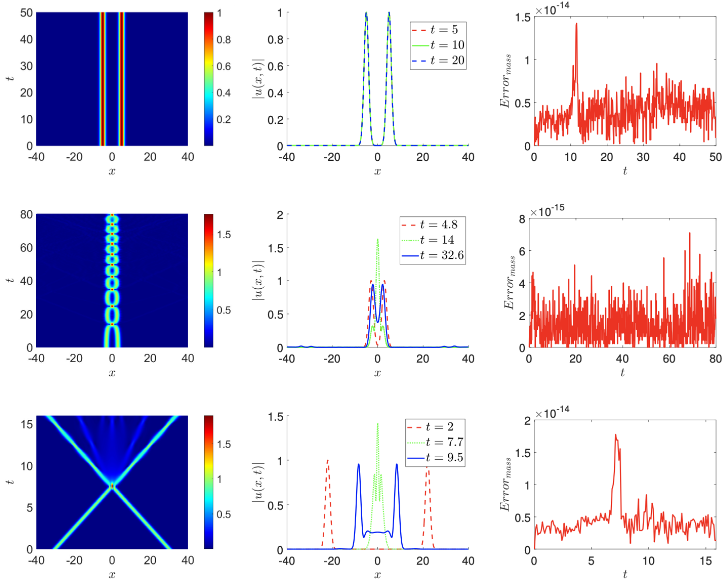


Figure 3.1: Plots of $|u(x,t)|$ (first column); $|u(x,t)|$ at different time (second column) and evolution of mass error (third column).

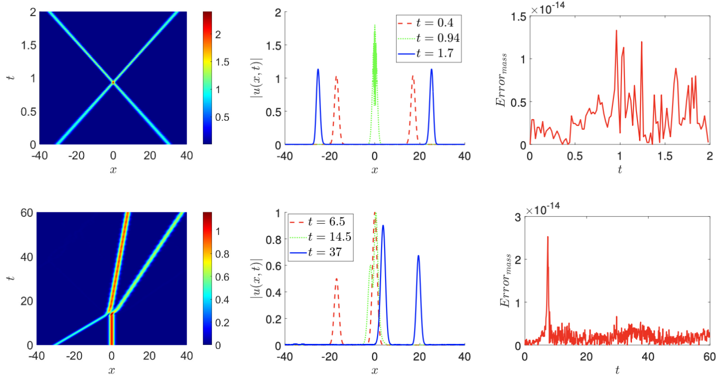


Figure 3.1: Plots of $|u(x,t)|$ (first column); $|u(x,t)|$ at different time (second column) and evolution of mass error (third column).

Summary

- The non-differentiability of logarithmic nonlinear term needs special care in discretisation.
- New tools were introduced for the analysis, though the results might not be the best at this moment.
- Exponential substitution is a feasible and simpler way to study and compute the positive solutions.
- Many issues are under-explored, e.g., fractional LogSE in space, time-fractional LogSE(?), ... Stay tuned!

THANK YOU!