Analysis and numerical approximation of space-fractional boundary value problems in one and two space dimensions

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- Conservation of mass $\partial u/\partial t + \partial q/\partial x = f$+Fick law $q = -\partial u/\partial x$ $\Longrightarrow \partial u/\partial t - \partial^2 u/\partial x^2 = f \Longrightarrow$ Steady state $-D^2 u(x) = f(x)$.
- Nonlocal Fick law on spatial domain $(0,1)$: $q = -I_r^{2-\alpha}\partial u/\partial x$ where

$$I_r^{2-\alpha} := r\,{}_0I_x^{2-\alpha} + (1-r)\,{}_xI_1^{2-\alpha}, \quad 1 < \alpha < 2, \quad 0 \le r \le 1$$

and

$${}_0I_x^{2-\alpha}g := \frac{1}{\Gamma(2-\alpha)}\int_0^x \frac{g(s)}{(x-s)^{\alpha-1}}ds, \quad {}_xI_1^{2-\alpha}g = \frac{1}{\Gamma(2-\alpha)}\int_x^1 \frac{g(s)}{(s-x)^{\alpha-1}}ds$$

$$\Longrightarrow -DI_r^{2-\alpha}Du(x) = f(x).$$

- When $r = 1/2$, $-DI_r^{2-\alpha}D$ corresponds to $(-\Delta)^{\alpha/2}$.
- Variable-coefficient problems:

$$q = -K(x)I_r^{2-\alpha}\partial u/\partial x \Longrightarrow -D\big[K(x)I_r^{2-\alpha}D\big]u(x) = f(x),$$

$$q = -I_r^{2-\alpha}\big(K(x)\partial u/\partial x\big) \Longrightarrow -DI_r^{2-\alpha}\big(K(x)Du(x)\big) = f(x).$$

- Boundary conditions: $u(0) = u(1) = 0$.

- Let $\omega^{(a,b)} = \omega^{(a,b)}(x) := (1-x)^a\, x^b$ for $a, b > -1$, $\{G_n^{a,b}(x)\}$ be Jacobi polynomials on $(0,1)$ and $\hat{G}_n^{a,b}(x) = G_n^{a,b}(x)/\|G_n^{a,b}(x)\|$. Let $L^2_\omega(0,1) := \left\{g(x) : \int_0^1 \omega(x)\, g(x)^2\, dx < \infty\right\}$, for weight function $\omega(x)(>0 \text{ on } (0,1))$ associated with the inner product $(f,g)_\omega := \int_0^1 \omega(x) f(x) g(x) dx$ and the norm $\|g\|_\omega := (g,g)_\omega^{1/2}$.
- Weighted space $H^r_{\omega^{(a,b)}} := \left\{v \,|\, v \text{ is measurable and } \|v\|_{r,\,\omega^{(a,b)}} < \infty\right\}$ for $r \in \mathbb{N}$ with associated norm and semi-norm

$$\|v\|_{r,\,\omega^{(a,b)}} := \left(\sum_{j=0}^{r} \|D^j v\|^2_{\omega^{(a+j,b+j)}}\right)^{1/2}, \quad |v|_{r,\,\omega^{(a,b)}} := \|D^r v\|_{\omega^{(a+r,b+r)}}.$$

  $H^r_{\omega^{(a,b)}}$ with non-integer r could be defined by K-method of interpolation.
- Equivalent space: For $r \in \mathbb{R}$, $H^r_{(a,b)} := \left\{v \in L^2_{\omega^{(a,b)}} \,|\, \|v\|_{r,(a,b)} < \infty\right\}$ where $\|v\|^2_{r,(a,b)} = \sum_{j=0}^{\infty} (1+j^2)^r v_j^2$ and $\{v_j\}$ are Fourier coefficients of v under the orthonormal basis $\{\hat{G}_j^{a,b}(x)\}$ of $L^2_{\omega^{(a,b)}}$.

