On spectral Petrov-Galerkin method for solving optimal control problem governed by fractional diffusion equations with fractional noise

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Background and motivation

Optimal Control Problmes

$$\min_{u \in U,\ q \in Q} J(u, q)$$

s.t.

$$e(u, q) = 0, \ c(u, q) \in \mathcal{K}.$$

state: $u \in U$, control: $q \in Q$; U, Q, Z, Y are Banach spaces
$J : W \to \mathbb{R}$ denotes an objective functional, $W = U \times Q$
$e : W \to Z$ and $c : W \to Y$ are operators, $\mathcal{K} \subset Y$ is a convex set.

Optimal Control with SDEs & SPDEs constraints

Control of a stochastic Burgers model of turbulence
[Prato-Debussche'99, SIAM J. Control Optim.]
Optimal aerodynamics design under uncertainties
[Schulz-Schillings'09, AIAA]
FEM of SOCP constrained by stochastic elliptic PDEs
[Hou-Lee-Manouzi'11, JMAA]
Stochastic collocation for OCP with stochastic PDE constraints
[Tiesler-Kirby-Xiu'12, SIAM J. Control Optim.]
Stochastic optimal Robin boundary control problems
[Chen-Quarteroni-Rozza'13, SIAM J. Numer. Anal.]
Solving stochastic optimal control problem by 2FBSDEs
[Zhao-Zhou-Tao'17, Commun. in Comput. Phys.]

Fractional Diffusion Equations

- anomalous diffusion • long-time memory • long-distance effect

Pollutants transport in groundwater

[Cushman-Ginn'93, Transport Porous Med.]

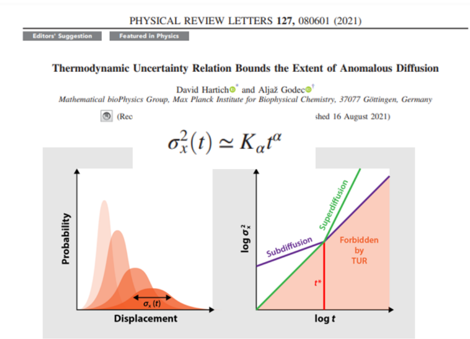
Image Processing

[Buades-Coll-Morel'10, SIAM Rev.

Viscoelasticity mechanics

[Mainardi'10, Imperial College Press]

Turbulence......



Consider an optimal control problem (OCP) governed by a space-fractional diffusion-reaction equation with additive fractional noise (SFDE):

$$\min_{q\in U_{ad}} J(u,q) = \mathbb{E}[\frac{1}{2}\|u-u_d\|^2_{L^2(I)} + \frac{\gamma}{2}\|q\|^2_{L^2(I)}] \tag{2.1}$$

subject to

$$\begin{cases} \mathcal{L}^\alpha_\theta u + \lambda u = \dot{W}^H(x) + q(x), & x \in I := (0,1), \\ u(x) = 0, & x \in \partial I, \end{cases} \tag{2.2}$$

where U_{ad} is an admissible set defined by

$$U_{ad} = \{q \in L^2(I) : \int_I q(x)\,dx \geq 0\}, \tag{2.3}$$

$\dot{W}^H$: formal derivative of fBm in x, $H \in (0,1)$;

γ, λ are constants, $\gamma > 0$, $\lambda \geq 0$;

q: a deterministic control, u_d: a given target function;

$\mathcal{L}^\alpha_\theta u := -[\theta\ {}_0D^\alpha_x + (1-\theta)\ {}_xD^\alpha_1]$: a general two-sided fractional operator, $\alpha \in (1,2)$, $\theta \in [0,1]$.

the operator $\mathcal{L}_\theta^\alpha$

Left and right Riemann-Liouville fractional derivatives:

$$
{}_0D_x^\alpha u(x) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_0^x \frac{u(s)}{(x-s)^{\alpha-1}} ds, \quad x > 0,
$$
$$
{}_xD_1^\alpha u(x) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_x^1 \frac{u(s)}{(s-x)^{\alpha-1}} ds, \quad x < 1.
$$

$\mathcal{L}_\theta^\alpha = -[\theta \; {}_0D_x^\alpha + (1-\theta) \; {}_xD_1^\alpha]$ **(nonlocal, singularity)**

Nonlocal: fast algorithm;
Singularity: graded mesh; correction term; non-polynomial bases
Compensate for the weak singularity⇒ **the weighted Sobolev space**
[Babuska-Guo'01, SIAM J. Numer. Anal.], [Guo-Wang'04, J. Approx. Theory], [Hao-Zhang'21, APNUM], [Li-Cao-Wang'22, Comput. Math. Appl.]

model of the bidirectional asymmetric anomalous diffusion:

- diffusion in hydrology [Benson-Wheatcraft'00, Water Resour. Res.]
- plasma turbulent transport [del-Castillo-Negrete'06, Phys. Plasmas]

$\mathcal{L}_{1/2}^\alpha \Longleftrightarrow (-\Delta)^{\frac{\alpha}{2}}$ (fractional Laplacian operator)$\Longleftrightarrow -\Delta$ $(\alpha = 2)$

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Definition and intrinsic properties of fBm

A fBm of Hurst index $H \in (0,1)$ denoted by $W_H(t),\ t \geq 0$ is a centered continuous Gaussian process, which satisfies:

$$\mathbb{E}[W_H(t)] = 0;$$

$$\mathrm{Cov}(s,t) := \mathbb{E}[W_H(t)\, W_H(s)] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

self-similarity:
$\mathrm{Cov}(\alpha t, \alpha s) = \alpha^{2H}\mathrm{Cov}(t,s)$ for $\alpha > 0,\ t,\ s \geq 0$;

stationary increments:
$W_H(t) - W_H(s) \sim W_H(t-s)$ for $t > s > 0$;

Hölder continuous property
$|W_H(t) - W_H(s)| \leq M|t-s|^{\gamma},\ \gamma < H$, a.s.,
where $M > 0$ is a constant, $t, s > 0$.

[A. N. Kolmogorov (1940); B. B. Mandelbrot, J. W. Van Ness, SIAM Review (1968)]

Describing the correlated random fluctuations

Table 1. Relation of R/σ and N for groups of phenomena

Phenomena	No. of cases	N years	R/σ	log N	log R/σ	K
(a) Group of 99 Cases						
River levels, discharges, and runoff	8	35	7·5	1·54	0·85	0·68
	8	45	8·9	1·65	0·94	0·70
	8	62	13·1	1·79	1·08	0·72
	9	108	16·4	2·02	1·19	0·69
Roda Nilometer	12	105	19·6	2·02	1·27	0·75
	10	208	36·5	2·32	1·54	0·77
	9	309	53·9	2·49	1·72	0·79
	8	420	60·3	2·56	1·77	0·78
	7	511	67·7	2·71	1·82	0·75
	6	613	81·3	2·79	1·89	0·77
	5	716	104	2·85	2·01	0·79
	4	820	122	2·91	2·08	0·80
	3	927	129	2·96	2·11	0·79
	2	1,040	130	3·02	2·12	0·78
Mean of 99 cases	—	—	—	2·28	1·48	0·75

- **H. E. Hurst (1951,1956)**
- **Reveal the long-term storage of reservoirs (find K)**
- **The Hurst index was named after him**

- **R. M. Pereira, et. al., J. Fluid Mech. (2016)**
- **Model of velocity field of turbulence: the Hurst index H=1/3**

velocity field representing a realistic local structure of turbulence:

$$\boldsymbol{u}^{\epsilon}(\boldsymbol{x}) = -\int_{\mathbb{R}^3} \varphi_L(\boldsymbol{x}-\boldsymbol{z}) \frac{\boldsymbol{x}-\boldsymbol{z}}{|\boldsymbol{x}-\boldsymbol{z}|_{\epsilon}^{5/2-H}} \wedge \mathrm{e}^{\gamma \boldsymbol{X}^{\epsilon}(\boldsymbol{z})} \boldsymbol{W}(\mathrm{d}z) \,, \qquad (1.4)$$

where $X^{\epsilon}(z)$ is an isotropic trace-free symmetric random matrix, which structure recall the one of the deformation field (1.3), and given explicitly by a tensor Wiener integral that we will specify later. The non dimensional constant γ governs the level of intermittency Let us finally remark that a crucial step of this construction, as dictated by the short-time dynamics of the Euler equations, is the intrinsically dependence of this statistically isotropic matrix X^{ϵ} on the vector white noise $\boldsymbol{W}$. We can see, given a Hurst exponent that we will take to be $H = 1/3$ to be consistent with K41 phenomenology, that the

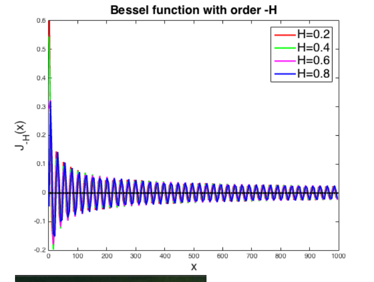
A representation of the fBm

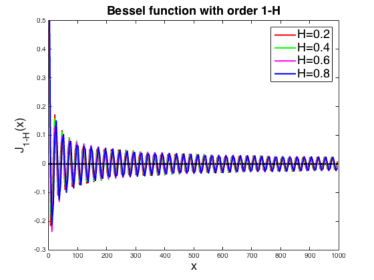
$$W^H = c_H\left(\sum_{k=1}^{\infty}\frac{\sin(\alpha_k x)}{\alpha_k^{1+H}J_{1-H}(\alpha_k)}\xi_k + \sum_{k=1}^{\infty}\frac{\cos(\beta_k x)}{\beta_k^{1+H}J_{-H}(\beta_k)}\zeta_k\right),\ x\in[0,1], \quad (2.4)$$

- $c_H = \sqrt{2/\pi}\,\Gamma^{1/2}(1+2H)\sin^{1/2}(\pi H)$
- α_k's, β_k's are the positive zeros of the Bessel function J_{-H} and J_{1-H}
- ξ_k's and ζ_k's are mutually independent standard Gaussian random variables.

[Dzhaparidze-Zanten'04, Probab. Theory Relat. Fields]

Bessel function

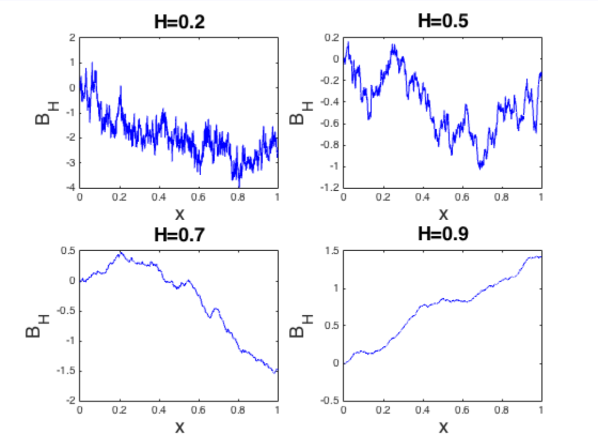




$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - \alpha^2)y = 0;$$

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}.$$

Simulation of the fBm



Lemma

For the Bessel function J_ν, where $\nu > -1$, we have

$$\begin{aligned} J^2_{1+\nu}(z) + J^2_\nu(z) &\approx \frac{2}{\pi z}, \text{ for large } |z|. \\ 2\nu J_\nu(x) &= xJ_{\nu+1}(x) + xJ_{\nu-1}(x). \end{aligned}$$

Let z_k be the positive zeros of the Bessel function J_ν. When k is large, we have

$$z_k = k\pi + \frac{\pi}{2}(\nu - \frac{1}{2}) - \frac{4\nu^2 - 1}{8(k\pi + \frac{\pi}{2}(\nu - \frac{1}{2}))} + O(\frac{1}{k^3}).$$

Lemma

Let α_n be the positive zeros of J_{-H} and β_n be the positive zeros of J_{1-H}. There exists a positive constant C independent of n such that

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{1}{4m^4}\left(\frac{1}{m\pi + \alpha_n} + \frac{1}{m\pi - \alpha_n}\right)^2 &\leq C\frac{1}{\alpha_n^4}, \\ \sum_{m=1}^{\infty} \frac{1}{4m^4}\left(\frac{1}{m\pi + \beta_n} + \frac{1}{m\pi - \beta_n}\right)^2 &\leq C\frac{1}{\beta_n^4}. \end{aligned}$$

Framework of the spectral-expansion-based algorithm

$$-\Delta u(x) + f(u(x)) = g(x) + \dot{W}^H(x), \quad x \in \mathcal{D} = [0,1],$$

with Dirichlet boundary condition $u(x) = 0,\ x \in \partial\mathcal{D}$,

Step 1. Choose an appropriate spectral expansion of the noises, e.g., for $\dot{W}^H(x)$

$$\dot{W}_M^H(x) = c_H \left(\sum_{k=1}^{M} \frac{\cos(\alpha_k x)}{\alpha_k^H J_{1-H}(\alpha_k)} \xi_k + \sum_{k=1}^{M} \frac{\sin(\beta_k x)}{\beta_k^H J_{-H}(\beta_k)} \zeta_k \right), \quad x \in [0,1],$$