- Coercivity: $(I_r^{2-\alpha} Du, Du) \geq c_0 \|u\|^2_{H^{\alpha/2}}$.
- Eigen-relation: $-DI_r^{2-\alpha} D(\omega^{(\alpha-\beta,\beta)} G_n^{\alpha-\beta,\beta}(x)) = \lambda_n G_n^{\beta,\alpha-\beta}(x)$ where $\alpha - 1 \leq \alpha - \beta, \beta \leq 1$ satisfies $(1-r)\sin(\pi\beta) = r\sin(\pi(\alpha-\beta))$ and

$$\lambda_n = \frac{-\sin(\pi\alpha)}{\sin(\pi(\alpha-\beta)) + \sin(\pi\beta)} \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)}.$$

- $DI_r^{2-\alpha}\omega^{(\alpha-\beta,\beta)} G_n^{(\alpha-\beta\,,\,\beta)}(x) = \lambda_n\, G_{n+1}^{(\beta-1\,,\,\alpha-\beta-1)}(x)$ where

$$\lambda_n = \frac{\sin(\pi\alpha)}{\sin(\pi(\alpha-\beta)) + \sin(\pi\beta)} \frac{\Gamma(n+\alpha)}{n!} \sim -(n+1)^{\alpha-1}.$$

- $\ker(DI_r^{2-\alpha}) = \mathrm{span}\{k(x) := (1-x)^{\alpha-\beta-1}\, x^{\beta-1}\}$.
- $DI_r^{2-\alpha}(x\,k(x)) = \lambda_{-1}$ and $DI_r^{2-\alpha}((1-x)\,k(x)) = -\lambda_{-1}$ where

$$\lambda_{-1} := -(1-r)\,\Gamma(\alpha)\frac{\sin(\pi\alpha)}{\sin(\pi(\alpha-\beta))}.$$

- Consider the homogeneous boundary value problem of $-DI_r^{2-\alpha}Dw = f$.
- $f(x) \in L^2_{\omega^{(\beta,\alpha-\beta)}}$ could be expressed as $f(x) = \sum_{i=0}^{\infty} f_i G_i^{(\beta,\alpha-\beta)}(x)$.
- If we assume $w(x) = \omega^{(\alpha-\beta,\beta)}(x) \sum_{i=0}^{\infty} w_i \, G_i^{(\alpha-\beta,\beta)}(x)$, we apply the relation $-DI_r^{2-\alpha}D(\omega^{(\alpha-\beta,\beta)}G_n^{\alpha-\beta,\beta}(x)) = \lambda_n G_n^{\beta,\alpha-\beta}(x)$ to obtain

$$-DI_r^{2-\alpha}Dw = -DI_r^{2-\alpha}D\sum_{i=0}^{\infty} w_i\big(\omega^{(\alpha-\beta,\beta)}(x)G_i^{(\alpha-\beta,\beta)}(x)\big)$$
$$= \sum_{i=0}^{\infty} w_i\lambda_i G_i^{(\beta,\alpha-\beta)}(x) = \sum_{i=0}^{\infty} f_i G_i^{(\beta,\alpha-\beta)}(x) \Longrightarrow w_i = \frac{f_i}{\lambda_i}.$$

- Spectral approximation of w: N-term truncation $w_N(x)$.
- Error estimate: $\|w - w_N\|_{\omega^{-(\alpha-\beta,\beta)}} \le QN^{-\alpha}\|f\|_{\omega^{(\beta,\alpha-\beta)}}$.
- **Questions** for $-D(KI_r^{2-\alpha}D)u = f$ and $-DI_r^{2-\alpha}(KDu) = f$:
  - FEM: Loss of coercivity of bilinear forms (Wang-Yang SINUM 13).
  - Spectral method: Loss of eigen-relations.

- The solution to $-DI_r^{2-\alpha}(KDu) = f$ could be expressed in terms of w

$$u(x) \;=\; C_1 \int_0^x \frac{\omega^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)}\, ds \;+\; \int_0^x \frac{Dw(s)}{K(s)}\, ds \qquad (1)$$

where C_1 is chosen such that $u(1) = 0$.