Step 2. Approximate the noise by the truncation of its spectral expansion in the equation to get a deterministic equations with random parameters, e.g.,

$$-\Delta u_M(x) + f(u_M(x)) = g(x) + \dot{W}_M^H(x), \quad x \in \mathcal{D} = [0,1],$$

Step 3. Analyze the consistency of the resulted approximated equation, e.g., ,

$$\mathbb{E}[\|u - u_{\mathrm{M}}\|^2] \le O(M^{-2H-2}).$$

Step 4. Construct a full-discrete scheme, e.g., the finite element method

$$\mathbb{E}[\left\|u - u_{\mathrm{M}}^h\right\|^2] \le Ch^{2H+2}, (by\ taking\ M = O(h^{-1})).$$

W. Cao, Z. Hao, Z. Zhang, J. Sci. Comput, 2022, DOI:10.1007/s10915-022-01779-x.

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Numerical methods for Elliptic SPDEs

$$-\Delta u(x) + f(u(x)) = g(x) + \xi(x), \quad x \in \mathcal{D},$$

with Dirichlet boundary condition $u(x) = 0, \; x \in \partial\mathcal{D}$, where $\mathcal{D} \subset \mathbb{R}^d$.

$\xi(x)$ denotes Gaussian noise $\dot{W}^Q(x)$, white noise when $Q = I$

- strong order $1-\varepsilon$ ($d=2, \; Q=I$) [*GyÖngy-Martinez'06, Stochastics*, *Cao-Yang-Yin'07, Numer. Math.*]
- strong order $2-d/2$ ($Q=I$) [*Zhang-Rozoviskii-Karniadakis'16, Numer. Math.*]
- strong order $2-d/2-\rho$ [*Cao-Hong-Liu'20, Commun. Math. Res.*]

$\xi(x)$ denotes fBm-type $\dot{W}^H(x)$, white noise when $H = 1/2$

- $H \in (0,1)$, strong order $H+1/2$ ($d=1$) [*Cao-Hong-Liu'17 (IMA J. Numer. Anal.)*]
- $H \in (0,1)$, strong order $H+1$ ($d=1$) [*Cao-Hao-Zhang'22 (J. Sci. Comput.)*]

$$(-\Delta)^{\frac{\alpha}{2}} u(x) + f(u(x)) = g(x) + \xi(x), \quad x \in \mathcal{D},$$

Weighted Sobolev space+spectral PG method [*Hao-Zhang'21, SIAM UQ*]

Aim of this work

- Consider regularity of the fractional noise $\dot{W}^H$ in weighted Sobolev space $H^r_{\omega^{\sigma^*,\sigma}}$ and error estimate of the spectral Petrov-Galerkin (PG) method for the state equation.
- Present a framework on constructing the regularity of the optimal control problem (2.1)-(2.3) in weighted Sobolev space by regarding the fractional noise as rough inputs.
- Develop a spectral PG approximation and give its error estimate for the optimal control problem (2.1)-(2.3) in weighted Sobolev space.

Jacobi polynomials

- $P_n^{\gamma,\beta}(t)$ denotes the classical Jacobi polynomial of degree n.

  where$\gamma, \beta > -1,\ n \in \mathbb{N}$ and $t \in [-1, 1]$,
- Let $t = 2x - 1$.

$$Q_n^{\gamma,\beta}(x) = P_n^{\gamma,\beta}(2x-1),\ \ x \in [0, 1].$$

Orthogonality. The Jacobi polynomials $Q_n^{\gamma,\beta}(x)$ are mutually orthogonal with respect to the weight $(1-x)^\gamma x^\beta$: for $\gamma, \beta > -1$,

$$\int_0^1 (1-x)^\gamma x^\beta Q_n^{\gamma,\beta}(x)\, Q_m^{\gamma,\beta}(x)\, dx = \delta_{mn} \| Q_n^{\gamma,\beta} \|^2_{\omega^{\gamma,\beta}}, \qquad (3.1)$$

where δ_{mn} is the Kronecker symbol and

$$\| Q_n^{\gamma,\beta} \|^2_{\omega^{\gamma,\beta}} = \frac{1}{2n+\gamma+\beta+1} \cdot \frac{\Gamma(n+\beta+1)\Gamma(n+\gamma+1)}{\Gamma(n+1)\Gamma(n+\gamma+\beta+1)} := h_n^{\gamma,\beta} = O(\frac{1}{n}).$$

Properties of Jacobi polynomials.

Lemma [*Ervin'18, Math. Comput.*] For the n-th order Jacobi polynomials $Q_n^{\sigma,\sigma^*}(x)$ and $Q_n^{\sigma^*,\sigma}(x)$, where $x \in [0,1]$, $\sigma^* + \sigma = \alpha$ and σ is determined by

$$\theta = \frac{\sin(\pi(\alpha-\sigma))}{\sin(\pi(\alpha-\sigma)) + \sin(\pi\sigma)},$$ (determine the weight index)

it holds that

$$\mathcal{L}_\theta^\alpha \left[(1-x)^\sigma x^{\sigma^*} Q_n^{\sigma,\sigma^*}(x)\right] = \lambda_{\theta,n}^\alpha Q_n^{\sigma^*,\sigma}(x), \tag{3.2}$$

in which

$$\lambda_{\theta,n}^\alpha = -\frac{\sin(\pi\alpha)}{\sin(\pi\sigma^*) + \sin(\pi\sigma)} \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)}.$$

Remark To ensure (19) uniquely solvable, we constrain $\sigma, \sigma^* \in (0,1]$.

For instance, $\sigma = \sigma^* = \alpha/2$ for $\theta = 1/2$ and $\sigma = 1, \sigma^* = \alpha - 1$ for $\theta = 1$.

Weighted Sobolev spaces

$\boldsymbol{L^2_{\omega^{\gamma,\beta}}(I)}$. Denote $\omega^{\gamma,\beta}(x) = (1-x)^\gamma x^\beta,\ \gamma,\beta > -1$. Then

$$L^2_{\omega^{\gamma,\beta}}(I) = \{v : \int_I \omega^{\gamma,\beta}(x) v^2(x)\, dx < \infty\}$$

$$(u,v)_{\omega^{\gamma,\beta}} = \int_I \omega^{\gamma,\beta} uv dx\ , \quad \|u\|_{\omega^{\gamma,\beta}} = \sqrt{(u,u)_{\omega^{\gamma,\beta}}}.$$

$\boldsymbol{H^s_{\omega^{\gamma,\beta}}(I)}$. The weighted Sobolev space with non-negative integer s is defined as

$$H^s_{\omega^{\gamma,\beta}}(I) = \Big\{ v : D^k v(x) \in L^2_{\omega^{\gamma+k,\beta+k}}(I), k = 0,1,\cdots,s \Big\},$$

$$\|v\|_{H^s_{\omega^{\gamma,\beta}}} = \Big(\sum_{k=0}^{s} |v|^2_{H^k_{\omega^{\gamma,\beta}}}\Big)^{1/2}, \quad |v|_{H^k_{\omega^{\gamma,\beta}}} = \|D^k v\|_{\omega^{\gamma+k,\beta+k}}.$$

- For $s \in \mathbb{R}^+$, $H^s_{\omega^{\gamma,\beta}}(I)$ can be defined by interpolation via the K-method.
- For $s < 0$, it is defined by the (weighted) L^2 duality.

Equivalent norm in $\boldsymbol{H^s_{\omega^{\gamma,\beta}}(I)}$. For $\forall s \in \mathbb{R}$,

$$\|v\|^2_{H^s_{\omega^{\gamma,\beta}}} = \sum_{n=0}^{\infty} (v_n^{\gamma,\beta})^2 h_n^{\gamma,\beta} (1+n^2)^s.$$

- $\gamma,\beta > -1$, $h_n^{\gamma,\beta} = \|Q_n^{\gamma,\beta}\|^2_{\omega^{\gamma,\beta}}$ and $v_n^{\gamma,\beta} = \frac{1}{h_n^{\gamma,\beta}} \int_I v(x) Q_n^{\gamma,\beta}(x) \omega^{\gamma,\beta}(x)\, dx.$

Regularity of the fBm

Lemma 1 For $0 < H < 1$, $\sigma, \sigma^* \in (0,1]$ determined by $\sigma + \sigma^* = \alpha$ and condition (19), it holds for any $\varepsilon > 0$ that

$$\mathbb{E}[\| \dot{W}^H \|^2_{H^{H-1-\varepsilon}_{\omega^{\sigma^*,\sigma}}}] < \infty. \tag{3.3}$$

Lemma 2 Let $a > -1$, we have

$$\cos(\alpha_k x) = \sum_{n=0}^{\infty} b^{a,1}_{n,k} Q^{a,a}_n(x), \;\; \sin(\beta_k x) = \sum_{n=0}^{\infty} b^{a,2}_{n,k} Q^{a,a}_n(x).$$

Then there exists a positive constant C independent of n and k, such that for $l > 0$

$$|b^{a,1}_{n,k}| + |b^{a,2}_{n,k}| \leq C n^{a+1-l} k^{l-a-1}.$$

Sketch of the proof of Lemma 1

$$\dot{W}^H(x) = \sum_{n=0}^{\infty} d^H_n Q^{\sigma^*,\sigma}_n,$$

$$\mathbb{E}[\| \dot{W}^H \|^2_{H^r_{\omega^{\sigma^*,\sigma}}}] = \sum_{n=0}^{\infty} \mathbb{E}[(d^H_n)^2] h^{\sigma^*,\sigma}_n (1+n^2)^r$$

$$\leq C \sum_{k=1}^{\infty} k^{2l-2H-1-2\delta} \sum_{n=1}^{\infty} n^{2\delta+2-2l} h^{\sigma^*,\sigma}_n (1+n^2)^r,$$

Semi-discrete state equation

$$\dot{W}_M^H = c_H \left(\sum_{k=1}^{M} \frac{\cos(\alpha_k x)}{\alpha_k^H J_{1-H}(\alpha_k)} \xi_k + \sum_{k=1}^{M} \frac{\sin(\beta_k x)}{\beta_k^H J_{-H}(\beta_k)} \zeta_k \right).$$

$$\begin{aligned} \mathcal{L}_\theta^\alpha y + \lambda y &= \dot{W}^H(x), \qquad x \in I := (0,1), \\ y(x) &= 0, \qquad x \in \partial I. \end{aligned} \tag{3.4}$$

Adopting the truncation of $\dot{W}^H(x)$, we approximate equation (3.4) by

$$\begin{aligned} \mathcal{L}_\theta^\alpha y_M + \lambda y_M &= \dot{W}_M^H(x), \qquad x \in I := (0,1), \\ y_M(x) &= 0, \qquad x \in \partial I, \end{aligned} \tag{3.5}$$

The spectral Petrov-Galerkin (PG) method of the state equation

Denote

$$a(u, v) := (u, \mathcal{L}^{\alpha}_{1-\theta}(\omega^{\sigma^*,\sigma} v)) + \lambda(u, v)_{\omega^{\sigma^*,\sigma}},$$

$$U_N = \{u | u = \omega^{\sigma,\sigma^*} v, v \in W_N\}, \ \ W_N = \mathrm{span}\{Q^{\sigma,\sigma^*}_m\}_{m=0}^N \subset H^{\alpha}_{\omega^{\sigma,\sigma^*}}(I),$$
$$Z_N = \{z | z = \omega^{\sigma^*,\sigma} v, v \in V_N\}, \ \ V_N = \mathrm{span}\{Q^{\sigma^*,\sigma}_m\}_{m=0}^N \subset H^{\alpha}_{\omega^{\sigma^*,\sigma}}(I).$$

and

$$\mathcal{L}^2_{\omega^{a,b}}(I) := L^2(\Omega; L^2_{\omega^{a,b}}(I)) = \{v \mid \mathbb{E}[\|v\|^2_{\omega^{a,b}}] < \infty\},$$
$$\mathcal{H}^r_{\omega^{a,b}}(I) := L^2(\Omega; H^r_{\omega^{a,b}}(I)) = \{v \mid \mathbb{E}[\|v\|^2_{H^r_{\omega^{a,b}}}] < \infty\}, \ \ a, b > -1.$$

The weak formulation of (3.5): to find $u \in \mathcal{L}^2_{\omega^{-\sigma,-\sigma^*}}(I)$ such that

$$a(u, v) = (\dot{W}^H + q, v)_{\omega^{\sigma^*,\sigma}}, \ \forall v \in \mathcal{H}^{\alpha}_{\omega^{\sigma^*,\sigma}}(I),$$

Then the spectral PG method can be used for discretization of (3.5) in physical space: To find $y_{M,N} \in U_N$ such that

$$a(y_{M,N}, v_N) = (\dot{W}^H_M(x), v_N)_{\omega^{\sigma^*,\sigma}}, \ \forall v_N \in V_N. \tag{3.6}$$