- A direct verification

$$-DI_r^{2-\alpha}(KDu) = -DI_r^{2-\alpha}\big(C_1\omega^{(\alpha-\beta-1,\beta-1)}(x) + Dw(x)\big) = 0 + f(x).$$

- $f(x) \in L^2_{\omega^{(\beta,\alpha-\beta)}} \Longrightarrow$ unique solution $u \in L^\infty$ and $\|u\|_{L^\infty} \le Q\|f\|_{\omega^{(\beta,\alpha-\beta)}}$.
- Approximation u_N of u is given by (1) with w replaced by w_N.
- $f \in H^t_{\omega^{(\beta,\alpha-\beta)}}$ for $t \ge 0$ and $K' \in L^2_{\omega^{(\alpha-\beta\,,\,\beta)}} \Longrightarrow$

$$\|u - u_N\|_{\omega^{(-(\alpha-\beta),-\beta)}} \le QN^{-\alpha-t}\|f\|_{t,\,\omega^{(\beta\,,\,\alpha-\beta)}},$$

$$\|D(u - u_N)\|_{\omega^{(-(\alpha-\beta)+1\,,\,-\beta+1)}} \le QN^{-(\alpha-1)-t}\|f\|_{t,\,\omega^{(\beta\,,\,\alpha-\beta)}}.$$

- Equivalent form of $-D(KI_r^{2-\alpha}D)u = f$ is
$$-I_r^{2-\alpha}Du = f_1 - Af_2$$
where $A := K(0)\big[I_r^{2-\alpha}Du(x)\big]\big|_{x=0}$, $f_1(x) = K^{-1}(x)\int_0^x f(y)dy$ and $f_2(x) = K^{-1}(x)$.
- Spectral expansion $u(x) = \omega^{(\alpha-\beta,\beta)}(x)\sum_{i=0}^{\infty} u_i\, G_i^{(\alpha-\beta,\beta)}(x)$.
- A useful relation $I_r^{2-\alpha}D\big(\omega^{(\alpha-\beta,\beta)}G_n^{(\alpha-\beta\,,\,\beta)}(x)\big) = \lambda_n\, G_{n+1}^{(\beta-1\,,\,\alpha-\beta-1)}(x)$.
- $f \in L^2_{\omega^{(\beta,\alpha-\beta)}} \Longrightarrow f_1, f_2 \in L^2_{\omega^{(\beta-1,\alpha-\beta-1)}} \Longrightarrow$ unique $u \in L^2_{\omega^{(-(\alpha-\beta),-\beta)}}$ such that
$$\|u\|_{\omega^{(-(\alpha-\beta),-\beta)}} \le C\|f\|_{\omega^{(\beta,\alpha-\beta)}}$$
and
$$A = \big(f_1, 1\big)_{\omega^{(\beta-1,\alpha-\beta-1)}}\big/\big(K^{-1}, 1\big)_{\omega^{(\beta-1,\alpha-\beta-1)}}.$$
- $f \in H^t_{\omega^{(\beta,\alpha-\beta)}}$ and $K \in W^{t+1}_\infty$ for $t \ge 0 \Longrightarrow$
$$\|u - u_N\|_{\omega^{(-(\alpha-\beta),-\beta)}} \le Q\|K\|_{W^{t+1}_\infty} N^{-\alpha-t}\big(\|f\|_{t,\omega^{(\beta,\alpha-\beta)}} + 1\big).$$

- Bilinear form of $-D(KI_r^{2-\alpha}D)u = f$ is $a(g, \hat{g}) := (KI_r^{2-\alpha}Dg, D\hat{g})$.
- Recall that this bilinear form is not coercive.
- Product rule for $-D\big(K \cdot (I_r^{2-\alpha}Du)\big)$:

$$-D(I_r^{2-\alpha}D)u - \frac{K'}{K}(I_r^{2-\alpha}D)u = \frac{f}{K}$$

- Weak coercivity (Garding inequality): $\|g\|_{L^2}^2 + a(g,g) \geq Q\|g\|_{H^{\alpha/2}}^2$
  $\Longrightarrow$ Error estimate: $\|u - u_h\|_{L^2} + h^{\sigma-\alpha/2}\|u - u_h\|_{H^{\alpha/2}} \leq Qh^{2\sigma-\alpha}$ where σ is determined by $\|u - I_h u\|_{H^{\alpha/2}} \leq Q\|u\|_{H^\sigma} h^{\sigma-\alpha/2}$.
- Application to $-DI_r^{2-\alpha}(KDu) = f$ by using $KDu = D(Ku) - K'u$
  $\Longrightarrow -D\big(I_r^{2-\alpha}D(Ku)\big) + DI_r^{2-\alpha}(K'u) = f$
  $\Longrightarrow$ For $v := Ku$ it holds $-D\big(I_r^{2-\alpha}Dv\big) + DI_r^{2-\alpha}(K'v/K) = f$.
- Extended to study high-dimensional problems.