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Then the spectral PG method can be used for discretization of (3.5) in physical space:
To find $y_{M,N} \in U_N$ such that

$$a(y_{M,N}, v_N) = (\dot{W}^H_M(x), v_N)_{\omega^{\sigma^*,\sigma}}, \;\forall v_N \in V_N. \tag{3.6}$$

Convergence of the spectral PG method

Lemma 3 For $0 < H < 1$ and any $\varepsilon > 0$, $s < H - 1$, we have that

$$\mathbb{E}[\| \dot{W}^H - \dot{W}_M^H \|^2_{H^s_{\omega^{\sigma^*,\sigma}}}] \le C\varepsilon^{-1} M^{-2(H-1-s-\varepsilon)}. \quad (3.7)$$

Theorem 1 Let y be the solution of (3.4) and $y_{M,N}$ be the solution of (3.6). Then there exists a number $N_0 > 0$, such that when $N > N_0$, we have

$$\mathbb{E}[\|y_{M,N}\|^2_{\omega^{-\sigma,-\sigma^*}}] \le C\mathbb{E}[\| \dot{W}_M^H \|^2_{H^{H-1-\varepsilon}_{\omega^{\sigma^*,\sigma}}}], \quad (3.8)$$

and

$$\mathbb{E}[\|y - y_{M,N}\|^2_{\omega^{-\sigma,-\sigma^*}}] \le C\varepsilon^{-1} M^{-2(H-1+\alpha-\varepsilon)} + CN^{-2(H-1+\alpha-\varepsilon)}.$$

The key step of the proof

$$\mathbb{E}[\|y - y_{M,N}\|^2_{\omega^{-\sigma,-\sigma^*}}] \le \underbrace{2\mathbb{E}[\|y - y_M\|^2_{\omega^{-\sigma,-\sigma^*}}]}_{\text{regularity of } \dot{W}^H + \text{stability estimate}} + \underbrace{2\mathbb{E}[\|y_M - y_{M,N}\|^2_{\omega^{-\sigma,-\sigma^*}}]}_{\text{(3.7)+spectral PG theory}}$$

The OCP with routh inputs

$$\min_{q\in U_{ad}} \tilde{J}(u,q) = \frac{1}{2}\|u-u_d\|^2_{L^2(I)} + \frac{\gamma}{2}\|q\|^2_{L^2(I)} \tag{4.1}$$

subject to

$$\begin{cases} \mathcal{L}^{\alpha}_{\theta} u + \lambda u = f(x) + q(x), & x \in I := (0,1), \\ u(x) = 0, & x \in \partial I, \end{cases} \tag{4.2}$$

- f and u_d are given deterministic functions.
- Note that the admissible set U_{ad} is closed and convex, and cost functional $\tilde{J}$ is strictly convex,;
- The control problem (4.1)-(4.2) admits a unique solution by a standard argument.

The first-order optimality condition

Theorem 2 Suppose that $q \in U_{ad}$ is an optimal control for the problem (4.1)-(4.2) and u is the associated state variable. Then there exists an adjoint state variable z, such that (u, z, q) satisfies the following optimality conditions

$$\begin{cases} \mathcal{L}_{\theta}^{\alpha} u + \lambda u = f(x) + q(x), & x \in I, \\ u(x) = 0, & x \in \partial I, \end{cases} \tag{4.3}$$

$$\begin{cases} \mathcal{L}_{1-\theta}^{\alpha} z + \lambda z = u(x) - u_d(x), & x \in I, \\ z(x) = 0, & x \in \partial I, \end{cases} \tag{4.4}$$

and the variational inequality

$$\int_I (\gamma q + z)(v - q)\, dx \geq 0, \quad v \in U_{ad}. \tag{4.5}$$

Remark *[Chen-Yi-Liu'08, SIAM J. Numer. Anal.]* The variational inequality (4.5) is equivalent to the following condition

$$\gamma q = \max\{0, \bar{z}\} - z, \tag{4.6}$$

in which $\bar{z} = \frac{1}{|I|} \int_I z(x)\, dx$ and $|I|$ denotes the length of interval I.

Regularity of (u, z, q) in (4.3)-(4.5)

Theorem 3 Let (u, z, q) be the solution of optimality system (4.3)-(4.5). If $f \in H^{r_1}_{\omega^{\sigma^*,\sigma}}(I)$, $u_d \in H^{r_2}_{\omega^{\sigma,\sigma^*}}(I)$ and $q \in L^2(I)$, $r_1, r_2 \geq -\alpha$, then the regularity of the state u, adjoint state z and control q satisfy

$$u \in H^{\min\{r_1+\alpha,\, r_2+2\alpha,\, s\}}_{\omega^{\sigma,\sigma^*}}(I),$$

$$z \in H^{\min\{r_1+2\alpha,\, r_2+\alpha,\, s\}}_{\omega^{\sigma^*,\sigma}}(I),$$

$$q \in H^{\min\{r_1+2\alpha,\, r_2+\alpha,\, s\}}_{\omega^{\sigma^*,\sigma}}(I),$$

respectively. Moreover, we have

$$\omega^{-\sigma,-\sigma^*} u \in H^{\min\{r_1,\, r_2+\alpha,\, s\}+\alpha}_{\omega^{\sigma,\sigma^*}}(I),$$

$$\omega^{-\sigma^*,-\sigma} z \in H^{\min\{r_1+\alpha, r_2,\, s\}+\alpha}_{\omega^{\sigma^*,\sigma}}(I),$$

$$\omega^{-\sigma^*,-\sigma} q \in H^{\min\{r_1+\alpha,\, r_2,\, s\}+\alpha}_{\omega^{\sigma^*,\sigma}}(I),$$

where $s = 3\min(\sigma, \sigma^*) + 1 - \varepsilon$.

Note that $\alpha \in (1, 2)$, $\sigma,\ \sigma* \in (0, 1)$, $\sigma + \sigma* = \alpha$.