- Regularity of $-D(I_r^{2-\alpha}D)w=f$: $f\in L^2 \Longrightarrow w\in H^{\min\{\beta,\alpha-\beta\}+1/2-\varepsilon}$ (see e.g. Ervin JDE 21).
- Equivalent form of $-D(KI_r^{2-\alpha}D)u=f$ is

$$-DI_r^{2-\alpha}Du = f_1' - Af_2'$$

where $A := K(0)\big[I_r^{2-\alpha}Du(x)\big]\big|_{x=0}$, $f_1(x)=K^{-1}(x)\int_0^x f(y)dy$ and $f_2(x)=K^{-1}(x) \Longrightarrow f\in L^2$ implies $u\in H^{\min\{\beta,\alpha-\beta\}+1/2-\varepsilon}$.

- In fact, the above conclusion also holds for $-D(I_r^{2-\alpha}KD)u=f$. Recall that

$$u(x) \;=\; C_1\int_0^x \frac{\omega^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)}\,ds \;+\; \int_0^x \frac{Dw(s)}{K(s)}\,ds.$$

$\Longrightarrow$ Key ingredient: $x^\gamma\in H^{\gamma+1/2-\varepsilon}$ for $0<\gamma<1$ by interpolation estimate.

- It was shown that the bilinear form of $-DI_r^{2-\alpha}(KD)$ is not coercive and the accuracy of FEM is relatively low, which motivates the (weighted) Petrov-Galerkin method.
- Let $\omega(x) := (1-x)^{\alpha-\beta}x^\beta$ and $\omega^*(x) := (1-x)^\beta x^{\alpha-\beta}$. Based on the structure of $u(x)$ ($u(x) = \omega(x)\phi(x)$ for some $\phi(x)$), we propose a weak formulation of $-DI_r^{2-\alpha}(KD) = f$ via $B : H_\omega^1 \times H_{\omega^*}^{\alpha-1} \to \mathbb{R}$

$$B(\phi, \psi) := (KD(\omega\phi), I_{1-r}^{2-\alpha}D(\omega^*\psi)) = (f, \omega^*\psi).$$

  Here we used the following adjoint property $(I_r^{2-\alpha}g, \hat{g}) = (g, I_{1-r}^{2-\alpha}\hat{g})$.
- Key difficulty: Prove the inf-sup condition in the Banach-Nečas-Babuška Theorem

$$\sup_{0\neq\psi\in H_{\omega^*}^{\alpha-1}} \frac{B(\phi,\psi)}{\|\psi\|_{H_{\omega^*}^{\alpha-1}}} \geq c_0\|\phi\|_{H_\omega^1}.$$

- Idea: Given $\phi \in H_\omega^1$, find $\psi \in H_{\omega^*}^{\alpha-1}$ such that

$$D(\omega\phi)\cdot\left(I_{1-r}^{2-\alpha}D(\omega^*\psi)\right) \geq 0 \text{ or even } D(\omega\phi) = I_{1-r}^{2-\alpha}D(\omega^*\psi),$$

  which allows moving K outside of the inner product by its low bound and then using the orthogonality of Jacobi polynomials.

- For $\phi(x) = \sum_{i=0}^{\infty} \phi_i \hat{G}_i^{(\alpha-\beta\,,\,\beta)}(x) \in H_\omega^1$, define $\Phi_i := -\phi_i/\mu_i$ where $\{\mu_i\}$ are given by

$$I_{1-r}^{2-\alpha} D\Big(\omega^{(\beta,\,\alpha-\beta)}(x)\, G_k^{(\beta\,,\,\alpha-\beta)}(x)\Big) = \mu_k\, G_{k+1}^{(\alpha-\beta-1\,,\,\beta-1)}(x), \quad k \geq 0.$$

- $\mu_k \sim -(k+1)^{\alpha-1}$ as $k \to \infty$.
- $\phi(x)$ could be rewritten as $\phi(x) = -\sum_{i=0}^{\infty} \mu_i \Phi_i \hat{G}_i^{(\alpha-\beta,\beta)}(x)$.
- It is also known that for $n \geq 0$

$$D\Big(\omega^{(\alpha-\beta,\,\beta)}(x) G_n^{(\alpha-\beta,\beta)}(x)\Big) = -(n+1)\omega^{(\alpha-\beta-1,\beta-1)} G_{n+1}^{(\alpha-\beta-1,\beta-1)}(x).$$

- Define $\psi(x)$ by $\psi(x) = \sum_{j=0}^{\infty}(j+1)\Phi_j \hat{G}_j^{(\beta,\alpha-\beta)}(x)$.
- As $\phi \in H_\omega^1$, $\|\phi\|_{H_\omega^1}^2 = \sum_{i=0}^{\infty}(1+i^2)^1 \left(\mu_i \Phi_i\right)^2 \simeq \sum_{i=0}^{\infty}(1+i^2)^\alpha \Phi_i^2 < \infty$, then $\|\psi\|_{H_{\omega^*}^{\alpha-1}}^2 = \sum_{j=0}^{\infty}(1+j^2)^{\alpha-1}\left((j+1)\,\Phi_j\right)^2 \simeq \sum_{j=0}^{\infty}(1+j^2)^\alpha \Phi_j^2 < \infty$, i.e., $\psi \in H_{\omega^*}^{\alpha-1}$ and $\|\psi\|_{H_{\omega^*}^{\alpha-1}} \simeq \|\phi\|_{H_\omega^1}$.

We invoke the above relations, $K(x) \ge k_0 > 0$ and

$$1 \le \frac{\|G_{i+1}^{(\alpha-\beta-1,\beta-1)}\|}{\|G_i^{(\beta,\alpha-\beta)}\|} \le 2$$

to find

$$\begin{aligned} B(\phi,\psi) = &\ \big(K(D\omega\phi(x)), I_{1-r}^{2-\alpha} D(\omega^*\psi(x))\big) \\ = &\Bigg(K\omega^{(\alpha-\beta-1,\beta-1)} \sum_{i=0}^{\infty} (i+1)\mu_i \Phi_i \frac{\|G_{i+1}^{(\alpha-\beta-1,\beta-1)}\|}{\|G_i^{(\alpha-\beta,\beta)}\|} \hat{G}_{i+1}^{(\alpha-\beta-1,\beta-1)}(x), \\ & \sum_{j=0}^{\infty} (j+1)\mu_j \Phi_j \frac{\|G_{j+1}^{(\alpha-\beta-1,\beta-1)}\|}{\|G_j^{(\beta,\alpha-\beta)}\|} \hat{G}_{j+1}^{(\alpha-\beta-1,\beta-1)}(x)\Bigg) \\ \ge &\ k_0 \sum_{i=0}^{\infty} (i+1)^2 \mu_i^2 \Phi_i^2 \frac{\|G_{i+1}^{(\alpha-\beta-1,\beta-1)}\|^2}{\|G_i^{(\beta,\alpha-\beta)}\|^2} \simeq\ k_0 \sum_{i=0}^{\infty} (i+1)^2\, \mu_i^2\, \Phi_i^2 \\ \simeq &\ k_0 \sum_{i=0}^{\infty} (1+i^2)\, \phi_i^2 \ \simeq\ k_0\, \|\phi\|_{H_\omega^1}^2 \ \simeq\ k_0 \|\phi\|_{H_\omega^1} \|\psi\|_{H_{\omega^*}^{\alpha-1}}. \end{aligned}$$

- Space-fractional diffusion equation in two space dimension

$$\nabla \cdot (-\Delta)^{\frac{\alpha-2}{2}} K(x) \nabla \widetilde{u}(x) = f(x)\,, \quad x \in \Omega\,,$$
$$\widetilde{u}(x) = 0\,, \quad x \in \mathbb{R}^2 \backslash \Omega\,.$$

- Ω is the unit disk in $\mathbb{R}^2$, $0 < \alpha < 2$, $K = \mathrm{diag}(k_1, k_2)$, and $(-\Delta)^{\frac{\alpha-2}{2}}$ is the Riesz Potential operator defined by

$$(-\Delta)^{-\frac{\beta}{2}} g(x) = \frac{1}{\gamma_d(\beta)} \int_{\mathbb{R}^d} \frac{g(x-y)}{|y|^{d-\beta}} \, dy, \quad 0 < \beta < d$$

with

$$\gamma_d(\beta) := 2^\beta \, \pi^{d/2} \, \Gamma(\beta/2) / \Gamma((d-\beta)/2)$$

- Note that this is different from the Riesz space-fractional diffusion equation

$$-k_1 \frac{\partial^\alpha u}{\partial |x_1|^\alpha} - k_2 \frac{\partial^\alpha u}{\partial |x_2|^\alpha} = f(x_1, x_2)\,, \quad (x_1, x_2) \in \Omega\,.$$

as $(-\Delta)^{\frac{\alpha-2}{2}} \cdot$ is not a one-dimensional fractional integral operator.