Refer to: *[Chen-Yi-Liu'08, SIAM J. Numer. Anal.]; [Ervin'20, arXiv:1911.03261]; [Hao-Zhang'21, APNUM]; [Li-Cao-Wang'22, Comput. Math. Appl.]*

The OCP with fractional noise

$\dot{W}^H \in \mathcal{H}^{H-1-\varepsilon}_{\omega^{\sigma*},\sigma}$. **Consider** $\dot{W}^H(x)$ **as rough inputs, we have**

$$\begin{cases} \mathcal{L}^{\alpha}_{\theta} u + \lambda u = \dot{W}^H(x) + q(x), & x \in I, \\ u(x) = 0, & x \in \partial I, \end{cases} \tag{4.7}$$

$$\begin{cases} \mathcal{L}^{\alpha}_{1-\theta} z + \lambda z = u(x) - u_d(x), & x \in I, \\ z(x) = 0, & x \in \partial I, \end{cases} \tag{4.8}$$

and the variational inequality $\hat{J}'(q)(v-q) \geq 0, \forall v \in U_{ad}$. The deterministic control q allows us to switch the order of expectation and Frechét derivation, thus we derive

$$\mathbb{E}[\int_I (\gamma q + z)(v-q)\,dx] \geq 0, \quad v \in U_{ad}. \Leftrightarrow \gamma q = \max\{0, \mathbb{E}[\bar{z}]\} - \mathbb{E}[z]. \tag{4.9}$$

We solve (4.7)-(4.9) to get the state u, the adjoint state z and the control q.

Error estimate of the OCP with fractional noise

Denote

$$b(z,w) := (z, \mathcal{L}_\theta^\alpha(\omega^{\sigma,\sigma^*} w)) + \lambda(z,w)_{\omega^{\sigma,\sigma^*}}.$$

The weak formulation of the first-order optimality condition: given $u_d \in \mathcal{H}^r_{\omega^{\sigma,\sigma^*}}(I)$, $r \geq -\alpha$, to find $(u,z,q) \in \mathcal{L}^2_{\omega^{-\sigma,-\sigma^*}}(I) \times \mathcal{L}^2_{\omega^{-\sigma^*,-\sigma}}(I) \times U_{ad}$ such that

$$\begin{cases} a(u,v) = (\dot{W}^H + q, v)_{\omega^{\sigma^*,\sigma}}, \ \forall v \in \mathcal{H}^\alpha_{\omega^{\sigma^*,\sigma}}(I), & (4.10a) \\ b(z,w) = (u - u_d, w)_{\omega^{\sigma,\sigma^*}}, \ \forall w \in \mathcal{H}^\alpha_{\omega^{\sigma,\sigma^*}}(I), & (4.10b) \\ \mathbb{E}[(\gamma q + z, v - q)] \geq 0, \ \forall v \in U_{ad}. & (4.10c) \end{cases}$$

The discrete first-order optimality condition (by truncated spectral expansion of $\dot{W}^H$ and the spectral PG method): given $u_d \in \mathcal{H}^r_{\omega^{\sigma,\sigma^*}}(I)$, $r \geq -\alpha$, to find $(u_{MN}, z_{MN}, q_{MN}) \in U_N \times Z_N \times U_{ad}$ such that

$$\begin{cases} a(u_{MN}, v_N) = (\dot{W}^H_M + q_{MN}, v_N)_{\omega^{\sigma^*,\sigma}}, \ \forall v_N \in V_N, & (4.11a) \\ b(z_{MN}, w_N) = (u_{MN} - u_d, w_N)_{\omega^{\sigma,\sigma^*}}, \ \forall w_N \in W_N, & (4.11b) \\ \mathbb{E}[(\gamma q_{MN} + z_{MN}, v - q_{MN})] \geq 0, \ \forall v \in U_{ad}, & (4.11c) \end{cases}$$

Error estimate of the OCP with fractional noise

Theorem 4 Let (u, z, q) and (u_{MN}, z_{MN}, q_{MN}) be the solution of (4.10) and (4.11), respectively. For $u_d \in \mathcal{H}^r_{\omega^{\sigma,\sigma^*}}(I)$, $r \geq -\alpha$, we have

$$\mathbb{E}[\|u - u_{MN}\|^2_{\omega^{-\sigma,-\sigma^*}}] + \mathbb{E}[\|z - z_{MN}\|^2_{\omega^{-\sigma^*,-\sigma}}] + \|q - q_{MN}\|^2$$
$$\leq C\varepsilon^{-1} M^{-2(H-1+\alpha-\varepsilon)} + CN^{-2\min\{H-1+\alpha-\varepsilon, r+\alpha\}}.$$

To obtain the error estimate, given $u_d \in \mathcal{H}^r_{\omega^{\sigma,\sigma^*}}(I)$, $r \geq -\alpha$, we have to introduce the following auxiliary system:

$$\begin{cases} a(u_{MN}(q), v_N) = (\dot{W}^H_M + q, v_N)_{\omega^{\sigma^*,\sigma}}, \ \forall v_N \in V_N, \\ b(z_{MN}(q), w_N) = (u_{MN}(q) - u_d, w_N)_{\omega^{\sigma,\sigma^*}}, \ \forall w_N \in W_N, \\ b(z_{MN}(u), w_N) = (u - u_d, w_N)_{\omega^{\sigma,\sigma^*}}, \ \forall w_N \in W_N. \end{cases}$$

Sketch of the proof

$$\mathbb{E}[\|u - u_{MN}\|^2_{\omega^{-\sigma,-\sigma^*}}]$$

$$\leq 2\mathbb{E}[\|u - u_{MN}(q)\|^2_{\omega^{-\sigma,-\sigma^*}}] \quad + \quad 2\mathbb{E}[\|u_{MN}(q) - u_{MN}\|^2_{\omega^{-\sigma,-\sigma^*}}]$$

$$C\varepsilon^{-1} M^{-2(H-1+\alpha-\varepsilon)} + CN^{-2\min\{H-1+\alpha-\varepsilon, r+\alpha\}} \qquad C\|q - q_{MN}\|^2$$

$$E[\|z - z_{MN}\|^2_{\omega^{-\sigma^*,-\sigma}}]$$

$$\leq 2\mathbb{E}[\|z - z_{MN}(u)\|^2_{\omega^{-\sigma^*,-\sigma}}] \quad + \quad 2\mathbb{E}[\|z_{MN}(u) - z_{MN}\|^2_{\omega^{-\sigma^*,-\sigma}}]$$

$$CN^{-2\min\{H-1+2\alpha-\varepsilon, r+\alpha, s+\alpha\}} \qquad C\mathbb{E}[\|u - u_{MN}\|^2_{\omega^{-\sigma,-\sigma^*}}]$$

$$\|q - q_{MN}\|^2 \leq \mathbb{E}[\|z - z_{MN}(q)\|_{\omega^{-\sigma^*,-\sigma}}]$$

$$\leq 2\mathbb{E}[\|z - z_{MN}(u)\|^2_{\omega^{-\sigma^*,-\sigma}}] \quad + \quad 2\mathbb{E}[\|z_{MN}(u) - z_{MN}(q)\|^2_{\omega^{-\sigma^*,-\sigma}}]$$