- The solid harmonic polynomials in $\mathbb{R}^d$ are the polynomials in d variables which satisfy Laplace equation. In $\mathbb{R}^2$ the solid harmonic polynomials of degree l can be conveniently written in polar coordinates, $((r,\,\varphi)\,:\,0 \le r < \infty\,,\;0 \le \varphi <\, 2\pi)$, as $\{r^l\cos(l\varphi)\;,\;r^l\sin(l\varphi)\}$. Define $\omega^\gamma\;:=\;(1-r^2)^\gamma$ and

$$V_{l,1}(x) := r^l\cos(l\varphi),\quad l \ge 0 \text{ and } V_{l,-1}(x) :=\; r^l\,\sin(l\varphi),\quad l \ge 1.$$

- In $\mathbb{R}^2$ an orthogonal basis for $L^2_{\omega^\gamma}(\Omega)$ is given via $V_{l,\mu}$ and the Jacobi polynomials $\{P_n^{(\gamma,l)}(\cdot)\}$ on $(-1,1)$

$$\left\{\cup_{l=0}^{\infty}\cup_{n=0}^{\infty}\left\{V_{l,1}(x)\,P_n^{(\gamma,l)}(2r^2-1)\right\}\right\}\cup\left\{\cup_{l=1}^{\infty}\cup_{n=0}^{\infty}\left\{V_{l,-1}(x)\,P_n^{(\gamma,l)}(2r^2-1)\right\}\right\}.$$

- We assume the solution $\widetilde{u}(x) = \omega^{\frac{\alpha}{2}}u(x)$ with

$$u(x) = \sum_{l\ge1,n\ge0,\mu\in\{1,-1\}} u_{l,n,\mu}\,V_{l,\mu}(x)\,P_n^{(\frac{\alpha}{2},l)}(2r^2-1)+\sum_{n\ge0}\frac{u_{0,n,1}}{2}V_{0,1}(x)\,P_n^{(\frac{\alpha}{2},0)}(2r^2-1)$$

and similarly express $f(x)$ with coefficients $f_{l,n,\mu}$.

- To determine $u_{l,n,\mu}$ by comparing both sides of the equation, we need to compute $\nabla\cdot(-\Delta)^{\frac{\alpha-2}{2}}K(x)\nabla\big((1-r^2)^{\frac{\alpha}{2}}V_{l,\mu}P_n^{(\frac{\alpha}{2},l)}(2r^2-1)\big)$.

- Compute $\nabla\big((1-r^2)^{\frac{\alpha}{2}} V_{l,\mu} P_n^{(\frac{\alpha}{2},l)}(2r^2-1)\big)$ by properties of Jacobi polynomials and translation between Cartesian and polar coordinates.
- To act $(-\Delta)^{\frac{\alpha-2}{2}}$ on the resulting expression of $K(x)\nabla\big((1-r^2)^{\frac{\alpha}{2}} V_{l,\mu} P_n^{(\frac{\alpha}{2},l)}(2r^2-1)\big)$, we employ the following relation.

Theorem (Dyda et al., Constr. Approx., 2017)

For $\delta = d + 2l$, s *an integer*, $\frac{\alpha}{2} - s > -1$,
$f(x) = (1-|x|^2)_+^{\frac{\alpha}{2}-s} V_{l,\mu}(x)\, P_n^{(\frac{\alpha}{2}-s\,,\,\frac{\delta}{2}-1)}(2|x|^2-1)$,

$$(-\Delta)^{\frac{\alpha-2}{2}} f(x) = (-1)^{1-s}\, 2^{\alpha-2}\, \frac{\Gamma(n+1-s+\frac{\alpha}{2})\,\Gamma(n-1+\frac{\delta+\alpha}{2})}{\Gamma(n+1)\,\Gamma(n+1-s+\frac{\delta}{2})} \times V_{l,\mu}(x)\, P_{n+1-s}^{(\frac{\alpha}{2}-2+s\,,\,\frac{\delta}{2}-1)}(2|x|^2-1)\,.$$

- We finally differentiate the resulting equation to obtain the expression of $\nabla\cdot(-\Delta)^{\frac{\alpha-2}{2}} K(x)\nabla\big((1-r^2)^{\frac{\alpha}{2}} V_{l,\mu} P_n^{(\frac{\alpha}{2},l)}(2r^2-1)\big)$.

By comparing the coefficients on both sides of the resulting equation and rescaling them by

$$d_{l,n} = \frac{\Gamma(n+1+\frac{\alpha}{2})}{\Gamma(n+1)} u_{l,n,1}, \quad \text{and} \quad \widetilde{f}_{l,n} = 2^{-(\alpha-2)} \frac{\Gamma(n+1+l)}{\Gamma(n+1+\frac{\alpha}{2}+l)} f_{l,n,1},$$

we obtain

$$\begin{aligned}
(n=0, l=0):&\ (k_1+k_2)\, d_{0,0} = \widetilde{f}_{0,0}\\
(n=0, l=1):&\ (2k_1 + (k_1+k_2))\, d_{1,0} = \widetilde{f}_{1,0}\\
\{(n=0, l)\}_{l\geq 2}:&\ 2(k_1+k_2)\, d_{l,0} + (k_1-k_2)\, d_{l-2,1} = \widetilde{f}_{l,0}\\
\{(n, l=0)\}_{n\geq 1}:&\ (k_1-k_2)\, d_{2,n-1} + (k_1+k_2)\, d_{0,n} = \widetilde{f}_{0,n}\\
\{(n, l=1)\}_{n\geq 1}:&\ (k_1-k_2)\, d_{3,n-1} + (2k_1 + (k_1+k_2))\, d_{1,n} = \widetilde{f}_{1,n}\\
\{(n, l)\}_{n\geq 1, l\geq 2}:&\ (k_1-k_2)\, d_{l+2,n-1} + 2(k_1+k_2)\, d_{l,n} + (k_1-k_2)\, d_{l-2,n+1} = \widetilde{f}_{l,n}\,.
\end{aligned}$$

A similar system is generated for the case $\mu = -1$. Thus we only consider the case $\mu = 1$ as above for representation.

Recall the most general case in the above 6 formulas

$$\{(n,l)\}_{n\geq 1, l\geq 2}:\ (k_1-k_2)\,d_{l+2,n-1} + 2(k_1+k_2)\,d_{l,n} + (k_1-k_2)\,d_{l-2,n+1} \ = \ \widetilde{f}_{l,n}\,.$$

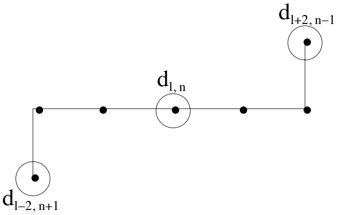

Figure: Stencil illustrating the coupling of the unknowns $d_{l,n}$.

This motivates us to decouple the system into infinite subsystems, each of which is composed by the relations starting from $d_{l+2,n-1} = d_{l_0,0},\, d_{l_0-2,1},\, d_{l_0-4,2}, \cdots$ until $l_0 - 2m = 0$ or $l_0 - 2m = 1$ for some m (Note that since $l_0 - 2m = 0$ and $l_0 - 2m = 1$ correspond to different formulas, thus the coefficient matrices corresponding to even and odd l_0 are slightly different).