$$C\|u - u_{MN}(q)\|_{\omega^{-\sigma,-\sigma^*}}$$

Sketch of the proof

$$\mathbb{E}[\|u - u_{MN}\|^2_{\omega^{-\sigma,-\sigma^*}}]$$

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$$C\varepsilon^{-1}M^{-2(H-1+\alpha-\varepsilon)} + CN^{-2\min\{H-1+\alpha-\varepsilon,r+\alpha\}} \qquad C\|q - q_{MN}\|^2$$

$$E[\|z - z_{MN}\|^2_{\omega^{-\sigma^*,-\sigma}}]$$

$$\leq 2\mathbb{E}[\|z - z_{MN}(u)\|^2_{\omega^{-\sigma^*,-\sigma}}] \quad + \quad 2\mathbb{E}[\|z_{MN}(u) - z_{MN}\|^2_{\omega^{-\sigma^*,-\sigma}}]$$

$$CN^{-2\min\{H-1+2\alpha-\varepsilon,r+\alpha,s+\alpha\}} \qquad C\mathbb{E}[\|u - u_{MN}\|^2_{\omega^{-\sigma,-\sigma^*}}]$$

$$\|q - q_{MN}\|^2 \leq \mathbb{E}[\|z - z_{MN}(q)\|_{\omega^{-\sigma^*,-\sigma}}]$$

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$$C\|u - u_{MN}(q)\|_{\omega^{-\sigma,-\sigma^*}}$$

Sketch of the proof

$$\mathbb{E}[\|u - u_{MN}\|^2_{\omega^{-\sigma,-\sigma^*}}]$$

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Sketch of the proof

$$\mathbb{E}[\|u-u_{MN}\|^2_{\omega^{-\sigma,-\sigma^*}}]$$

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Sketch of the proof

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$C\varepsilon^{-1} M^{-2(H-1+\alpha-\varepsilon)} + CN^{-2\min\{H-1+\alpha-\varepsilon, r+\alpha\}} \qquad C\|q - q_{MN}\|^2$

$E[\|z - z_{MN}\|^2_{\omega^{-\sigma^*,-\sigma}}]$

$\leq 2\mathbb{E}[\|z - z_{MN}(u)\|^2_{\omega^{-\sigma^*,-\sigma}}] \quad + \quad 2\mathbb{E}[\|z_{MN}(u) - z_{MN}\|^2_{\omega^{-\sigma^*,-\sigma}}]$

$CN^{-2\min\{H-1+2\alpha-\varepsilon, r+\alpha, s+\alpha\}} \qquad C\mathbb{E}[\|u - u_{MN}\|^2_{\omega^{-\sigma,-\sigma^*}}]$

$\|q - q_{MN}\|^2 \leq \mathbb{E}[\|z - z_{MN}(q)\|_{\omega^{-\sigma^*,-\sigma}}]$

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Sketch of the proof

$$\mathbb{E}[\|u - u_{MN}\|^2_{\omega^{-\sigma,-\sigma^*}}]$$

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Sketch of the proof

$$\mathbb{E}[\|u - u_{MN}\|^2_{\omega^{-\sigma,-\sigma^*}}]$$

$$\leq 2\mathbb{E}[\|u - u_{MN}(q)\|^2_{\omega^{-\sigma,-\sigma^*}}] \quad + \quad 2\mathbb{E}[\|u_{MN}(q) - u_{MN}\|^2_{\omega^{-\sigma,-\sigma^*}}]$$

$$C\varepsilon^{-1}M^{-2(H-1+\alpha-\varepsilon)} + CN^{-2\min\{H-1+\alpha-\varepsilon,r+\alpha\}} \qquad C\|q - q_{MN}\|^2$$

$$E[\|z - z_{MN}\|^2_{\omega^{-\sigma^*,-\sigma}}]$$

$$\leq 2\mathbb{E}[\|z - z_{MN}(u)\|^2_{\omega^{-\sigma^*,-\sigma}}] \quad + \quad 2\mathbb{E}[\|z_{MN}(u) - z_{MN}\|^2_{\omega^{-\sigma^*,-\sigma}}]$$

$$CN^{-2\min\{H-1+2\alpha-\varepsilon,r+\alpha,s+\alpha\}} \qquad C\mathbb{E}[\|u - u_{MN}\|^2_{\omega^{-\sigma,-\sigma^*}}]$$

$$\|q - q_{MN}\|^2 \leq \mathbb{E}[\|z - z_{MN}(q)\|_{\omega^{-\sigma^*,-\sigma}}]$$

$$\leq 2\mathbb{E}[\|z - z_{MN}(u)\|^2_{\omega^{-\sigma^*,-\sigma}}] \quad + \quad 2\mathbb{E}[\|z_{MN}(u) - z_{MN}(q)\|^2_{\omega^{-\sigma^*,-\sigma}}]$$

$$C\|u - u_{MN}(q)\|_{\omega^{-\sigma,-\sigma^*}}$$

Sketch of the proof

$$\mathbb{E}[\|u - u_{MN}\|^2_{\omega^{-\sigma,-\sigma^*}}]$$

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$$\leq 2\mathbb{E}[\|z - z_{MN}(u)\|^2_{\omega^{-\sigma^*,-\sigma}}] \quad + \quad 2\mathbb{E}[\|z_{MN}(u) - z_{MN}(q)\|^2_{\omega^{-\sigma^*,-\sigma}}]$$

$$C\|u - u_{MN}(q)\|_{\omega^{-\sigma,-\sigma^*}}$$

Numerical examples

- Errors are measured in the following way ($M_1 = 10^5$, $M_r = N_r = 4096$):

$$E_{MN}^{a,b}(p) = \left(\frac{1}{M_1} \sum_{i=1}^{M_1} \frac{\|p_{MN}(\omega_i) - p_{M_rN_r}(\omega_i)\|_{\omega^{a,b}}^2}{\|p_{M_rN_r}(\omega_i)\|_{\omega^{a,b}}^2} \right)^{1/2} \tag{5.1}$$

and

$$E_{MN} = E_{MN}^{-\sigma,-\sigma^*}(u) + E_{MN}^{-\sigma^*,-\sigma}(z) + E_{MN}^{0,0}(q).$$

Example 1. Take $\gamma = 1$, $u_d = \xi(1-x)^\beta x^\beta \cos x$ in the OCP with fractional noise (2.1)-(2.3), $\xi \sim N(0,1)$.