Following the aforementioned rule, we obtain decoupled linear systems of equations of sizes: 1, 1, 2, 2, 3, 3, ..., m, m, ... with the coefficient matrices of $m \times m$ $(m \geq 2)$ given as

$$A_m = \begin{bmatrix} 2(k_1+k_2) & (k_1-k_2) & & & \\ (k_1-k_2) & 2(k_1+k_2) & (k_1-k_2) & & \\ & \ddots & & & \\ & & \ddots & & \\ & & (k_1-k_2) & 2(k_1+k_2) & (k_1-k_2) \\ & & & (k_1-k_2) & 2(k_1+k_2) \end{bmatrix}$$

and

$$A_{m,*} = \begin{bmatrix} 2(k_1+k_2) & (k_1-k_2) & & & \\ (k_1-k_2) & 2(k_1+k_2) & (k_1-k_2) & & \\ & \ddots & & & \\ & & \ddots & & \\ & & (k_1-k_2) & 2(k_1+k_2) & (k_1-k_2) \\ & & & (k_1-k_2) & 2k_1+(k_1+k_2) \end{bmatrix}.$$

- As A_m and $A_{m,*}$ are real, symmetric matrices, and hence have real eigenvalues (singular values), a simple application of Gerschgorin's theorem establishes that there exists constants c_{min} and c_{max} such that the minimum and maximum eigenvalues of the matrices satisfy $0 < c_{min} \le \lambda_{min}, \lambda_{max} \le c_{max} < \infty$. Hence A_m and $A_{m,*}$ are uniformly invertible with $\|A_m^{-1}\|_2, \|A_{m,*}^{-1}\|_2 \le c_{min}^{-1}$, i.e. the system is uniquely solvable.
- To estimate the regularity of u, we introduce the weighted function space

$$\mathbf{B}^{s_1,s_2}_{\omega^\gamma}(\Omega) \ := \ \left\{ v \,|\, v \in L^2_{\omega^\gamma}(\Omega) \ \text{ and } \ |v|_{\mathbf{B}^{s_1,s_2}_{\omega^\gamma}(\Omega)} < \infty \right\}, \ s_1, \, s_2 > 0$$

where the semi-norm $|\cdot|_{\mathbf{B}^{s_1,s_2}_{\omega^\gamma}(\Omega)}$ is defined by

$$|v|^2_{\mathbf{B}^{s_1,s_2}_{\omega^\gamma}(\Omega)} \ = \ \sum_{l,n,\mu} \left(l^{2s_1} \, + \, n^{2s_2} \right) \, v^2_{l,n,\mu} \, h^2_{l,n}.$$

Here $h_{l,n}$ refers to the norm of the basis function $V_{l,1}(x) \, P_n^{(\gamma,l)}(2r^2 - 1)$ and the norm is then defined as

$$\|v\|^2_{\mathbf{B}^{s_1,s_2}_{\omega^\gamma}(\Omega)} \ = \ \sum_{l,n,\mu} \left(1 \, + \, l^{2s_1} \, + \, n^{2s_2} \right) \, v^2_{l,n,\mu} \, h^2_{l,n} \, .$$

To bound

$$\sum_{l=1}^{\infty}\sum_{n=0}^{\infty}(1 + l^{2s_1} + n^{2s_2})\, d_{l,n}^2\, h_{l,n}^2$$

we introduce a diagonal matrix W with entries $w_{ii} > 0$ such that

$$W\,A\,W^{-1}\,W\mathbf{d} = W\mathbf{f}, A = A_m \text{ or } A_{m,*}.$$

With $\mathcal{A} = W\,A\,W^{-1}$, note that $\mathcal{A}$ is symmetric, positive definite, and has the same eigenvalues as A. Thus, $c_{max}^{-1} \leq \|\mathcal{A}^{-1}\|_2 \leq c_{min}^{-1}$, and $\|W\mathbf{d}\|_2^2 \lesssim \|W\mathbf{f}\|_2^2$, which can be used to for weighted estimates.

Theorem

For $f(x) \in \mathbf{B}_{\frac{\alpha}{2}}^{s_1,s_2}(\Omega)$ there exists a unique solution $\widetilde{u}(x) = \omega^{\frac{\alpha}{2}}\,u(x)$ to the proposed model with $u(x) \in \mathbf{B}_{\frac{\alpha}{2}}^{s_1+\frac{\alpha}{2},\,s_2+\alpha}(\Omega)$. Furthermore, for $\widetilde{g} = \omega^{\frac{\alpha}{2}}g$ with $g \in L^2_{\omega^{\frac{\alpha}{2}}}(\Omega)$, we have

$$(-\Delta)^{\frac{\alpha}{2}}\widetilde{g} = -\nabla\cdot(-\Delta)^{\frac{\alpha-2}{2}}\nabla\widetilde{g}.$$

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Thank You

for Your Attention!