- Recall $\dot{W}^H \in \mathcal{H}_{\omega^{\sigma^*},\sigma}^{H-1-\varepsilon}$ and note that $u_d \in \mathcal{H}_{\omega^{\sigma,\sigma*}}^{2\beta+\min\{\sigma,\sigma*\}+1-\varepsilon}$.

Table: The numerical values of (σ, σ^*) corresponding to different α and θ (cited from *[Hao-Lin-Zhang'20, AMC]*).

θ	α=1.2	α=1.4	α=1.6	α=1.8
0.5	(0.6000, 0.6000)	(0.7000, 0.7000)	(0.8000, 0.8000)	(0.9000, 0.9000)
0.7	(0.8829, 0.3171)	(0.8602, 0.5398)	(0.8900, 0.7100)	(0.9411, 0.8589)
1.0	(1.0000, 0.2000)	(1.0000, 0.4000)	(1.0000, 0.6000)	(1.0000, 0.8000)

Table 1. Errors and convergence orders for solving (4.7)-(4.9) with $\theta = 0.7$ and $\beta = -0.8$. The expected convergence order is $\min\{r+\alpha,\ H-1+\alpha-\varepsilon\}$.

		H=0.9		H=0.6		H=0.4		H=0.2	
α	$M = N$	E_{MN}	order	E_{MN}	order	E_{MN}	order	E_{MN}	order
1.2	8	2.25e-01	*	3.92e-01	*	5.31e-01	*	7.40e-01	*
	16	1.16e-01	0.95	2.38e-01	0.72	3.67e-01	0.53	5.79e-01	0.35
	32	5.62e-02	1.05	1.37e-01	0.79	2.40e-01	0.61	4.23e-01	0.45
	64	2.69e-02	1.06	7.89e-02	0.80	1.56e-01	0.62	3.11e-01	0.45
Expected order			1.1		0.8		0.6		0.4
1.4	8	1.37e-01	*	2.58e-01	*	3.51e-01	*	5.12e-01	*
	16	6.27e-02	1.13	1.41e-01	0.88	2.20e-01	0.68	3.66e-01	0.49
	32	2.62e-02	1.26	7.20e-02	0.97	1.29e-01	0.76	2.45e-01	0.58
	64	1.07e-02	1.29	3.63e-02	0.99	7.41e-02	0.81	1.59e-01	0.62
Expected order			1.3		1.0		0.8		0.6
1.6	8	9.20e-02	*	1.81e-01	*	2.45e-01	*	3.73e-01	*
	16	3.77e-02	1.29	8.86e-02	1.03	1.38e-01	0.82	2.41e-01	0.63
	32	1.40e-02	1.43	4.03e-02	1.14	7.26e-02	0.93	1.45e-01	0.73
	64	5.04e-03	1.47	1.78e-02	1.17	3.66e-02	0.99	8.32e-02	0.80
Expected order			1.5		1.2		1.0		0.8
1.8	8	6.47e-02	*	1.31e-01	*	1.76e-01	*	2.77e-01	*
	16	2.39e-02	1.44	5.75e-02	1.18	8.93e-02	0.98	1.62e-01	0.77
	32	7.88e-03	1.60	2.32e-02	1.31	4.16e-02	1.10	8.70e-02	0.90
	64	2.50e-03	1.66	9.03e-03	1.36	1.85e-02	1.17	4.41e-02	0.98
Expected order			1.7		1.4		1.2		1.0

Table 2: Errors and convergence orders for solving (4.7)-(4.9) with $\theta = 0.7$ and $u_d = \xi \cos x$. The expected convergence order is $H - 1 + \alpha - \varepsilon$.

		H=0.9		H=0.6		H=0.4		H=0.2	
α	$M = N$	E_{MN}	order	E_{MN}	order	E_{MN}	order	E_{MN}	order
1.2	8	1.80e-01	*	3.65e-01	*	5.24e-01	*	7.57e-01	*
	16	9.29e-02	0.96	2.23e-01	0.71	3.61e-01	0.54	5.81e-01	0.38
	24	6.14e-02	1.02	1.64e-01	0.77	2.83e-01	0.59	4.86e-01	0.44
	32	4.51e-02	1.07	1.31e-01	0.78	2.38e-01	0.61	4.26e-01	0.46
Expected order			1.1		0.8		0.6		0.4
1.4	8	1.21e-01	*	2.49e-01	*	3.51e-01	*	5.28e-01	*
	16	5.61e-02	1.11	1.37e-01	0.86	2.20e-01	0.68	3.72e-01	0.50
	24	3.45e-02	1.20	9.38e-02	0.94	1.62e-01	0.75	2.94e-01	0.58
	32	2.40e-02	1.26	7.13e-02	0.96	1.30e-01	0.78	2.46e-01	0.62
Expected order			1.3		1.0		0.8		0.6
1.6	8	8.56e-02	*	1.78e-01	*	2.47e-01	*	3.84e-01	*
	16	3.55e-02	1.27	8.80e-02	1.02	1.40e-01	0.82	2.47e-01	0.64
	24	2.02e-02	1.39	5.61e-02	1.11	9.61e-02	0.92	1.84e-01	0.73
	32	1.34e-02	1.43	4.05e-02	1.13	7.33e-02	0.94	1.47e-01	0.78
Expected order			1.5		1.2		1.0		0.8
1.8	8	6.27e-02	*	1.31e-01	*	1.79e-01	*	2.85e-01	*
	16	2.32e-02	1.43	5.79e-02	1.17	9.07e-02	0.98	1.66e-01	0.78
	24	1.23e-02	1.56	3.44e-02	1.29	5.86e-02	1.08	1.16e-01	0.89
	32	7.75e-03	1.62	2.36e-02	1.31	4.27e-02	1.10	8.82e-02	0.96
Expected order			1.7		1.4		1.2		1.0

Concluding remarks

- Determine that the regularity index of the fractional noise in weighted Sobolev space $H^r_{\omega^{\sigma*},\sigma}$ and $r = H - 1 - \varepsilon$;
- Present error estimates for the approximated solution of the state equation, which is produced by adopting a spectral truncation of the fractional noise and the spectral PG method;
- To incorporate the weak singularity near boundaries, we construct a framework of regularity for the optimal control problem with fractional noise (rough inputs) in weighted Sobolev space;
- Develop the spectral Petrov-Galerkin method for the OCP with fractional noise and give the error estimates.
- **Future work:**

  Multi-dimensional problem;

  Problems with general non-local operator;

  More types of noises and weak convergence analysis;

  Fast algorithms.

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Thank you for your attention!

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